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A Computational Approach to a Quasi- Minimal Bezier Surface for Computer Graphics

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ABSTRACT
In computer science, the algorithms related to geometry can be exploited in computer aided geometric design, a field in computational geometry. Bézier surfaces are restricted class of surfaces used in computer science, computer graphics and the allied disciplines of science. In this work, a computational approach for finding the Béziersurface related minimal surfaces as the extremal of mean curvature functional is presented. A minimal surface is the surface that locally minimizes its area and has zero mean curvature everywhere. The vanishing mean curvature results in a non-linear partial differential equation for a minimal surface spanned by the boundary of interest and the solution of the partial differential equation does exist for very few special cases whereas, the vanishing condition of the gradient of the area functional is in general not possible as it involves square-root in its integrand. Instead, we find the vanishing condition of the gradient of the mean curvature for a related problem, Plateau-Bézier problem that gives the constraints on the interior control points in terms of boundary control points of the prescribed border. The emerging Bézier surface is the quasi-minimal surface as the extremal of mean curvature.

KEYWORDS
Minimal surfaces, Bézier surfaces, Mean Curvature variation

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1. INTRODUCTION

The Plateau's problem [1–3] (named after the Belgian physicist Joseph A. Plateau consists of finding a minimal surface amongst all the surfaces spanned by the same given boundary, is one of the earliest optimization problems of calculus of variations. The Plateau's problem, the problem of finding a minimal surface has attracted many mathematicians in the field of optimization and they have contributed significantly in the field, Schwarz [4], Riemann [2], Garnier [5] and Weierstrass [2] are few to name in the early 20th century. Lagrange studied the variational problem of determining the surface z = z(x, y) of least area stretched across a given closed contour in 1762, which gave rise to minimal surface theory, the Euler-Lagrange equation for a surface in the form z = z(x, y) is given by $\left(1+z_x^2\right)z_{yy}-2z_xz_yz_{xy}+\left(1+z_y^2\right)z_{xx}=0$, however, the equation has solutions for the special cases not the general solution z = z(x, y), which could be called the minimal surface. Classical examples of minimal surfaces include, the plane, the catenoid obtained by rotating a catenary once around its directrix, the helicoid which is a surface sweptout by a line spinning at a constant speed around an axis perpendicular to the line and travelling at a constant speed along the axis. The early work, however, was a bit confined to minimal surfaces with specific boundaries, until in 1931, Douglas [6] and Rad´o [7] independently proved the existence a of minimal surface over closed contour of curves by finding the extremal of Dirichlet functional rather than the non-linear area functional, the integrand of the area functional involves its square root and in general it is hard to solve the integral. With the use of various numerical techniques in the last century, quasi-minimal surfaces are also used as the solution for Plateau problem. Other energy functionals different from the area functional can be used to find the quasi-minimal surfaces for a certain specific class of surfaces such as B´ezier surfaces which rely upon the Bernstein bases functions and the other forms of bases functions, which are called modified Bezier surfaces. A B´ezier surface is one of the restricted class of surfaces defined for a control net of points $\mathrm{P}=\{P_{pq}\}_{p,q=0}^{m,n}$,

$$\mathbf{x}(u,v)=\sum_{j=0}^{m}\sum_{k=0}^{n}B_j^m(u)B_k^n(v)P_{jk}, \tag{1.1}$$

where $B_j^m(u)=\binom{m}{j}u^j(1-u)^{m-j}$, $j\in[0,1]$, are the Bernstein polynomials, P_{jk} are usually called the control points of Bezier surface. Bezier surface models are used as appropriate tools to describe the physical phenomena and they can assist prediction based computational models in addition to the other possible models meant for effective machine learning of available data [8-11]. A notable problem in geometry is the Plateau Bezier problem to the Bezier surface of least area among all the Bezier surfaces spanned by the given control points of its prescribed border. Monterde [12] exploited the discrete version of Dirichlet functional for obtaining the corresponding minimal Bezier surface as the extremal of the Dirichlet functional. Chen et al. [13] and Hao et al. [14] studied the Plateau-Bezier problem for its solution for Bezier surfaces spanned by boundary curves of higher degree polynomials as the extremal of extended Dirichlet functional.

Monterde and Ugail [15] proposed a general biquadratic functional whichincludes functionals already found in literature for the surfaces as the extremal of this functional, namely the Farinand Hansford functional [16], standard biharmonic functional of Schneider and Kobbelt [17] and Bloor and Wilson's modified biharmonic functional [18]. Ahmad and Masud [19-21] suggested an ansatz for an iterative scheme to find variationally improved surfaces as the extremal of rms of the mean curvature of the surface, in particular a Coons patch, by an iterative scheme to minimize the mean curvature functional. Xu et al. [22] gave quasi-harmonic Bezier surfaces as the better approximation for Plateau-Bezier problem by making use of the fact that minimal surfaces can be associated to harmonic surfaces since a harmonic surface with isothermal parametrization is a minimal surface. The extremals of energy functionals namely the Dirichlet functional, extended Dirichlet functional, quasi-harmonic functional, extended quasi-harmonic and Willmore energy functional [12, 13, 22-26] are useful to obtain the quasi-minimal surfaces for one or the other desired feature of a surface. In our work, we intend to find the corresponding minimal Beezier surface as the extremal of utilize the mean curvature functional by finding the vanishing condition for the gradient of the mean curvature functional that results in constraints on the interior control points as the linear combination of the boundary control points. Since a minimal surface has a zero mean curvature identically, it motivated us to find the extremal of mean curvature functional of a Bezier surface with respect to its inner unspecified control points to obtain a system of equations of control points of a quasi-minimal Bezier surface. The paper is organized as follows: The related preliminary terminology for the Bezier surfaces and useful properties are provided in section 2. In section 3, we discuss the mean curvature functional for a surface and show that how it can be utilized for generating a minimal surface and in the section 4, main scheme of the iterative process is discussed, and conclusion and final remarks are given in section 5.

2. B´EZIER SURFACE AND SOME RELATED PRELIMINARY TERMS

B´ezier surfaces are quite frequently used in computer aided geometric design(CAGD) for its intuitive geometric properties. Let us state few basic results related to Bernstein polynomials, B´ezier surfaces and related integral properties of Bernstein polynomials. The Bernstein polynomial of mth degree form a complete basis function over [0, 1] and are defined as,

$$\mathcal{B}_j^m(u) = \binom{m}{j} u^j (1-u)^{n-j}, \quad j \in [0,1], \tag{2.1}$$

where the binomial coefficient are $\binom{m}{j} = \dfrac{m!}{j!(m-j)!}$, is a polynomial in Bernstein form, named after Sergei Natanovich Bernstein [27]. For instance, the Bernstein polynomials of degree $m=5$, namely $\mathrm{B}_0^5(u), \mathrm{B}_1^5(u), \mathrm{B}_2^5(u), \mathrm{B}_3^5(u), \mathrm{B}_4^5(u)$ and $\mathrm{B}_5^5(u)$ are shown in Figure 1. A Bezier curve is a parametric curve which is used in computer graphics and related fields [12, 28]. The Bezier curve depends on Bernstein polynomials (2.1), used as the blending functions or the basis of a B_ezier curve with a a set of ($n+1$) control points (also called Bezier points) denoted by $\mathrm{P}_0, \mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_n$. A Bezier curve of degree n is given in the form

$$\mathrm{x}(u) = \sum_{j=0}^{m} \mathcal{B}_j^m(u) P_m, \tag{2.2}$$

the coefficients P_m are called Bernstein coefficients or Bezier coefficients and $\mathrm{B}_j^m(u)$ (eq. (2.2)) is the Bernstein operator of order m for $j, m \in z$ ($0 \le j \le m$) for $u \in [0,1]$.

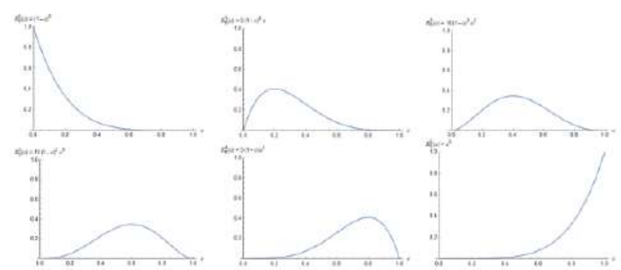

FIG. 1: Bernstein polynomials of degree $n = 5$, $B_0^5(u) = (1-u)^5, B_1^5(u) = 5u(1-u)^4, B_2^5(u) = 10(1-u)^3u^2, B_3^5(u) = 10(1-u)^2u^3, B_4^5(u) = 5(1-u)u^4 and B_5^5(u) = u^5$ are shown.

Bezier surfaces $\mathbf{x}(s,t)$ (eq. (1.1)) are the higher dimensional generalization of B¥'{e}zier curves (2.2 for a given set of $n+1, m+1$ control points $\{P_{jk}\}_{j,k=0}^{m,n}$ for the blending functions $\mathrm{B}_j^n(s)\mathrm{B}_k^m(t) = \mathrm{B}_{j,k}^{n,m}(s,t) : \mathrm{R}^2 \to \mathrm{R}$ where $\mathrm{B}_j^n(s)$ and $\mathrm{B}_k^m(t)$ are Bernstein basis functions given by the eq. (2.1) for $0 \le s,t \le 1$. The product of two Bernstein basis functions $B_j^m(u)$ and $B_k^n(u)$ can be written in terms of the Bernstein basis function of higher degree,

$$B_j^m(u) B_k^n(u) = \frac{\binom{m}{j}\binom{n}{k}}{\binom{m+n}{j+k}} B_{j+k}^{m+n}(u). \tag{2.3}$$

The first derivative of the m^{th} degree Bernstein polynomial is a polynomial,

$$D\left[B_j^m(u)\right] = m\left(B_{j-1}^{m-1}(u) - B_j^{m-1}(u)\right), D \equiv d/du \tag{2.4}$$

of lower degree and it is of degree $m-1$. The second derivative of the m^{th} degree Bernstein polynomial is

$$\frac{d^2}{du^2}\left[B_j^m(u)\right] = m(m-1)\left[B_{j-2}^{m-2}(u) - 2B_{j-1}^{m-2}(u) + B_j^{m-2}(u)\right]. \tag{2.5}$$

We can find the higher derivatives of the Bernstein polynomials by utilizing the eq. (2.4) and (2.5) by the following generalized formula,

$$\mathbf{D}^p B_i^n(u) = \frac{n!}{(n-p)!}\sum_{k=max(0,i+p-n)}^{min(i,p)}(-1)^{k+p}\binom{p}{k}B_{i-k}^{n-p}(u). \tag{2.6}$$

Indefinite integral of Bernstein basis is given as

$$\int B_j^m(u)\,du = \frac{1}{m+1}\sum_{k=j+1}^{m+1}B_k^{m+1}(u), \tag{2.7}$$

whereas all the Bernstein basis function of same order have the same definite integral over the interval $[0,1]$ and it is given by the following expression,

$$\int_0^1 B_j^m(u)\,du = \frac{1}{m+1} \tag{2.8}$$

For a control net of points $\mathrm{P} = \{P_{pq}\}_{p,q=0}^{m,n}$, the *Bézier* surfaces are definedby the eq.(1.1) for $B_j^m(u) = \binom{m}{j}u^j(1-u)^{m-j},\ j\in[0,1]$, the Bernstein polynomials given by eq.(2.1). We need to find the partial derivatives of the *Bézier* patch eq.(1.1) w.r.t its surface parameters u, v and the control points $\left(x_{pq}^a\right)$, $a = 1,2,3$ for the gradient of the mean curvature functional eq.(3.10). Let us write the expression for the partial derivative $\frac{\partial \mathbf{x}_u}{\partial x_{pq}^a}$,

$$\frac{\partial \mathbf{x}_u}{\partial x_{pq}^a} = \frac{\partial}{\partial u}\left(\frac{\partial}{\partial x_{pq}^a}(\mathbf{x})\right) = \frac{\partial}{\partial u}\left(\sum_{p,q=0}^{m,n}B_p^m(u)B_q^n(v)\frac{\partial P_{pq}}{\partial x_{pq}^a}\right), \tag{2.9}$$

where

$$\frac{\partial P_{jk}}{\partial x_{pq}^a} = \frac{\partial}{\partial x_{pq}^a}\left(x_{jk}^a\right) = \frac{\partial}{\partial x_{pq}^a}\left(x_{jk}^1, x_{jk}^2, x_{jk}^3\right) = \begin{cases} e^a, & \forall\ j = p \text{ and } k = q \text{ (basis vectors)} \\ \mathbf{0}, & \forall\ j \neq p \text{ or } k \neq q \text{ (zero vector)} \end{cases}. \tag{2.10}$$

For $\frac{\partial \mathbf{x}_u}{\partial x_{pq}^a} = \frac{\partial}{\partial u}\frac{\partial \mathbf{x}}{\partial x_{pq}^a}$, and using eqs. (1.1) and (2.10), we can express $\frac{\partial \mathbf{x}_u}{\partial x_{pq}^a}$ in terms of Bernstein polynomials as follows

$$\frac{\partial \mathbf{x}_u}{\partial x_{pq}^a} = \frac{\partial}{\partial u}\frac{\partial}{\partial x_{pq}^a}\left(\sum_{j=0}^{m}\sum_{k=0}^{n}B_j^m(u)B_k^n(v)P_{jk}\right) = \frac{\partial}{\partial u}\left(B_p^m(u)B_q^n(v)e^a\right), \forall j = p, k = q, \tag{2.11}$$

knowing that $\frac{\partial}{\partial u}\left(B_p^m(u)\right) = m\left(B_{p-1}^{m-1}(u) - B_p^{m-1}(u)\right)$, the last equation (2.11) can be written as

$$\frac{\partial \mathbf{x}_u}{\partial x_{pq}^a} = mB_q^n(v)\left(B_{p-1}^{m-1}(u) - B_p^{m-1}(u)\right)e^a. \tag{2.12}$$

Similarly the partial derivative $\frac{\partial \mathbf{x}_v}{\partial x_{pq}^a}$ of the *Bézier* patch can be written as

$$\frac{\partial \mathbf{x}_v}{\partial x_{pq}^a} = nB_p^m(u)\left(B_{q-1}^{n-1}(u) - B_q^{n-1}(u)\right)e^a, \tag{2.13}$$

and one of the second order partial derivatives is

$$\begin{aligned}
\frac{\partial \mathbf{x}_{uu}}{\partial x_{pq}^a} &= \frac{\partial}{\partial x_{pq}^a}\left(\sum_{j=0}^{m}\sum_{k=0}^{n}\left(B_j^m(u)\right)_{uu}B_k^n(v)P_{jk}\right) \\
&= \frac{\partial}{\partial x_{pq}^a}\left(\sum_{j=0}^{m}\sum_{k=0}^{n}\left(m(m-1)\left[B_{j-2}^{m-2}(u) - 2B_{j-1}^{m-2}(u) + B_j^{m-2}(u)\right]\right)B_k^n(v)P_{jk}\right) \\
&= m(m-1)\sum_{j=0}^{m}\sum_{k=0}^{n}\left(\left[B_{j-2}^{m-2}(u) - 2B_{j-1}^{m-2}(u) + B_j^{m-2}(u)\right]\right)B_k^n(v)\frac{\partial P_{jk}}{\partial x_{pq}^a} \\
&= m(m-1)\left(\left[B_{p-2}^{m-2}(u) - 2B_{p-1}^{m-2}(u) + B_p^{m-2}(u)\right]\right)B_q^n(v)e^a,
\end{aligned} \tag{2.14}$$

similarly other second order partial derivatives can be computed. Note that, the forward differences of P_{jk} are

$$\begin{aligned}
&\Delta^{10}(P_{j,k}) = P_{j+1,k} - P_{j,k}, \text{ and } \Delta^{01}(P_{j,k}) = P_{j,k+1} - P_{j,k} \\
&\Delta^{20}P_{j,k} = \Delta^{10}(\Delta^{10}(P_{j,k})) = \Delta^{10}(P_{j+1,k} - P_{j,k}) = \Delta^{10}(P_{j+1,k}) - \Delta^{10}(P_{j,k}) = P_{j+2,k} - 2P_{j+1,k} + P_{j,k} \\
&\Delta^{02}P_{j,k} = \Delta^{01}(\Delta^{01}(P_{j,k})) = \Delta^{01}(P_{j,k+1} - P_{j,k}) = \Delta^{01}(P_{j,k+1}) - \Delta^{01}(P_{j,k}) = P_{j,k+2} - 2P_{j,k+1} + P_{j,k} \\
&\Delta^{11}(P_{j,k}) = \Delta^{10}(\Delta^{01}(P_{j,k})) = \Delta^{10}(P_{j,k+1} - P_{j,k}) = \Delta^{10}(P_{j,k+1}) - \Delta^{10}(P_{j,k}) \\
&\Delta^{11}(P_{j,k}) = P_{j+1,k+1} - P_{j,k+1} - P_{j+1,k} + P_{j,k}
\end{aligned} \tag{2.15}$$

For the partial derivatives of the *Bézier* patch w.r.t the surface parameters u and v, note that

$$\begin{aligned}
\mathbf{x}_u(u,v) &= \sum_{j=0}^{m}\sum_{k=0}^{n}\left(B_j^m(u)\right)_u B_k^n(v)P_{jk} \\
&= m\sum_{j=0}^{m}\sum_{k=0}^{n}\left(B_{j-1}^{m-1}(u) - B_j^{m-1}(u)\right)B_k^n(v)P_{jk} \\
&= m\sum_{j=0}^{m-1}\sum_{k=0}^{n}B_j^{m-1}(u)B_k^n(v)\Delta^{10}P_{jk},
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
\mathbf{x}_v(u,v) &= \sum_{g=0}^{m}\sum_{h=0}^{n}B_g^m(u)\left(B_h^n(v)\right)P_{gh} \\
&= n\sum_{g=0}^{m}\sum_{h=0}^{n}B_g^m(u)\left(B_{h-1}^{n-1}(v) - B_h^{n-1}(v)\right)P_{gh} \\
&= n\sum_{g=0}^{m}\sum_{h=0}^{n-1}B_g^m(u)B_h^{n-1}(v)\Delta^{01}P_{gh},
\end{aligned} \tag{2.17}$$

Thus, we have

$$\begin{aligned}
\mathbf{x}_u(u,v) &= m\sum_{j=0}^{m-1}\sum_{k=0}^{n}B_j^{m-1}(u)B_k^n(v)\Delta^{10}P_{jk} \\
\mathbf{x}_v(u,v) &= n\sum_{g=0}^{m}\sum_{h=0}^{n-1}B_g^m(u)B_h^{n-1}(v)\Delta^{01}P_{gh} \\
\mathbf{x}_{uu}(u,v) &= m(m-1)\sum_{r=0}^{m-2}\sum_{l=0}^{n}B_r^{m-2}(u)B_l^n(v)\Delta^{20}P_{rl} \\
\mathbf{x}_{uv}(u,v) &= mn\sum_{r=0}^{m-1}\sum_{l=0}^{n-1}B_r^{m-1}(u)B_l^{n-1}(v)\Delta^{11}P_{rl} \\
\mathbf{x}_{vv}(u,v) &= n(n-1)\sum_{r=0}^{m}\sum_{l=0}^{n-2}B_r^m(u)B_l^{n-2}(v)\Delta^{02}P_{rl}
\end{aligned} \tag{2.18}$$

The partial derivatives given in eqs. (2.18) that of *Bézier* surface eq. (1.1) give us the fundamental coefficients

$$g_{11}(u,v) = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = m^2 \left\langle \left(\sum_{j=0}^{m-1} \sum_{k=0}^{n} B_j^{m-1}(u) B_k^n(v) \right) \Delta^{10} P_{jk}, \left(\sum_{j=0}^{m-1} \sum_{k=0}^{n} B_j^{m-1}(u) B_k^n(v) \right) \Delta^{10} P_{jk} \right\rangle$$
$$= m^2 \left(\sum_{j=0}^{m-1} \sum_{k=0}^{n} B_j^{m-1}(u) B_k^n(v) \right) \left(\sum_{j=0}^{m-1} \sum_{k=0}^{n} B_j^{m-1}(u) B_k^n(v) \right) \langle \Delta^{10} P_{jk}, \Delta^{10} P_{jk} \rangle \tag{2.19}$$

to get

$$\begin{aligned} g_{11}(u,v) &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = m^2 \left(\sum_{j=0}^{m-1} \sum_{k=0}^{n} B_j^{m-1}(u) B_k^n(v) \right)^2 \langle \Delta^{10} P_{jk}, \Delta^{10} P_{jk} \rangle, \\ g_{12}(u,v) &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = mn \left(\sum_{j=0}^{m-1} \sum_{k=0}^{n} B_j^{m-1}(u) B_k^n(v) \right) \left(\sum_{g=0}^{m} \sum_{h=0}^{n-1} B_g^m(u) B_h^{n-1}(v) \right) \langle \Delta^{10} P_{jk}, \Delta^{01} P_{gh} \rangle, \\ g_{22}(u,v) &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = n^2 \left(\sum_{g=0}^{m} \sum_{h=0}^{n-1} B_g^m(u) B_h^{n-1}(v) \right)^2 \langle \Delta^{01} P_{gh}, \Delta^{01} P_{gh} \rangle. \end{aligned} \tag{2.20}$$

Using eqs. (2.18), normal to the *Bézier* surface eq. (1.1) at a point on the surface, in terms of Bernstein polynomials can be

expressed as

$$\mathbf{x}_u \times \mathbf{x}_v = mn \left(\sum_{j=0}^{m-1} \sum_{k=0}^{n} B_j^{m-1}(u) B_k^n(v) \right) \left(\sum_{g=0}^{m} \sum_{h=0}^{n-1} B_g^m(u) B_h^{n-1}(v) \right) (\Delta^{10} P_{jk} \times \Delta^{01} P_{gh}), \tag{2.21}$$

which can be re-written in the form,

$$\mathbf{x}_u \times \mathbf{x}_v = mn \sum_{j,k=0}^{m-1,n} \sum_{g,h=0}^{m,n-1} B_j^{m-1}(u) B_k^n(v) B_g^m(u) B_h^{n-1}(v) (\Delta^{10} P_{jk} \times \Delta^{01} P_{gh}). \tag{2.22}$$

In the section below, we find the quasi-minimal Be´zier surface as the extremal of the mean curvature functional given in the form eq. (3.10).

3. MEAN CURVATURE FUNCTIONAL FOR A SURFACE

Minimal surface problem or Plateau's problem consists of finding a surface with least area among all the possible surfaces spanned by the given closed contour. The basic objective is to extremize the area functional, however the area functional is non-linear as it involves square root in its integrand. Therefore, we aim to extremize a functional with the numerator of the mean curvature H as its integrand. This functional is more convenient than minimizing directly the area functional

$$\mathcal{A}(x) = \int\int_{\mathcal{D}} |\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)| \, du dv, \tag{3.1}$$

where $\mathrm{D} \subset \mathbb{R}^2$ is the parametric domain for the surface $\mathbf{x}(u,v)$, with the boundary condition $\mathbf{x}(\partial \mathrm{D}) = \Gamma$ for $0 \le u, v \le 1$ and $\mathbf{x}_u(u,v)$ and $\mathbf{x}_v(u,v)$ are the partial derivatives of $\mathbf{x}(u,v)$ with respect to u and v. It is known ¥cite{docarmo} that the first variation of $\mathrm{A}(\mathbf{x})$ vanishes if and only if the mean curvature H of $\mathbf{x}(u,v)$ is zero everywhere. Thus a minimal surface is also a surface of least (zero) mean curvature spanning the given boundary. Thus we can use the mean curvature functional for the same surface in place of area functional for the least area. For a locally parameterized surface $\mathbf{x} = \mathbf{x}(u,v)$, the first and the second fundamental coefficients are,

$$g_{11} = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \quad g_{12} = \langle \mathbf{x}_u, \mathbf{x}_v \rangle \quad and \quad g_{22} = \langle \mathbf{x}_v, \mathbf{x}_v \rangle, \tag{3.2}$$

$$b_{11} = \langle \mathbf{n}, \mathbf{x}_{uu} \rangle, \quad b_{12} = \langle \mathbf{n}, \mathbf{x}_{uv} \rangle \quad and \quad b_{22} = \langle \mathbf{n}, \mathbf{x}_{vv} \rangle, \tag{3.3}$$

$$\mathbf{n}(u,v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}, \tag{3.4}$$

is the unit normal to the surface $\mathbf{x}(u,v)$ and

$$H(u,v) = \frac{g_{22} b_{11} - 2 g_{12} b_{12} + g_{11} b_{22}}{2(g_{11} g_{22} - g_{12}^2)}, \tag{3.5}$$

is the mean curvature. The mean curvature(3.5) by virtue of the eqs. (3.2), (3.3) and (3.4) can be written in the following form involving the partial derivatives of the surface,

$$H(u,v) = \frac{\langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{uu} \rangle \langle \mathbf{x}_v, \mathbf{x}_v \rangle - 2 \langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{uv} \rangle \langle \mathbf{x}_u, \mathbf{x}_v \rangle + \langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{vv} \rangle \langle \mathbf{x}_u, \mathbf{x}_u \rangle}{2(g_{11} g_{22} - g_{12}^2)} \tag{3.6}$$

where, the denominator $g_{11} g_{22} - g_{12}^2$ of the mean curvature (eq. (3.5) or (3.6)) is always greater than zero for a surface for the real parameters u and v. For a minimal surface, the mean curvature (eq. (3.5) or (3.6)) vanishes everywhere on the surface which is possible only when the numerator of the mean curvature is zero. Thus, the extremals of the numerator of the mean curvature eq. (3.6) gives a surface of least area. We aim at finding the quasi-minimal *Bézier* surface $\mathbf{x}(u,v)$ (eq. (1.1)) as the extremal of the functional,

$$\mathcal{J}(\mathcal{P}) = \int_0^1 \int_0^1 \mu(u,v) \, du dv, \tag{3.7}$$

where $\mu(u,v)$ is the numerator of the mean curvature eq. (3.6),

$$\mu(u,v) = \mu_1(u,v) - 2\mu_2(u,v) + \mu_3(u,v), \tag{3.8}$$

and $\mu_1(u,v)$, $\mu_2(u,v)$ and $\mu_3(u,v)$ denote its constituent parts for convenience,

$$\mu_1(u,v) = \langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{uu} \rangle \langle \mathbf{x}_v, \mathbf{x}_v \rangle, \quad \mu_2(u,v) = \langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{uv} \rangle \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \quad \mu_3(u,v) = \langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{vv} \rangle \langle \mathbf{x}_u, \mathbf{x}_u \rangle. \tag{3.9}$$

The eq. (3.7) along with the eq. (3.8), is then written in the following convenient form,

$$\mathcal{J}(\mathcal{P}) = \int_R (\mu_1(u,v) - 2\mu_2(u,v) + \mu_3(u,v)) \, du dv. \tag{3.10}$$

The above mentioned mean curvature functional is tested for the special class of surfaces, namely *Bézier* surfaces to get a quasi-minimal *Bézier* surface as the solution of Plateau *Bézier* problem by solving the vanishing condition of gradient of mean curvature functional for the interior control points.

4. EXTREMAL OF MEAN CURVATURE FUNCTIONAL FOR A BEZIER SURFACE

In this section, we find the minimality conditions for a B´ezier surface where the interior control points are controlled by the prescribed boundary control points of the surface patch by utilizing the functional (3.10) and the constraint equation is given in the following theorem.

Theorem 4.1. For the prescribed boundary control points P_{0j}, P_{mj}, P_{i0} and P_{in} in R^3 of the two dimensional B´ezier surface with the coefficients,

$$\beta^{m,p}_{j,r,g_1,g_2} = C^{m-1}_j C^{m-1}_r C^m_{g_1} C^m_{g_2} \left(\frac{C^{m-1}_{p-1}}{C^{5m-3}_{j+r+g_1+g_2+p-1}} - \frac{C^{m-1}_p}{C^{5m-3}_{j+r+g_1+g_2+p}} \right), \tag{4.1}$$

$$\beta^{n,q}_{h,l,k_1,k_2} = C^{n-1}_h C^{n-1}_l C^n_{k_1} C^n_{k_2} \left(\frac{C^{n-1}_{q-1}}{C^{5n-3}_{h+l+k_1+k_2+q-1}} - \frac{C^{n-1}_q}{C^{5n-3}_{h+l+k_1+k_2+q}} \right), \tag{4.2}$$

$$\gamma^{m,p}_{r,g_1,g_2,g_3} = C^{m-2}_r C^m_{g_1} C^m_{g_2} C^m_{g_3} \left(\frac{C^{m-1}_{p-1}}{C^{5m-3}_{r+g_1+g_2+g_3+p-1}} - \frac{C^{m-1}_p}{C^{5m-3}_{r+g_1+g_2+g_3+p}} \right), \tag{4.3}$$

$$\eta^{m,p}_{g_1,g_2,g_3,j} = C^m_{g_1} C^m_{g_2} C^m_{g_3} C^{m-1}_j \left(\frac{C^{m-2}_{p-2}}{C^{5m-3}_{g_1+g_2+g_3+j+p-2}} - 2\frac{C^{m-2}_{p-1}}{C^{5m-3}_{g_1+g_2+g_3+j+p-1}} + \frac{C^{m-2}_p}{C^{5m-3}_{g_1+g_2+g_3+j+p}} \right), \tag{4.4}$$

$$\lambda^{m,p}_{p,j,g_1,g_2,r} = \frac{C^m_p C^{m-1}_j C^m_{g_1} C^m_{g_1} C^{m-2}_r}{C^{5m-3}_{p+j+g_1+g_2+r}}, \alpha^{m,p}_{g,r,j_1,j_2} = \frac{C^m_g C^{m-1}_r C^{m-1}_{j_1} C^{m-1}_{j_2} C^m_p}{C^{5m-3}_{g+r+j_1+j_2+p}}, \alpha^{n,q}_{k,l,h_1,h_2} = \frac{C^n_k C^{n-1}_l C^{n-1}_{h_1} C^{n-1}_{h_2} C^n_q}{C^{5n-3}_{k+l+h_1+h_2+q}}, \tag{4.5}$$

the *Bézier* surface $B^{n,m}_{ij}(u,v)$ is quasi-minimal *Bézier* surface if the inner control points $\{P_{ij}\}^{n-1,m-1}_{i,j}$ satisfy the following constraint the following constraint

$$\begin{aligned}
0 = (n(m-1) &\sum_{j,k=0}^{m-1,n} \sum_{g_1,h_1=0}^{m,n-1} \sum_{g_2,h_2=0}^{m,n-1} \sum_{g_3,h_3=0}^{m,n-1} \sum_{j_1,k_1=0}^{m-1,n} \sum_{j_2,k_2=0}^{m-1,n} \sum_{j_3,k_3=0}^{m-1,n} \sum_{g,h=0}^{m,n-1} \Bigg(\sum_{r,l=0}^{m-2,n} \langle \lambda^{m,p}_{p,j,g_1,g_2,r} \beta^{n,q}_{h_1,h_2,k,l} 2 \langle e^a, \Delta^{01} P_{g_2h_2} \rangle \\
&\langle \Delta^{10} P_{jk} \times \Delta^{01} P_{g_1h_1}, \Delta^{20} P_{rl} \rangle + \alpha^{n,q}_{k,l,h_1,h_2,h_3} \eta^{m,p}_{g_1,g_2,g_3,j} \langle e^a, \Delta^{10} P_{jk} \times \Delta^{01} P_{g_1h_1} \rangle \langle \Delta^{01} P_{g_2h_2}, \Delta^{01} P_{g_3h_3} \rangle + \gamma^{m,p}_{r,g_1,g_2,g_3} \\
&\alpha^{n,q}_{l,h_1,h_2,h_3} \langle \Delta^{20} P_{rl}, e^a \times \Delta^{01} P_{g_1h_1} \rangle \langle \Delta^{01} P_{g_2h_2}, \Delta^{01} P_{g_3h_3} \rangle + \lambda^{m,p}_{p,j,g_1,g_2,r} \beta^{n,q}_{h_1,h_2,k,l} \langle \Delta^{20} P_{rl}, \Delta^{10} P_{jk} \times e^a \rangle \times \\
&\langle \Delta^{01} P_{g_1h_1}, \Delta^{01} P_{g_2h_2} \rangle) - 2mn \sum_{r,l=0}^{m-1,n-1} \left(\alpha^{n,q}_{k,l,h_1,h_2} \beta^{m,p}_{j,r,g_1,g_2} \left(\langle \Delta^{10} P_{jk} \times \Delta^{01} P_{g_1h_1}, \Delta^{11} P_{rl} \rangle \langle e^a, \Delta^{01} P_{g_2h_2} \rangle + \right. \right. \\
&\langle \Delta^{11} P_{rl}, e^a \times \Delta^{01} P_{g_1h_1} \rangle \langle \Delta^{10} P_{jk}, \Delta^{01} P_{g_2h_2} \rangle) + \alpha^{m,p}_{g,r,j_1,j_2} \beta^{n,q}_{h,l,k_1,k_2} \left(\langle \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh}, \Delta^{11} P_{rl} \rangle \langle \Delta^{10} P_{j_2k_2}, e^a \rangle \right. \\
&+ \langle \Delta^{11} P_{rl}, \Delta^{10} P_{j_1k_1} \times e^a \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{01} P_{gh} \rangle) + \beta^{m,p}_{j_1,j_2,g_1,g_2} \beta^{n,q}_{h_1,h_2,k_1,k_2} \langle e^a, \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{g_1h_1} \rangle \times \\
&\langle \Delta^{10} P_{j_2k_2}, \Delta^{01} P_{g_2h_2} \rangle) + m(n-1) \sum_{r,l=0}^{m,n-2} \left(\alpha^{m,p}_{j_1,j_2,j_3,g,p} \eta^{n,q}_{k_1,k_2,k_3,h} \langle e^a, \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{10} P_{j_3k_3} \rangle + \right. \\
&\lambda^{n,q}_{l,h,k_1,k_2} \beta^{m,p}_{j_1,j_2,g,r} \left(\langle \Delta^{02} P_{rl}, e^a \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{j_1k_1}, \Delta^{10} P_{j_2k_2} \rangle + 2 \langle \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh}, \Delta^{02} P_{rl} \rangle \langle e^a, \Delta^{10} P_{j_2k_2} \rangle \right) + \\
&\alpha^{m,p}_{j_1,j_2,j_3,r,p} \gamma^{n,q}_{l,h_1,k_2,k_3} \langle \Delta^{02} P_{rl}, \Delta^{10} P_{j_1k_1} \times e^a \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{10} P_{j_3k_3} \rangle))
\end{aligned} \tag{4.6}$$

Proof:

Let us compute the gradient of this mean curvature functional eq. (3.10) with respect to the coordinates of the inner control points $P_{pq} = (x^1_{pq}, x^2_{pq}, x^3_{pq})$. For any $a \in \{1,2,3\}$, $p \in \{1,...,m-1\}$, $q \in \{1,...,n-1\}$, the gradient of the mean curvature functional is given by

$$\frac{\partial \mathcal{J}(\mathcal{P})}{\partial x^a_{pq}} = \frac{\partial}{\partial x^a_{pq}} \int_R (\mu_1(u,v) - 2\mu_2(u,v) + \mu_3(u,v))\, du dv, \tag{4.7}$$

where $\mu_1(u,v), \mu_2(u,v)$ and $\mu_3(u,v)$ are given in eq. (3.9). We can rewrite the expression for the gradient of the numerator of the mean curvature as

$$\begin{aligned}
\frac{\partial \mathcal{J}(\mathcal{P})}{\partial x^a_{pq}} = &\int_R \left(\frac{\partial(\langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{uu} \rangle)}{\partial x^a_{pq}} \langle \mathbf{x}_v, \mathbf{x}_v \rangle + \langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{uu} \rangle \frac{\partial(\langle \mathbf{x}_v, \mathbf{x}_v \rangle)}{\partial x^a_{pq}} \right) du dv - \\
&2 \int_R \left(\frac{\partial(\langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{uv} \rangle)}{\partial x^a_{pq}} \langle \mathbf{x}_u, \mathbf{x}_v \rangle + \langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{uv} \rangle \frac{\partial(\langle \mathbf{x}_u, \mathbf{x}_v \rangle)}{\partial x^a_{pq}} \right) du dv + \\
&\int_R \left(\frac{\partial(\langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{vv} \rangle)}{\partial x^a_{pq}} \langle \mathbf{x}_u, \mathbf{x}_u \rangle + \langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{vv} \rangle \frac{\partial(\langle \mathbf{x}_u, \mathbf{x}_u \rangle)}{\partial x^a_{pq}} \right) du dv.
\end{aligned} \tag{4.8}$$

The gradient of the first fundamental coefficients of the ***Bézier*** patch w.r.t. the control points $(x_{pq})^a = \{x^1_{pq}, x^2_{pq}, x^3_{pq}\}$ is given by

$$\begin{aligned}
\frac{\partial}{\partial x^a_{pq}} \langle \mathbf{x}_u, \mathbf{x}_u \rangle &= 2m(B^{m-1}_{p-1}(u) - B^{m-1}_p(u)) B^n_q(v) \langle e^a, m \sum_{j,k=0}^{m-1,n} B^{m-1}_j(u) B^n_k(v) \Delta^{10} P_{jk} \rangle, \\
&= 2m^2 \sum_{j,k=0}^{m-1,n} B^{m-1}_j(u) B^n_k(v) (B^{m-1}_{p-1}(u) - B^{m-1}_p(u)) B^n_q(v) \langle e^a, \Delta^{10} P_{jk} \rangle, \\
\frac{\partial}{\partial x^a_{pq}} \langle \mathbf{x}_u, \mathbf{x}_v \rangle &= mn((B^{m-1}_{p-1}(u) - B^{m-1}_p(u)) B^n_q(v) \sum_{g,h=0}^{m,n-1} B^m_g(u) B^{n-1}_h(v) \langle e^a, \Delta^{01} P_{gh} \rangle + \\
&(B^{n-1}_{q-1}(v) - B^{n-1}_q(v)) B^m_p(u) \sum_{j,k=0}^{m-1,n} B^{m-1}_j(u) B^n_k(v) \langle \Delta^{10} P_{jk}, e^a \rangle), \\
\frac{\partial}{\partial x^a_{pq}} \langle \mathbf{x}_v, \mathbf{x}_v \rangle &= 2n(B^{n-1}_{q-1}(v) - B^{n-1}_q(v)) B^m_p(u) \langle e^a, n \sum_{g,h=0}^{m,n-1} B^m_g(u) B^{n-1}_h(v) \Delta^{01} P_{gh} \rangle.
\end{aligned} \tag{4.9}$$

where e^a, $a \in \{1,2,3\}$, denotes the a^{th} vector of the standard basis, i.e. $e^1 = \{1,0,0\}, e^2 = \{0,1,0\}, e^3 = \{0,0,1\}$. The gradient of the normal vector can be computed as,

$$\frac{\partial}{\partial x^a_{pq}} (\mathbf{x}_u \times \mathbf{x}_v) = \frac{\partial}{\partial u} \left(\frac{\partial \mathbf{x}}{\partial x^a_{pq}} \right) \times \mathbf{x}_v + \mathbf{x}_u \times \frac{\partial}{\partial v} \left(\frac{\partial \mathbf{x}}{\partial x^a_{pq}} \right) \tag{4.10}$$

Plugging in the expressions $\partial \mathbf{x} / \partial x^a_{pq} = B^m_p(u) B^n_q(v) e^a$, $\partial B^m_p(u) / \partial u = m\left(B^{m-1}_{p-1}(u) - B^{m-1}_p(u)\right)$ and $\partial B^n_q(v) / \partial v = n\left(B^{n-1}_{q-1}(v) - B^{n-1}_q(v)\right)$ along with the partial derivatives of Bernstein polynomials in above eq. (4.10), reduces it to

$$\begin{aligned}
\frac{\partial(\mathbf{x}_u \times \mathbf{x}_v)}{\partial x^a_{pq}} &= \frac{\partial}{\partial u} \left(\frac{\partial x}{\partial x^a_{pq}} \right) \times \mathbf{x}_v + \mathbf{x}_u \times \frac{\partial}{\partial v} \left(\frac{\partial x}{\partial x^a_{pq}} \right) \\
&= \frac{\partial}{\partial u} \left(B^m_p(u) B^n_q(v) e^a \right) \times \left(n \sum_{g,h=0}^{m,n-1} B^m_g(u) B^{n-1}_h(v) \Delta^{01} P_{gh} \right) \\
&+ \left(m \sum_{j,k=0}^{m-1,n} B^{m-1}_j(u) B^n_k(v) \Delta^{10} P_{jk} \right) \times \frac{\partial}{\partial v} \left(B^m_p(u) B^n_q(v) e^a \right)
\end{aligned} \tag{4.11}$$

this implies,

$$\frac{\partial}{\partial x^a_{pq}} (\mathbf{x}_u \times \mathbf{x}_v) = mn \begin{pmatrix} (B^{m-1}_{p-1}(u) - B^{m-1}_p(u)) B^n_q(v) \sum_{g,h=0}^{m,n-1} B^m_g(u) B^{n-1}_h(v) (e^a \times \Delta^{01} P_{gh}) \\ + (B^{n-1}_{q-1}(v) - B^{n-1}_q(v)) B^m_p(u) \sum_{j,k=0}^{m-1,n} B^{m-1}_j(u) B^n_k(v) (\Delta^{10} P_{jk} \times e^a) \end{pmatrix}. \tag{4.12}$$

The gradient of the term $\left\langle \mathbf{x}_{uu}, \mathbf{x}_u \times \mathbf{x}_v \right\rangle$ can be computed as

$$\frac{\partial}{\partial x^a_{pq}}\left(\langle \mathbf{x}_{uu}, \mathbf{x}_u \times \mathbf{x}_v\rangle\right) = \left\langle \frac{\partial \mathbf{x}_{uu}}{\partial x^a_{pq}}, \mathbf{x}_u \times \mathbf{x}_v \right\rangle + \left\langle \mathbf{x}_{uu}, \frac{\partial \mathbf{x}_u \times \mathbf{x}_v}{\partial x^a_{pq}} \right\rangle. \tag{4.13}$$

By inserting $\partial \mathbf{x}_{uu} / \partial x^a_{pq} = m(m-1)\left(B^{m-2}_{p-2}(u) - 2B^{m-2}_{p-1}(u) + B^{m-2}_{p}(u)\right)B^n_q(v)$ along with the gradient of normal vector as computed in eq. (4,12), above equation reduces to

$$\frac{\partial\left(\langle \mathbf{x}_{uu}, \mathbf{x}_u \times \mathbf{x}_v\rangle\right)}{\partial x^a_{pq}} = m^2(m-1)n \begin{pmatrix} \left(B^{m-2}_{p-2}(u) - 2B^{m-2}_{p-1}(u) + B^{m-2}_{p}(u)\right) B^n_q(v) \sum\limits_{j,k=0}^{m-1,n} B^{m-1}_j(u)\, B^n_k(v) \sum\limits_{g,h=0}^{m,n-1} B^m_g(u) \times \\ B^{n-1}_h(v) \langle e^a, \Delta^{10}P_{jk} \times \Delta^{01}P_{gh}\rangle + \sum\limits_{r,l=0}^{m-2,n} B^{m-2}_r(u)\, B^n_l(v) \left(B^{m-1}_{p-1}(u) - B^{m-1}_p(u)\right) \times \\ B^n_q(v) \sum\limits_{g,h=0}^{m,n-1} B^m_g(u) B^{n-1}_h(v) \langle \Delta^{20}P_{rl}, e^a \times \Delta^{01}P_{gh}\rangle + \sum\limits_{r,l=0}^{m-2,n} B^{m-2}_r(u)\, B^n_l(v) \times \\ B^m_p(u) \left(B^{n-1}_{q-1}(v) - B^{n-1}_q(v)\right) \sum\limits_{j,k=0}^{m-1,n} B^{m-1}_j(u)\, B^n_k(v) \langle \Delta^{20}P_{rl}, \Delta^{10}P_{jk} \times e^a\rangle \end{pmatrix}. \tag{4.14}$$

Knowing that,

$$\frac{\partial \mathbf{x}_{uv}}{\partial x^a_{pq}} = mne^a \left(B^{n-1}_{q-1}(v) - B^{n-1}_q(v)\right)\left(B^{m-1}_{p-1}(u) - B^{m-1}_p(u)\right), \tag{4.15}$$

helps us to find,

$$\frac{\partial\left(\langle \mathbf{x}_{uv}, \mathbf{x}_u \times \mathbf{x}_v\rangle\right)}{\partial x^a_{pq}} = m^2n^2 \begin{pmatrix} \left(B^{n-1}_{q-1}(v) - B^{n-1}_q(v)\right)\left(B^{m-1}_{p-1}(u) - B^{m-1}_p(u)\right) \sum\limits_{j,k=0}^{m-1,n} B^{m-1}_j(u) B^n_k(v) \sum\limits_{g,h=0}^{m,n-1} B^m_g(u) B^{n-1}_h(v) \\ \langle e^a, \Delta^{10}P_{jk} \times \Delta^{01}P_{gh}\rangle + \sum\limits_{r,l=0}^{m-1,n-1} B^{m-1}_r(u)\, B^{n-1}_l(v) \left(B^{m-1}_{p-1}(u) - B^{m-1}_p(u)\right) B^n_q(v) \\ \sum\limits_{g,h=0}^{m,n-1} B^m_g(u) B^{n-1}_h(v) \langle \Delta^{11}P_{rl}, e^a \times \Delta^{01}P_{gh}\rangle + \sum\limits_{r,l=0}^{m-1,n-1} B^{m-1}_r(u) B^{n-1}_l(v) \left(B^{n-1}_{q-1}(v) - B^{n-1}_q(v)\right) \\ B^m_p(u) \sum\limits_{j,k=0}^{m-1,n} B^{m-1}_j(u) B^n_k(v) \langle \Delta^{11}P_{rl}, \Delta^{10}P_{jk} \times e^a\rangle \end{pmatrix} \tag{4.16}$$

Similarly, knowing that

$$\partial \mathbf{x}_{vv} / \partial x^a_{pq} = n(n-1)\left(B^{n-2}_{q-2}(v) - 2B^{n-2}_{q-1}(v) + B^{n-2}_q(v)\right) B^m_p(u) \tag{4.17}$$

gives us

$$\frac{\partial\left(\langle \mathbf{x}_{vv}, \mathbf{x}_u \times \mathbf{x}_v\rangle\right)}{\partial x^a_{pq}} = n^2(n-1)m \begin{pmatrix} \left(B^{n-2}_{q-2}(v) - 2B^{n-2}_{q-1}(v) + B^{n-2}_q(v)\right) B^m_p(u) \sum\limits_{j,k=0}^{m-1,n} B^{m-1}_j(u)\, B^n_k(v) \sum\limits_{g,h=0}^{m,n-1} B^m_g(u) \times \\ B^{n-1}_h(v) \langle e^a, \Delta^{10}P_{jk} \times \Delta^{01}P_{gh}\rangle + \sum\limits_{r,l=0}^{m,n-2} B^m_r(u)\, B^{n-2}_l(v) \left(B^{m-1}_{p-1}(u) - B^{m-1}_p(u)\right) \times \\ B^n_q(v) \left(\sum\limits_{g,h=0}^{m,n-1} B^m_g(u) B^{n-1}_h(v)\right) \langle \Delta^{02}P_{rl}, e^a \times \Delta^{01}P_{gh}\rangle + \sum\limits_{r,l=0}^{m,n-2} B^m_r(u)\, B^{n-2}_l(v)\, B^m_p(u) \\ \left(B^{n-1}_{q-1}(v) - B^{n-1}_q(v)\right) \sum\limits_{j,k=0}^{m-1,n} B^{m-1}_j(u)\, B^n_k(v) \langle \Delta^{02}P_{rl}, \Delta^{10}P_{jk} \times e^a\rangle \end{pmatrix} \tag{4.18}$$

Therefore, gradient of the first term of the mean curvature functional, eq. (4.8) can be computed by substituting the eqs . (4.14) and (4.9) in it, so that

$$\frac{\partial}{\partial x^a_{pq}}\mu_1(u,v) = \langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{uu}\rangle \frac{\partial(\langle \mathbf{x}_v, \mathbf{x}_v\rangle)}{\partial x^a_{pq}} + \langle \mathbf{x}_v, \mathbf{x}_v\rangle \frac{\partial(\langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{uu}\rangle)}{\partial x^a_{pq}} \tag{4.19}$$

and thus,

$$\begin{aligned} &\frac{\partial}{\partial x^a_{pq}}\mu_1(u,v) = \\ &2n^3m^2(m-1)\left(B^{n-1}_{q-1}(v) - B^{n-1}_q(v)\right) B^m_p(u) \sum_{j,k=0}^{m-1,n} B^{m-1}_j(u)\, B^n_k(v) \left(\sum_{g,h=0}^{m,n-1} B^m_g(u)\, B^{n-1}_h(v)\right)^2 \sum_{r,l=0}^{m-2,n} B^{m-2}_r(u)\, B^n_l(v) \\ &\langle \Delta^{10}P_{jk} \times \Delta^{01}P_{gh}, \Delta^{20}P_{rl}\rangle \langle e^a, \Delta^{01}P_{gh}\rangle + m^2(m-1)n^3 \left(\sum_{g,h=0}^{m,n-1} B^m_g(u) B^{n-1}_h(v)\right)^3 \left(B^{m-2}_{p-2}(u) - 2B^{m-2}_{p-1}(u) + B^{m-2}_p(v)\right) \\ &B^n_q(v) \sum_{j,k=0}^{m-1,n} B^{m-1}_j(u)\, B^n_k(v) \langle e^a, \Delta^{10}P_{jk} \times \Delta^{01}P_{gh}\rangle \langle \Delta^{01}P_{gh}, \Delta^{01}P_{gh}\rangle + \sum_{r,l=0}^{m-2,n} B^{m-2}_r(u)\, B^n_l(v) \left(B^{m-1}_{p-1}(u) - B^{m-1}_p(u)\right) \\ &B^n_q(v) \left(\sum_{g,h=0}^{m,n-1} B^m_g(u) B^{n-1}_h(v)\right)^3 \langle \Delta^{20}P_{rl}, e^a \times \Delta^{01}P_{gh}\rangle \langle \Delta^{01}P_{gh}, \Delta^{01}P_{gh}\rangle + \sum_{r,l=0}^{m-2,n} B^{m-2}_r(u)\, B^n_l(v)\, B^m_p(u) \\ &\left(B^{n-1}_{q-1}(v) - B^{n-1}_q(v)\right) \sum_{j,k=0}^{m-1,n} B^{m-1}_j(u)\, B^n_k(v) \left(\sum_{g,h=0}^{m,n-1} B^m_g(u) B^{n-1}_h(v)\right)^2 \langle \Delta^{20}P_{rl}, \Delta^{10}P_{jk} \times e^a\rangle \langle \Delta^{01}P_{gh}, \Delta^{01}P_{gh}\rangle \end{aligned} \tag{4.20}$$

and thus,

$$\begin{aligned} &\frac{\partial}{\partial x^a_{pq}}\mu_1(u,v) = \\ &2n^3m^2(m-1)\left(B^{n-1}_{q-1}(v) - B^{n-1}_q(v)\right) B^m_p(u) \sum_{j,k=0}^{m-1,n} B^{m-1}_j(u)\, B^n_k(v) \sum_{g_1,h_1=0}^{m,n-1} B^m_{g_1}(u)\, B^{n-1}_{h_1}(v) \sum_{g_2,h_2=0}^{m,n-1} B^m_{g_2}(u)\, B^{n-1}_{h_2}(v) \\ &\sum_{r,l=0}^{m-2,n} B^{m-2}_r(u)\, B^n_l(v) \langle \Delta^{10}P_{jk} \times \Delta^{01}P_{g_1h_1}, \Delta^{20}P_{rl}\rangle \langle e^a, \Delta^{01}P_{g_2h_2}\rangle + m^2(m-1)n^3 \sum_{g_1,h_1=0}^{m,n-1} B^m_{g_1}(u)\, B^{n-1}_{h_1}(v) \sum_{g_2,h_2=0}^{m,n-1} B^m_{g_2}(u) \times \\ &B^{n-1}_{h_2}(v) \sum_{g_3,h_3=0}^{m,n-1} B^m_{g_3}(u)\, B^{n-1}_{h_3}(v) \left(B^{m-2}_{p-2}(u) - 2B^{m-2}_{p-1}(u) + B^{m-2}_p(v)\right) B^n_q(v) \sum_{j,k=0}^{m-1,n} B^{m-1}_j(u)\, B^n_k(v) \langle e^a, \Delta^{10}P_{jk} \times \Delta^{01}P_{g_1h_1}\rangle \\ &\langle \Delta^{01}P_{g_2h_2}, \Delta^{01}P_{g_3h_3}\rangle + \sum_{r,l=0}^{m-2,n} B^{m-2}_r(u)\, B^n_l(v) \left(B^{m-1}_{p-1}(u) - B^{m-1}_p(u)\right) B^n_q(v) \sum_{g_1,h_1=0}^{m,n-1} B^m_{g_1}(u)\, B^{n-1}_{h_1}(v) \sum_{g_2,h_2=0}^{m,n-1} B^m_{g_2}(u)\, B^{n-1}_{h_2}(v) \\ &\sum_{g_3,h_3=0}^{m,n-1} B^m_{g_3}(u)\, B^{n-1}_{h_3}(v) \langle \Delta^{20}P_{rl}, e^a \times \Delta^{01}P_{g_1h_1}\rangle \langle \Delta^{01}P_{g_2h_2}, \Delta^{01}P_{g_3h_3}\rangle + \sum_{r,l=0}^{m-2,n} B^{m-2}_r(u)\, B^n_l(v)\, B^m_p(u) \left(B^{n-1}_{q-1}(v) - B^{n-1}_q(v)\right) \\ &\sum_{j,k=0}^{m-1,n} B^{m-1}_j(u)\, B^n_k(v) \sum_{g_1,h_1=0}^{m,n-1} B^m_{g_1}(u)\, B^{n-1}_{h_1}(v) \sum_{g_2,h_2=0}^{m,n-1} B^m_{g_2}(u)\, B^{n-1}_{h_2}(v) \langle \Delta^{20}P_{rl}, \Delta^{10}P_{jk} \times e^a\rangle \langle \Delta^{01}P_{g_1h_1}, \Delta^{01}P_{g_2h_2}\rangle \end{aligned} \tag{4.21}$$

Let us integrate the above expressions by using the property of Bernsteinbasis polynomials as mentioned in eqs. (2.3) and (2.8), we get

$$\begin{aligned} &\int\limits_R \frac{\partial}{\partial x^a_{pq}}\mu_1(u,v)\,du\,dv = \\ &\frac{n^3(m-1)m^2}{(5m-2)(5n-2)}\Bigg(\sum_{j,k=0}^{m-1,n} \sum_{g_1,h_1=0}^{m,n-1} \sum_{g_2,h_2=0}^{m,n-1} \sum_{r,l=0}^{m-2,n} \frac{\binom{m}{p}\binom{m-1}{j}\binom{m}{g_1}\binom{m}{g_2}\binom{m-2}{r}}{\binom{5m-3}{p+j+g_1+g_2+r}} \binom{n}{k}\binom{n-1}{h_1}\binom{n-1}{h_2}\binom{n}{l} \times \\ &\left(\frac{\binom{n-1}{q-1}}{\binom{5n-3}{k+h_1+h_2+l+q-1}} - \frac{\binom{n-1}{q}}{\binom{5n-3}{k+h_1+h_2+l+q}}\right) 2\langle \Delta^{10}P_{jk} \times \Delta^{01}P_{g_1h_1}, \Delta^{20}P_{rl}\rangle \langle e^a, \Delta^{01}P_{g_2h_2}\rangle + \sum_{g_1,h_1=0}^{m,n-1} \sum_{g_2,h_2=0}^{m-1,n} \\ &\sum_{g_3,h_3=0}^{m,n-1} \sum_{j,k=0}^{m-1,n} \binom{m}{g_1}\binom{m}{g_2}\binom{m}{g_3}\binom{m-1}{j} \frac{\binom{n-1}{h_1}\binom{n-1}{h_2}\binom{n-1}{h_3}\binom{n}{k}\binom{n}{q}}{\binom{5n-3}{h_1+h_2+h_3+k+q}} \Bigg(\frac{\binom{m-2}{p-2}}{\binom{5m-3}{g_1+g_2+g_3+j+p-2}} - \\ &2 \times \frac{\binom{m-2}{p-1}}{\binom{5m-3}{g_1+g_2+g_3+j+p-1}} + \frac{\binom{m-2}{p}}{\binom{5m-3}{g_1+g_2+g_3+j+p}}\Bigg) \langle e^a, \Delta^{10}P_{jk} \times \Delta^{01}P_{g_1h_1}\rangle \langle \Delta^{01}P_{g_2h_2}, \Delta^{01}P_{g_3h_3}\rangle + \\ &\sum_{r,l=0}^{m-2,n} \sum_{g_1,h_1=0}^{m,n-1} \sum_{g_2,h_2=0}^{m,n-1} \sum_{g_3,h_3=0}^{m,n-1} \binom{m-2}{r}\binom{m}{g_1}\binom{m}{g_2}\binom{m}{g_3}\binom{n}{l}\binom{n-1}{h_1}\binom{n-1}{h_2}\binom{n-1}{h_3}\binom{n}{q} \Bigg(\frac{\binom{m-1}{p-1}}{\binom{5m-3}{r+g_1+g_2+g_3+p-1}} - \\ &\frac{\binom{m-1}{p}}{\binom{5m-3}{r+g_1+g_2+g_3+p}}\Bigg) \langle \Delta^{20}P_{rl}, e^a \times \Delta^{01}P_{g_1h_1}\rangle \langle \Delta^{01}P_{g_2h_2}, \Delta^{01}P_{g_3h_3}\rangle + \sum_{r,l=0}^{m-2,n} \sum_{j,k=0}^{m-1,n} \sum_{g_1,h_1=0}^{m,n-1} \sum_{g_2,h_2=0}^{m,n-1} \binom{m-2}{r}\binom{m-1}{j} \\ &\binom{m}{g_1}\binom{m}{g_2}\binom{m}{p}\binom{n}{l}\binom{n}{k}\binom{n-1}{h_1}\binom{n-1}{h_2} \left(\frac{\binom{n-1}{q-1}}{\binom{5n-3}{l+k+h_1+h_2+q-1}} - \frac{\binom{n-1}{q}}{\binom{5n-3}{l+k+h_1+h_2+q}}\right) \\ &\langle \Delta^{20}P_{rl}, \Delta^{10}P_{jk} \times e^a\rangle \langle \Delta^{01}P_{g_1h_1}, \Delta^{01}P_{g_2h_2}\rangle\Bigg) \end{aligned} \tag{4.22}$$

and hence

$$\begin{aligned}
&\int_R \frac{\partial}{\partial x^a_{pq}} \mu_1(u,v)\,dudv = \\
&\frac{n^3(m-1)m^2}{(5m-2)(5n-2)} \sum_{j,k=0}^{m-1,n}\sum_{r,l=0}^{m-2,n}\sum_{g_1,h_1=0}^{m,n-1}\sum_{g_2,h_2=0}^{m,n-1}\sum_{g_3,h_3=0}^{m,n-1} \Bigg(\frac{\binom{m}{p}\binom{m-1}{j}\binom{m}{g_1}\binom{m}{g_2}\binom{m-2}{r}}{\binom{5m-3}{p+j+g_1+g_2+r}} \binom{n}{k}\binom{n-1}{h_1}\binom{n-1}{h_2}\binom{n}{l} \\
&\left(\frac{\binom{n-1}{q-1}}{\binom{5n-3}{k+h_1+h_2+l+q-1}} - \frac{\binom{n-1}{q}}{\binom{5n-3}{k+h_1+h_2+l+q}}\right) 2\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1},\Delta^{20}P_{rl}\rangle\langle e^a,\Delta^{01}P_{g_2h_2}\rangle + \binom{m}{g_1}\binom{m}{g_2}\times \\
&\binom{m}{g_3}\binom{m-1}{j}\frac{\binom{n-1}{h_1}\binom{n-1}{h_2}\binom{n-1}{h_3}\binom{n}{k}\binom{n}{q}}{\binom{5n-3}{h_1+h_2+h_3+k+q}}\left(\frac{\binom{m-2}{p-2}}{\binom{5m-3}{g_1+g_2+g_3+j+p-2}} - 2\frac{\binom{m-2}{p-1}}{\binom{5m-3}{g_1+g_2+g_3+j+p-1}} + \right. \\
&\left.\frac{\binom{m-2}{p}}{\binom{5m-3}{g_1+g_2+g_3+j+p}}\right)\langle e^a,\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle + \binom{m-2}{r}\binom{m}{g_1}\binom{m}{g_2}\binom{m}{g_3}\binom{n}{l}\binom{n-1}{h_1}\times \\
&\frac{\binom{n-1}{h_2}\binom{n-1}{h_3}\binom{n}{q}}{\binom{5n-3}{l+h_1+h_2+h_3+q}}\left(\frac{\binom{m-1}{p-1}}{\binom{5m-3}{r+g_1+g_2+g_3+p-1}} - \frac{\binom{m-1}{p}}{\binom{5m-3}{r+g_1+g_2+g_3+p}}\right)\langle\Delta^{20}P_{rl},e^a\times\Delta^{01}P_{g_1h_1}\rangle \\
&\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle + \frac{\binom{m-2}{r}\binom{m-1}{j}\binom{m}{g_1}\binom{m}{g_2}\binom{m}{p}}{\binom{5m-3}{r+j+g_1+g_2+g_3+p}}\binom{n}{l}\binom{n}{k}\binom{n-1}{h_1}\binom{n-1}{h_2}\left(\frac{\binom{n-1}{q-1}}{\binom{5n-3}{l+k+h_1+h_2+q-1}} - \right. \\
&\left.\frac{\binom{n-1}{q}}{\binom{5n-3}{l+k+h_1+h_2+q}}\right)\langle\Delta^{20}P_{rl},\Delta^{10}P_{jk}\times e^a\rangle\langle\Delta^{01}P_{g_1h_1},\Delta^{01}P_{g_2h_2}\rangle\Bigg)
\end{aligned} \qquad (4.23)$$

Above equations may be re-written as

$$\begin{aligned}
&\int_R \frac{\partial}{\partial x^a_{pq}} \mu_1(u,v)\,dudv = \\
&\frac{n^3(m-1)m^2}{(5m-2)(5n-2)} \sum_{j,k=0}^{m-1,n}\sum_{g_1,h_1=0}^{m,n-1}\sum_{g_2,h_2=0}^{m,n-1}\sum_{g_3,h_3=0}^{m,n-1}\sum_{r,l=0}^{m-2,n} \Big(C^{n-1}_{h_1}C^{n-1}_{h_2}C^{n}_{k_1}C^{n}_{k_2}\frac{C^m_pC^{m-1}_jC^m_{g_1}C^m_{g_2}C^{m-2}_r}{C^{5m-3}_{p+j+g_1+g_2+r}}\left(\frac{C^{n-1}_{q-1}}{C^{5n-3}_{h_1+h_2+k_1+k_2+q-1}}\right. \\
&\left.-\frac{C^{n-1}_{q}}{C^{5n-3}_{h_1+h_2+k_1+k_2+q}}\right)2\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1},\Delta^{20}P_{rl}\rangle\langle e^a,\Delta^{01}P_{g_2h_2}\rangle + \frac{C^n_kC^{n-1}_{h_1}C^{n-1}_{h_2}C^{n-1}_{h_3}C^n_q}{C^{5n-3}_{k+h_1+h_2+h_3+q}}C^m_{g_1}C^m_{g_2}C^m_{g_3}C^{m-1}_j \\
&\left(\frac{C^{m-2}_{p-2}}{C^{5m-3}_{g_1+g_2+g_3+j+p-2}} - 2\frac{C^{m-2}_{p-1}}{C^{5m-3}_{g_1+g_2+g_3+j+p-1}} + \frac{C^{m-2}_{p}}{C^{5m-3}_{g_1+g_2+g_3+j+p}}\right)\langle e^a,\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle + \\
&C^{m-2}_rC^m_{g_1}C^m_{g_2}C^m_{g_3}\left(\frac{C^{m-1}_{p-1}}{C^{5m-3}_{r+g_1+g_2+g_3+p-1}} - \frac{C^{m-1}_{p}}{C^{5m-3}_{r+g_1+g_2+g_3+p}}\right)\frac{C^n_kC^{n-1}_lC^{n-1}_{h_1}C^{n-1}_{h_2}C^n_q}{C^{5n-3}_{k+l+h_1+h_2+q}}\langle\Delta^{20}P_{rl},e^a\times\Delta^{01}P_{g_1h_1}\rangle\times \\
&\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle + \frac{C^m_pC^{m-1}_jC^m_{g_1}C^m_{g_2}C^{m-2}_r}{C^{5m-3}_{p+j+g_1+g_2+r}}C^{n-1}_{h_1}C^{n-1}_{h_2}C^n_{k_1}C^n_{k_2}\left(\frac{C^{n-1}_{q-1}}{C^{5n-3}_{h_1+h_2+k_1+k_2+q-1}} - \frac{C^{n-1}_{q}}{C^{5n-3}_{h_1+h_2+k_1+k_2+q}}\right)\times \\
&\langle\Delta^{20}P_{rl},\Delta^{10}P_{jk}\times e^a\rangle\langle\Delta^{01}P_{g_1h_1},\Delta^{01}P_{g_2h_2}\rangle\Big)
\end{aligned} \qquad (4.24)$$

so that

$$\begin{aligned}
&\int_R \frac{\partial}{\partial x^a_{pq}} \mu_1(u,v)\,dudv = \\
&\frac{n^3(m-1)m^2}{(5m-2)(5n-2)} \sum_{j,k=0}^{m-1,n}\sum_{g_1,h_1=0}^{m,n-1}\sum_{g_2,h_2=0}^{m,n-1}\sum_{g_3,h_3=0}^{m,n-1}\sum_{r,l=0}^{m-2,n}\Big[\lambda^{m,p}_{p,j,g_1,g_2,r}\beta^{n,q}_{h_1,h_2,k,l}2\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1},\Delta^{20}P_{rl}\rangle\langle e^a,\Delta^{01}P_{g_2h_2}\rangle \\
&+\alpha^{n,q}_{k,h_1,h_2,h_3}\eta^{m,p}_{g_1,g_2,g_3,j}\langle e^a,\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle + \gamma^{m,p}_{r,g_1,g_2,g_3}\alpha^{n,q}_{l,h_1,h_2,h_3}\langle\Delta^{20}P_{rl},e^a\times\Delta^{01}P_{g_1h_1}\rangle\times \\
&\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle + \lambda^{m,p}_{p,j,g_1,g_2,r}\beta^{n,q}_{h_1,h_2,k,l}\langle\Delta^{20}P_{rl},\Delta^{10}P_{jk}\times e^a\rangle\langle\Delta^{01}P_{g_1h_1},\Delta^{01}P_{g_2h_2}\rangle\Big]
\end{aligned} \qquad (4.25)$$

reduces to

$$\begin{aligned}
&\int_R \frac{\partial}{\partial x^a_{pq}} \mu_1(u,v)\,dudv = \\
&\frac{n^3(m-1)m^2}{(5m-2)(5n-2)} \sum_{j,k=0}^{m-1,n}\sum_{g_1,h_1=0}^{m,n-1}\sum_{g_2,h_2=0}^{m,n-1}\sum_{g_3,h_3=0}^{m,n-1}\sum_{r,l=0}^{m-2,n}\Big(\lambda^{m,p}_{p,j,g_1,g_2,r}\beta^{n,q}_{h_1,h_2,k,l}\big(2\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1},\Delta^{20}P_{rl}\rangle\langle e^a,\Delta^{01}P_{g_2h_2}\rangle + \\
&\langle\Delta^{20}P_{rl},\Delta^{10}P_{jk}\times e^a\rangle\langle\Delta^{01}P_{g_1h_1},\Delta^{01}P_{g_2h_2}\rangle\big) + \alpha^{n,q}_{k,h_1,h_2,h_3}\eta^{m,p}_{g_1,g_2,g_3,j}\langle e^a,\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle + \\
&\gamma^{m,p}_{r,g_1,g_2,g_3}\alpha^{n,q}_{l,h_1,h_2,h_3}\langle\Delta^{20}P_{rl},e^a\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle\Big)
\end{aligned} \qquad (4.26)$$

where the coefficients $\alpha^{m,p}_{g,r,j_1,j_2}$, $\beta^{m,p}_{j_1,j_2,g_1,g_2}$, $\gamma^{m,p}_{r,g_1,g_2,g_3}$, $\lambda^{m,p}_{p,j,g_1,g_2,r}$ and $\eta^{m,p}_{g_1,g_2,g_3,j}$ as given in the eqs. (4.1) to (4.5). Let us find now the gradient of the second term of the mean curvature functional eq. (4.8),

$$\frac{\partial}{\partial x^a_{pq}}\mu_2(u,v) = \langle \mathbf{x}_u\times\mathbf{x}_v,\mathbf{x}_{uv}\rangle\frac{\partial(\langle\mathbf{x}_u,\mathbf{x}_v\rangle)}{\partial x^a_{pq}} + \langle\mathbf{x}_u,\mathbf{x}_v\rangle\frac{\partial(\langle\mathbf{x}_u\times\mathbf{x}_v,\mathbf{x}_{uv}\rangle)}{\partial x^a_{pq}} \qquad (4.27)$$

and hence

$$\begin{aligned}
&\frac{\partial}{\partial x^a_{pq}}\mu_2(u,v) = \\
&m^3n^3\Big(\sum_{j,k=0}^{m-1,n}B^{m-1}_j(u)B^n_k(v)\sum_{r,l=0}^{m-1,n-1}B^{m-1}_r(u)B^{n-1}_l(v)\big((B^{m-1}_{p-1}(u)-B^{m-1}_p(u)\big)B^n_q(v)\Big(\sum_{g,h=0}^{m,n-1}B^m_g(u)B^{n-1}_h(v)\Big)^2 \\
&\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{gh},\Delta^{11}P_{rl}\rangle\langle e^a,\Delta^{01}P_{gh}\rangle + \sum_{g,h=0}^{m,n-1}B^m_g(u)B^{n-1}_h(v)\sum_{r,l=0}^{m-1,n-1}B^{m-1}_r(u)B^{n-1}_l(v)\big(B^{n-1}_{q-1}(v)-B^{n-1}_q(v)\big) \\
&B^m_p(u)\Big(\sum_{j,k=0}^{m-1,n}B^{m-1}_j(u)B^n_k(v)\Big)^2\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{gh},\Delta^{11}P_{rl}\rangle\langle\Delta^{10}P_{jk},e^a\rangle + \big(B^{n-1}_{q-1}(v)-B^{n-1}_q(v)\big)\big(B^{m-1}_{p-1}(u) \\
&-B^{m-1}_p(u)\big)\Big(\sum_{j,k=0}^{m-1,n}B^{m-1}_j(u)B^n_k(v)\Big)^2\Big(\sum_{g,h=0}^{m,n-1}B^m_g(u)B^{n-1}_h(v)\Big)^2\langle e^a,\Delta^{10}P_{jk}\times\Delta^{01}P_{gh}\rangle\langle\Delta^{10}P_{jk},\Delta^{01}P_{gh}\rangle + \\
&\sum_{r,l=0}^{m-1,n-1}B^{m-1}_r(u)B^{n-1}_l(v)\big(B^{m-1}_{p-1}(u)-B^{m-1}_p(u)\big)B^n_q(v)\sum_{j,k=0}^{m-1,n}B^{m-1}_j(u)B^n_k(v)\Big(\sum_{g,h=0}^{m,n-1}B^m_g(u)B^{n-1}_h(v)\Big)^2 \\
&\langle\Delta^{11}P_{rl},e^a\times\Delta^{01}P_{gh}\rangle\langle\Delta^{10}P_{jk},\Delta^{01}P_{gh}\rangle + \sum_{r,l=0}^{m-1,n-1}B^{m-1}_r(u)B^{n-1}_l(v)\big(B^{n-1}_{q-1}(v)-B^{n-1}_q(v)\big)B^m_p(u)\sum_{g,h=0}^{m,n-1}B^m_g(u) \\
&B^{n-1}_h(v)\Big(\sum_{j,k=0}^{m-1,n}B^{m-1}_j(u)B^n_k(v)\Big)^2\langle\Delta^{11}P_{rl},\Delta^{10}P_{jk}\times e^a\rangle\langle\Delta^{10}P_{jk},\Delta^{01}P_{gh}\rangle\Big)
\end{aligned} \qquad (4.28)$$

$$\begin{aligned}
&\frac{\partial}{\partial x^a_{pq}}\mu_2(u,v) = \\
&m^3n^3\sum_{j,k=0}^{m-1,n}\sum_{r,l=0}^{m-1,n-1}\sum_{g,h=0}^{m,n-1}\sum_{g_1,h_1=0}^{m,n-1}\sum_{g_2,h_2=0}^{m,n-1}\sum_{j_1,k_1=0}^{m-1,n}\sum_{j_2,k_2=0}^{m-1,n}\big(B^{m-1}_j(u)B^n_k(v)B^{m-1}_r(u)B^{n-1}_l(v)B^m_{g_1}(u)B^{n-1}_{h_1}(v)B^m_{g_2}(u)B^{n-1}_{h_2}(v) \\
&B^n_q(v)(B^{m-1}_{p-1}(u)-B^{m-1}_p(u))\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{gh},\Delta^{11}P_{rl}\rangle\langle e^a,\Delta^{01}P_{gh}\rangle + B^m_g(u)B^{n-1}_h(v)B^{m-1}_r(u)B^{n-1}_l(v)B^{m-1}_{j_1}(u) \\
&B^n_{k_1}(v)B^{m-1}_{j_2}(v)(B^{n-1}_{q-1}(v)-B^{n-1}_q(v))B^m_p(u)\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{gh},\Delta^{11}P_{rl}\rangle\langle\Delta^{10}P_{jk},e^a\rangle + (B^{n-1}_{q-1}(v)-B^{n-1}_q(v))\times \\
&(B^{m-1}_{p-1}(u)-B^{m-1}_p(u))B^{m-1}_{j_1}(u)B^n_{k_1}(v)B^{m-1}_{j_2}(u)B^n_{k_2}(v)B^m_{g_1}(u)B^{n-1}_{h_1}(v)B^m_{g_2}(u)B^{n-1}_{h_2}(v)\langle e^a,\Delta^{10}P_{jk}\times\Delta^{01}P_{gh}\rangle\times \\
&\langle\Delta^{10}P_{jk},\Delta^{01}P_{gh}\rangle + (B^{m-1}_{p-1}(u)-B^{m-1}_p(u))B^n_q(v)B^{m-1}_r(u)B^{n-1}_l(v)B^{m-1}_j(u)B^n_k(v)B^m_{g_1}(u)B^{n-1}_{h_1}(v)B^m_{g_2}(u)B^{n-1}_{h_2}(v) \\
&\langle\Delta^{11}P_{rl},e^a\times\Delta^{01}P_{gh}\rangle\langle\Delta^{10}P_{jk},\Delta^{01}P_{gh}\rangle + (B^{n-1}_{q-1}(v)-B^{n-1}_q(v))B^m_p(u)B^m_g(u)B^{n-1}_h(v)B^{m-1}_r(u)B^{n-1}_l(v)B^{m-1}_{j_1}(u) \\
&B^n_{k_1}(v)B^{m-1}_{j_2}(u)B^n_{k_2}(v)\langle\Delta^{11}P_{rl},\Delta^{10}P_{jk}\times e^a\rangle\langle\Delta^{10}P_{jk},\Delta^{01}P_{gh}\rangle\big)
\end{aligned} \qquad (4.29)$$

$$\begin{aligned}
&\int_R \frac{\partial}{\partial x^a_{pq}} \mu_2(u,v)\,dudv = \\
&\frac{m^3n^3}{(5n-2)(5m-2)}\sum_{j,k=0}^{m-1,n}\sum_{r,l=0}^{m-1,n-1}\sum_{g,h=0}^{m,n-1}\sum_{g_1,h_1=0}^{m,n-1}\sum_{g_2,h_2=0}^{m,n-1}\sum_{j_1,k_1=0}^{m-1,n}\sum_{j_2,k_2=0}^{m-1,n}\Big(\binom{m-1}{j}\binom{m-1}{r}\binom{m}{g_1}\binom{m}{g_2}\frac{\binom{n}{k}\binom{n-1}{l}\binom{n-1}{h_1}}{\binom{5n-3}{k+l+h_1+h_2+q}} \\
&\binom{n-1}{h_2}\binom{n}{q}\left(\frac{\binom{m-1}{p-1}}{\binom{5m-3}{j+r+g_1+g_2+p-1}} - \frac{\binom{m-1}{p}}{\binom{5m-3}{j+r+g_1+g_2+p}}\right)\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1},\Delta^{11}P_{rl}\rangle\langle e^a,\Delta^{01}P_{g_2h_2}\rangle + \binom{m}{g} \\
&\frac{\binom{m-1}{r}\binom{m}{p}\binom{m-1}{j_1}\binom{m-1}{j_2}}{\binom{5m-3}{g+r+p+j_1+j_2}}\binom{n-1}{h}\binom{n-1}{l}\binom{n}{k_1}\binom{n}{k_2}\left(\frac{\binom{n-1}{q-1}}{\binom{5n-3}{h+l+k_1+k_2+q-1}} - \frac{\binom{n-1}{q}}{\binom{5n-3}{h+l+k_1+k_2+q}}\right) \\
&\langle\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{gh},\Delta^{11}P_{rl}\rangle\langle\Delta^{10}P_{j_2k_2},e^a\rangle + \binom{m-1}{j_1}\binom{m-1}{j_2}\binom{m}{g_1}\binom{m}{g_2}\binom{n}{k_1}\binom{n}{k_2}\binom{n-1}{h_1}\binom{n-1}{h_2} \\
&\left(\frac{\binom{n-1}{q-1}}{\binom{5n-3}{k_1+k_2+h_1+h_2+q-1}} - \frac{\binom{n-1}{q}}{\binom{5n-3}{k_1+k_2+h_1+h_2+q}}\right)\left(\frac{\binom{m-1}{p-1}}{\binom{5m-3}{j_1+j_2+g_1+g_2+p-1}} - \frac{\binom{m-1}{p}}{\binom{5m-3}{j_1+j_2+g_1+g_2+p}}\right) \\
&\langle e^a,\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{10}P_{j_2k_2},\Delta^{01}P_{g_2h_2}\rangle + \binom{n-1}{l}\binom{n}{q}\binom{n}{k}\binom{n-1}{h_1}\binom{n-1}{h_2}\binom{m-1}{r}\binom{m-1}{j}\binom{m}{g_1}\binom{m}{g_2} \\
&\left(\frac{\binom{m-1}{p-1}}{\binom{5m-3}{r+j+g_1+g_2+p-1}} - \frac{\binom{m-1}{p}}{\binom{5m-3}{r+j+g_1+g_2+p}}\right)\langle\Delta^{11}P_{rl},e^a\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{10}P_{jk},\Delta^{01}P_{g_2h_2}\rangle + \binom{m-1}{r}\binom{m}{p}\binom{m}{g} \\
&\binom{m-1}{j_1}\binom{m-1}{j_2}\binom{n-1}{l}\binom{n-1}{h}\binom{n}{k_1}\binom{n}{k_2}\left(\frac{\binom{n-1}{q-1}}{\binom{5n-3}{l+h+k_1+k_2+q-1}} - \frac{\binom{n-1}{q}}{\binom{5n-3}{l+h+k_1+k_2+q}}\right) \\
&\langle\Delta^{11}P_{rl},\Delta^{10}P_{j_1k_1}\times e^a\rangle\langle\Delta^{10}P_{j_2k_2},\Delta^{01}P_{gh}\rangle\Big),
\end{aligned} \qquad (4.30)$$

which can be written as

$$\begin{aligned}
&\int_R \frac{\partial}{\partial x^a_{pq}} \mu_2(u,v)\,dudv = \\
&\frac{m^3n^3}{(5n-2)(5m-2)}\sum_{j,k=0}^{m-1,n}\sum_{r,l=0}^{m-1,n-1}\sum_{g,h=0}^{m,n-1}\sum_{g_1,h_1=0}^{m,n-1}\sum_{g_2,h_2=0}^{m,n-1}\sum_{j_1,k_1=0}^{m-1,n}\sum_{j_2,k_2=0}^{m-1,n}\big(C^{m-1}_jC^{m-1}_rC^m_{g_1}C^m_{g_2}C^n_kC^{n-1}_lC^{n-1}_{h_1}C^{n-1}_{h_2}C^n_q \\
&\left(\frac{C^{m-1}_{p-1}}{C^{5m-3}_{j+r+g_1+g_2+p-1}} - \frac{C^{m-1}_{p}}{C^{5m-3}_{j+r+g_1+g_2+p}}\right)\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1},\Delta^{11}P_{rl}\rangle\langle e^a,\Delta^{01}P_{g_2h_2}\rangle + \frac{C^m_gC^m_pC^{m-1}_rC^{m-1}_{j_1}C^{m-1}_{j_2}}{C^{5m-3}_{g+p+r+j_1+j_2}} \\
&C^{n-1}_hC^{n-1}_lC^n_{k_1}C^n_{k_2}\left(\frac{C^{n-1}_{q-1}}{C^{5n-3}_{h+l+k_1+k_2+q-1}} - \frac{C^{n-1}_{q}}{C^{5n-3}_{h+l+k_1+k_2+q}}\right)\langle\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{gh},\Delta^{11}P_{rl}\rangle\langle\Delta^{10}P_{j_2k_2},e^a\rangle + C^{m-1}_{j_1}C^{m-1}_{j_2} \\
&C^m_{g_1}C^m_{g_2}C^n_{k_1}C^n_{k_2}C^{n-1}_{h_1}C^{n-1}_{h_2}\left(\frac{C^{n-1}_{q-1}}{C^{5n-3}_{k_1+k_2+h_1+h_2+q-1}} - \frac{C^{n-1}_{q}}{C^{5n-3}_{k_1+k_2+h_1+h_2+q}}\right)\left(\frac{C^{m-1}_{p-1}}{C^{5m-3}_{j_1+j_2+g_1+g_2+p-1}} - \frac{C^{m-1}_{p}}{C^{5m-3}_{j_1+j_2+g_1+g_2+p}}\right)\times \\
&\langle e^a,\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{10}P_{j_2k_2},\Delta^{01}P_{g_2h_2}\rangle + C^{n-1}_lC^{m-1}_rC^m_{g_1}C^m_{g_2}C^n_kC^{n-1}_{h_1}C^{n-1}_{h_2}C^n_q\left(\frac{C^{m-1}_{p-1}}{C^{5m-3}_{j+r+g_1+g_2+p-1}}\right. \\
&\left.-\frac{C^{m-1}_{p}}{C^{5m-3}_{j+r+g_1+g_2+p}}\right)\langle\Delta^{11}P_{rl},e^a\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{10}P_{jk},\Delta^{01}P_{g_2h_2}\rangle + \frac{C^m_gC^m_pC^{m-1}_rC^{m-1}_{j_1}C^{m-1}_{j_2}}{C^{5m-3}_{g+p+r+j_1+j_2}}C^{n-1}_hC^{n-1}_lC^n_{k_1}C^n_{k_2} \\
&\left(\frac{C^{n-1}_{q-1}}{C^{5n-3}_{h+l+k_1+k_2+q-1}} - \frac{C^{n-1}_{q}}{C^{5n-3}_{h+l+k_1+k_2+q}}\right)\langle\Delta^{11}P_{rl},\Delta^{10}P_{j_1k_1}\times e^a\rangle\langle\Delta^{10}P_{j_2k_2},\Delta^{01}P_{gh}\rangle\big)
\end{aligned} \qquad (4.31)$$

$$
\begin{aligned}
&\int_R \frac{\partial}{\partial x^a_{pq}} \mu_2(u,v)\,du\,dv = \\
&\frac{m^3 n^3}{(5n-2)(5m-2)} \sum_{j,k=0}^{m-1,n} \sum_{r,l=0}^{m-1,n-1} \sum_{g,h=0}^{m,n-1} \sum_{g_1,h_1=0}^{m,n-1} \sum_{g_2,h_2=0}^{m,n-1} \sum_{j_1,k_1=0}^{m-1,n} \sum_{j_2,k_2=0}^{m-1,n} (C_j^{m-1} C_r^{m-1} C_{g_1}^m C_{g_2}^m C_k^n C_l^{n-1} C_{h_1}^{n-1} C_{h_2}^{n-1} C_q^n \\
&\left(\frac{C_{p-1}^{m-1}}{C^{5m-3}_{j+r+g_1+g_2+p-1}} - \frac{C_p^{m-1}}{C^{5m-3}_{j+r+g_1+g_2+p}} \right) (\langle \Delta^{10} P_{jk} \times \Delta^{01} P_{g_1h_1}, \Delta^{11} P_{rl} \rangle \langle e^a, \Delta^{01} P_{g_2h_2} \rangle + \langle \Delta^{11} P_{rl}, e^a \times \Delta^{01} P_{g_1h_1} \rangle \times \\
&\langle \Delta^{10} P_{jk}, \Delta^{01} P_{g_2h_2} \rangle) + \frac{C_g^m C_p^m C_r^{m-1} C_{j_1}^{m-1} C_{j_2}^{m-1}}{C^{5m-3}_{g+p+r+j_1+j_2}} C_h^{n-1} C_l^{n-1} C_{k_1}^n C_{k_2}^n \left(\frac{C_{q-1}^{n-1}}{C^{5n-3}_{h+l+k_1+k_2+q-1}} - \frac{C_q^{n-1}}{C^{5n-3}_{h+l+k_1+k_2+q}} \right) \\
&(\langle \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh}, \Delta^{11} P_{rl} \rangle \langle \Delta^{10} P_{j_2k_2}, e^a \rangle + \langle \Delta^{11} P_{rl}, \Delta^{10} P_{j_1k_1} \times e^a \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{01} P_{gh} \rangle) + C_{j_1}^{m-1} C_{j_2}^{m-1} \\
&C_{g_1}^m C_{g_2}^m C_{k_1}^n C_{k_2}^n C_{h_1}^{n-1} C_{h_2}^{n-1} \left(\frac{C_{q-1}^{n-1}}{C^{5n-3}_{k_1+k_2+h_1+h_2+q-1}} - \frac{C_q^{n-1}}{C^{5n-3}_{k_1+k_2+h_1+h_2+q}} \right) \left(\frac{C_{p-1}^{m-1}}{C^{5m-3}_{j_1+j_2+g_1+g_2+p-1}} - \frac{C_p^{m-1}}{C^{5m-3}_{j_1+j_2+g_1+g_2+p}} \right) \times \\
&\langle e^a, \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{g_1h_1} \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{01} P_{g_2h_2} \rangle)
\end{aligned} \tag{4.32}
$$

this implies,

$$
\begin{aligned}
&\int_R \frac{\partial}{\partial x^a_{pq}} \mu_2(u,v)\,du\,dv = \\
&\frac{m^3 n^3}{(5n-2)(5m-2)} \sum_{j,k=0}^{m-1,n} \sum_{r,l=0}^{m-1,n-1} \sum_{g,h=0}^{m,n-1} \sum_{g_1,h_1=0}^{m,n-1} \sum_{g_2,h_2=0}^{m,n-1} \sum_{j_1,k_1=0}^{m-1,n} \sum_{j_2,k_2=0}^{m-1,n} (\alpha^{n,q}_{k,l,h_1,h_2} \beta^{m,p}_{j,r,g_1,g_2} (\langle \Delta^{10} P_{jk} \times \Delta^{01} P_{g_1h_1}, \Delta^{11} P_{rl} \rangle \times \\
&\langle e^a, \Delta^{01} P_{g_2h_2} \rangle + \langle \Delta^{11} P_{rl}, e^a \times \Delta^{01} P_{g_1h_1} \rangle \langle \Delta^{10} P_{jk}, \Delta^{01} P_{g_2h_2} \rangle) + \alpha^{m,p}_{g,r,j_1,j_2} \beta^{n,q}_{h,l,k_1,k_2} (\langle \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh}, \Delta^{11} P_{rl} \rangle \times \\
&\langle \Delta^{10} P_{j_2k_2}, e^a \rangle + \langle \Delta^{11} P_{rl}, \Delta^{10} P_{j_1k_1} \times e^a \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{01} P_{gh} \rangle) + \beta^{m,p}_{j_1,j_2,g_1,g_2} \beta^{n,q}_{h_1,h_2,k_1,k_2} \langle e^a, \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{g_1h_1} \rangle \times \\
&\langle \Delta^{10} P_{j_2k_2}, \Delta^{01} P_{g_2h_2} \rangle)
\end{aligned} \tag{4.33}
$$

where the coefficients $\alpha^{m,p}_{g,r,j_1,j_2}$, $\beta^{m,p}_{j_1,j_2,g_1,g_2}$ are given in the eqs. (4.1) to (4.5). Finally, we determine the gradient of the third term of mean curvature functional,

$$
\frac{\partial}{\partial x^a_{pq}} \mu_3(u,v) = \frac{\partial(\langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{vv} \rangle)}{\partial x^a_{pq}} \langle \mathbf{x}_u, \mathbf{x}_u \rangle + \langle \mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{vv} \rangle \frac{\partial(\langle \mathbf{x}_u, \mathbf{x}_u \rangle)}{\partial x^a_{pq}}, \tag{4.34}
$$

So that we can find

$$
\begin{aligned}
&\frac{\partial}{\partial x^a_{pq}} \mu_3(u,v) = \\
&m^3 n^2 (n-1) (\left(\sum_{j,k=0}^{m-1,n} B_j^{m-1}(u) B_k^n(v) \right)^3 \sum_{g,h=0}^{m,n-1} B_g^m(u) B_h^{n-1}(v) ((B_{q-2}^{n-2}(v) - 2B_{q-1}^{n-2}(v) + B_q^{n-2}(v)) B_p^m(u) \times \\
&\langle e^a, \Delta^{10} P_{jk} \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{jk}, \Delta^{10} P_{jk} \rangle + \sum_{g,h=0}^{m,n-1} B_g^m(u) B_h^{n-1}(v) \sum_{r,l=0}^{m,n-2} B_r^m(u) B_l^{n-2}(v) (B_{p-1}^{m-1}(u) - B_p^{m-1}(u)) \\
&B_q^n(v) \left(\sum_{j,k=0}^{m-1,n} B_j^{m-1}(u) B_k^n(v) \right)^2 \langle \Delta^{02} P_{rl}, e^a \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{jk}, \Delta^{10} P_{jk} \rangle + B_p^m(u) (B_{q-1}^{n-1}(v) - B_q^{n-1}(v)) \times \\
&\left(\sum_{j,k=0}^{m-1,n} B_j^{m-1}(u) B_k^n(v) \right)^3 \sum_{r,l=0}^{m,n-2} B_r^m(u) B_l^{n-2}(v) \langle \Delta^{02} P_{rl}, \Delta^{10} P_{jk} \times e^a \rangle \langle \Delta^{10} P_{jk}, \Delta^{10} P_{jk} \rangle + \sum_{r,l=0}^{m,n-2} B_r^m(u) B_l^{n-2}(v) \\
&(B_{p-1}^{m-1}(u) - B_p^{m-1}(u)) B_q^n(v) \left(\sum_{j,k=0}^{m-1,n} B_j^{m-1}(u) B_k^n(v) \right)^2 \sum_{g,h=0}^{m,n-1} B_g^m(u) B_h^{n-1}(v) 2 \langle \Delta^{10} P_{jk} \times \Delta^{01} P_{gh}, \Delta^{02} P_{rl} \rangle \times \\
&\langle e^a, \Delta^{10} P_{jk} \rangle)
\end{aligned} \tag{4.35}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial x^a_{pq}} \mu_3(u,v) = \\
&m^3 n^2 (n-1) \sum_{j_1,k_1=0}^{m-1,n} \sum_{j_2,k_2=0}^{m-1,n} \sum_{j_3,k_3=0}^{m-1,n} \sum_{g,h=0}^{m,n-1} \sum_{r,l=0}^{m,n-2} (B_{j_1}^{m-1}(u) B_{k_1}^n(v) B_{j_2}^{m-1}(u) B_{k_2}^n(v) B_{j_3}^{m-1}(u) B_{k_3}^n(v) B_g^m(u) B_h^{n-1}(v) B_p^m(u) \\
&(B_{q-2}^{n-2}(v) - 2B_{q-1}^{n-2}(v) + B_q^{n-2}(v)) \langle e^a, \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{10} P_{j_3k_3} \rangle + B_g^m(u) B_h^{n-1}(v) B_r^m(u) B_l^{n-2}(v) \\
&B_{j_1}^{m-1}(u) B_{k_1}^n(v) B_{j_2}^{m-1}(u) B_{k_2}^n(v) (B_{p-1}^{m-1}(u) - B_p^{m-1}(u)) B_q^n(v) \langle \Delta^{02} P_{rl}, e^a \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{j_1k_1}, \Delta^{10} P_{j_2k_2} \rangle + B_{j_1}^{m-1}(u) \\
&B_{k_1}^n(v) B_{j_2}^{m-1}(u) B_{k_2}^n(v) B_{j_3}^{m-1}(u) B_{k_3}^n(v) B_r^m(u) B_l^{n-2}(v) B_p^m(u) (B_{q-1}^{n-1}(v) - B_q^{n-1}(v)) \langle \Delta^{02} P_{rl}, \Delta^{10} P_{j_1k_1} \times e^a \rangle \times \\
&\langle \Delta^{10} P_{j_2k_2}, \Delta^{10} P_{j_3k_3} \rangle + B_r^m(u) B_l^{n-2}(v) B_{j_1}^{m-1}(u) B_{k_1}^n(v) B_{j_2}^{m-1}(u) B_{k_2}^n(v) B_g^m(u) B_h^{n-1}(v) (B_{p-1}^{m-1}(u) - B_p^{m-1}(u)) \\
&B_q^n(v)\, 2 \times \langle \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh}, \Delta^{02} P_{rl} \rangle \langle e^a, \Delta^{10} P_{j_2k_2} \rangle)
\end{aligned} \tag{4.36}
$$

$$
\begin{aligned}
&\int_R \frac{\partial}{\partial x^a_{pq}} \mu_3(u,v)\,du\,dv = \\
&\frac{m^3 n^2 (n-1)}{(5n-2)(5m-2)} \sum_{j_1,k_1=0}^{m-1,n} \sum_{j_2,k_2=0}^{m-1,n} \sum_{j_3,k_3=0}^{m-1,n} \sum_{g,h=0}^{m,n-1} \sum_{r,l=0}^{m,n-2} \left(\frac{\binom{m-1}{j_1}\binom{m-1}{j_2}\binom{m-1}{j_3}\binom{m}{g}\binom{m}{p}}{\binom{5m-3}{j_1+j_2+j_3+g+p}} \binom{n}{k_1}\binom{n}{k_2}\binom{n}{k_3}\binom{n-1}{h} \right. \\
&\left(\frac{\binom{n-2}{q-2}}{\binom{5n-3}{k_1+k_2+k_3+h+q-2}} - 2\frac{\binom{n-2}{q-1}}{\binom{5n-3}{k_1+k_2+k_3+h+q-1}} + \frac{\binom{n-2}{q}}{\binom{5n-3}{k_1+k_2+k_3+h+q}} \right) \langle e^a, \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh} \rangle \\
&\langle \Delta^{10} P_{j_2k_2}, \Delta^{10} P_{j_3k_3} \rangle + \binom{m-1}{j_1}\binom{m-1}{j_2}\binom{m}{g}\binom{m}{r} \frac{\binom{n}{k_1}\binom{n}{k_2}\binom{n-2}{l}\binom{n-1}{h}\binom{n}{q}}{\binom{5n-3}{k_1+k_2+l+h+q}} \\
&\left(\frac{\binom{m-1}{p-1}}{\binom{5m-3}{j_1+j_2+g+r+p-1}} + \frac{\binom{m-1}{p}}{\binom{5m-3}{j_1+j_2+g+r+p}} \right) \langle \Delta^{02} P_{rl}, e^a \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{j_1k_1}, \Delta^{10} P_{j_2k_2} \rangle + \binom{n}{k_1}\binom{n}{k_2} \\
&\binom{n}{k_3}\binom{n-2}{l} \frac{\binom{m-1}{j_1}\binom{m-1}{j_2}\binom{m-1}{j_3}\binom{m}{r}\binom{m}{p}}{\binom{5m-3}{j_1+j_2+j_3+r+p}} \left(\frac{\binom{n-1}{q-1}}{\binom{5n-3}{k_1+k_2+k_3+l+q-1}} + \frac{\binom{n-1}{q}}{\binom{5n-3}{k_1+k_2+k_3+l+q}} \right) \\
&\langle \Delta^{02} P_{rl}, \Delta^{10} P_{j_1k_1} \times e^a \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{10} P_{j_3k_3} \rangle + \binom{m-1}{j_1}\binom{m-1}{j_2}\binom{m}{g}\binom{m}{r} \frac{\binom{n}{k_1}\binom{n}{k_2}\binom{n-2}{l}\binom{n-1}{h}\binom{n}{q}}{\binom{5n-3}{k_1+k_2+l+h+q}} \\
&\left. \left(\frac{\binom{m-1}{p-1}}{\binom{5m-3}{j_1+j_2+g+r+p-1}} + \frac{\binom{m-1}{p}}{\binom{5m-3}{j_1+j_2+g+r+p}} \right) 2 \times \langle \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh}, \Delta^{02} P_{rl} \rangle \langle e^a, \Delta^{10} P_{j_2k_2} \rangle \right)
\end{aligned} \tag{4.37}
$$

$$
\begin{aligned}
&\int_R \frac{\partial}{\partial x^a_{pq}} \mu_3(u,v)\,du\,dv = \\
&\frac{m^3 n^2 (n-1)}{(5n-2)(5m-2)} \sum_{j_1,k_1=0}^{m-1,n} \sum_{j_2,k_2=0}^{m-1,n} \sum_{j_3,k_3=0}^{m-1,n} \sum_{g,h=0}^{m,n-1} \sum_{r,l=0}^{m,n-2} \left(\frac{C_{j_1}^{m-1} C_{j_2}^{m-1} C_{j_3}^{m-1} C_g^m C_p^m}{C^{5m-3}_{j_1+j_2+j_3+g+p}} C_{k_1}^n C_{k_2}^n C_{k_3}^n C_h^{n-1} \left(\frac{C_{q-2}^{n-2}}{C^{5n-3}_{k_1+k_2+k_3+h+q-2}} - 2 \times \right. \right. \\
&\left. \frac{C_{q-1}^{n-2}}{C^{5n-3}_{k_1+k_2+k_3+h+q-1}} + \frac{C_q^{n-2}}{C^{5n-3}_{k_1+k_2+k_3+h+q}} \right) \langle e^a, \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{10} P_{j_3k_3} \rangle + \frac{C_{k_1}^n C_{k_2}^n C_l^{n-2} C_h^{n-1} C_q^n}{C^{5n-3}_{k_1+k_2+l+h+q}} C_{j_1}^{m-1} C_{j_2}^{m-1} \\
&C_g^m C_r^m \left(\frac{C_{p-1}^{m-1}}{C^{5m-3}_{j_1+j_2+g+r+p-1}} - \frac{C_p^{m-1}}{C^{5m-3}_{j_1+j_2+g+r+p}} \right) \langle \Delta^{02} P_{rl}, e^a \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{j_1k_1}, \Delta^{10} P_{j_2k_2} \rangle + \frac{C_{j_1}^{m-1} C_{j_2}^{m-1} C_{j_3}^{m-1} C_r^m C_p^m}{C^{5m-3}_{j_1+j_2+j_3+r+p}} C_{k_1}^n \\
&C_{k_2}^n C_{k_3}^n C_l^{n-2} \left(\frac{C_{q-1}^{n-1}}{C^{5n-3}_{k_1+k_2+k_3+l+q-1}} - \frac{C_q^{n-1}}{C^{5n-3}_{k_1+k_2+k_3+l+q}} \right) \langle \Delta^{02} P_{rl}, \Delta^{10} P_{j_1k_1} \times e^a \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{10} P_{j_3k_3} \rangle + C_{j_1}^{m-1} C_{j_2}^{m-1} C_g^m C_r^m \\
&\left. \frac{C_{k_1}^n C_{k_2}^n C_l^{n-2} C_h^{n-1} C_q^n}{C^{5n-3}_{k_1+k_2+l+h+q}} \left(\frac{C_{p-1}^{m-1}}{C^{5m-3}_{j_1+j_2+g+r+p-1}} - \frac{C_p^{m-1}}{C^{5m-3}_{j_1+j_2+g+r+p}} \right) 2 \times \langle \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh}, \Delta^{02} P_{rl} \rangle \langle e^a, \Delta^{10} P_{j_2k_2} \rangle \right)
\end{aligned} \tag{4.38}
$$

$$
\begin{aligned}
&\int_R \frac{\partial}{\partial x^a_{pq}} \mu_3(u,v)\,du\,dv = \\
&\frac{m^3 n^2 (n-1)}{(5n-2)(5m-2)} \sum_{j_1,k_1=0}^{m-1,n} \sum_{j_2,k_2=0}^{m-1,n} \sum_{j_3,k_3=0}^{m-1,n} \sum_{g,h=0}^{m,n-1} \sum_{r,l=0}^{m,n-2} (\alpha^{m,p}_{j_1,j_2,j_3,g,p} \eta^{n,q}_{k_1,k_2,k_3,h} \langle e^a, \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{10} P_{j_3k_3} \rangle + \\
&\lambda^{n,q}_{l,h,k_1,k_2} \beta^{m,p}_{j_1,j_2,g,r} \langle \Delta^{02} P_{rl}, e^a \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{j_1k_1}, \Delta^{10} P_{j_2k_2} \rangle + \alpha^{m,p}_{j_1,j_2,j_3,r,p} \eta^{n,q}_{l,k_1,k_2,k_3} \langle \Delta^{02} P_{rl}, \Delta^{10} P_{j_1k_1} \times e^a \rangle \\
&\langle \Delta^{10} P_{j_2k_2}, \Delta^{10} P_{j_3k_3} \rangle + \lambda^{n,q}_{l,h,k_1,k_2} \beta^{m,p}_{j_1,j_2,g,r} 2 \times \langle \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh}, \Delta^{02} P_{rl} \rangle \langle e^a, \Delta^{10} P_{j_2k_2} \rangle)
\end{aligned} \tag{4.39}
$$

$$
\begin{aligned}
&\int_R \frac{\partial}{\partial x^a_{pq}} \mu_3(u,v)\,du\,dv = \\
&\frac{m^3 n^2 (n-1)}{(5n-2)(5m-2)} \sum_{j_1,k_1=0}^{m-1,n} \sum_{j_2,k_2=0}^{m-1,n} \sum_{j_3,k_3=0}^{m-1,n} \sum_{g,h=0}^{m,n-1} \sum_{r,l=0}^{m,n-2} (\alpha^{m,p}_{j_1,j_2,j_3,g,p} \eta^{n,q}_{k_1,k_2,k_3,h} \langle e^a, \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{10} P_{j_3k_3} \rangle + \\
&\lambda^{n,q}_{l,h,k_1,k_2} \beta^{m,p}_{j_1,j_2,g,r} (\langle \Delta^{02} P_{rl}, e^a \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{j_1k_1}, \Delta^{10} P_{j_2k_2} \rangle + 2 \times \langle \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh}, \Delta^{02} P_{rl} \rangle \langle e^a, \Delta^{10} P_{j_2k_2} \rangle) + \\
&\alpha^{m,p}_{j_1,j_2,j_3,r,p} \eta^{n,q}_{l,k_1,k_2,k_3} \langle \Delta^{02} P_{rl}, \Delta^{10} P_{j_1k_1} \times e^a \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{10} P_{j_3k_3} \rangle)
\end{aligned} \tag{4.40}
$$

Substituting eq. (4.26), eq. (4.33) and (4.40) in eq. (4.7), we find that

$$
\begin{aligned}
&\frac{\partial \mathcal{J}(\mathcal{P})}{\partial x^a_{pq}} = \\
&\frac{n^3 (m-1) m^2}{(5m-2)(5n-2)} \sum_{j,k=0}^{m-1,n} \sum_{g_1,h_1=0}^{m,n-1} \sum_{g_2,h_2=0}^{m,n-1} \sum_{g_3,h_3=0}^{m,n-1} \sum_{r,l=0}^{m-2,n} \left(\lambda^{m,p}_{p,j,g_1,g_2,r} \beta^{n,q}_{h_1,h_2,k,l} 2 \langle \Delta^{10} P_{jk} \times \Delta^{01} P_{g_1h_1}, \Delta^{20} P_{rl} \rangle \langle e^a, \Delta^{01} P_{g_2h_2} \rangle \right. \\
&+ \alpha^{n,q}_{k,h_1,h_2,h_3} \gamma^{m,p}_{g_1,g_2,g_3,j} \langle e^a, \Delta^{10} P_{jk} \times \Delta^{01} P_{g_1h_1} \rangle \langle \Delta^{01} P_{g_2h_2}, \Delta^{01} P_{g_3h_3} \rangle + \gamma^{m,p}_{r,g_1,g_2,g_3} \alpha^{n,q}_{l,h_1,h_2,h_3} \langle \Delta^{20} P_{rl}, e^a \times \Delta^{01} P_{g_1h_1} \rangle \times \\
&\left. \langle \Delta^{01} P_{g_2h_2}, \Delta^{01} P_{g_3h_3} \rangle + \lambda^{m,p}_{p,j,g_1,g_2,r} \beta^{n,q}_{h_1,h_2,k,l} \langle \Delta^{20} P_{rl}, \Delta^{10} P_{jk} \times e^a \rangle \langle \Delta^{01} P_{g_1h_1}, \Delta^{01} P_{g_2h_2} \rangle \right) - 2 \frac{m^3 n^3}{(5n-2)(5m-2)} \\
&\sum_{j,k=0}^{m-1,n} \sum_{r,l=0}^{m-1,n-1} \sum_{g,h=0}^{m,n-1} \sum_{g_1,h_1=0}^{m,n-1} \sum_{g_2,h_2=0}^{m,n-1} \sum_{j_1,k_1=0}^{m-1,n} \sum_{j_2,k_2=0}^{m-1,n} \left(\alpha^{n,q}_{k,l,h_1,h_2} \beta^{m,p}_{j,r,g_1,g_2} (\langle \Delta^{10} P_{jk} \times \Delta^{01} P_{g_1h_1}, \Delta^{11} P_{rl} \rangle \langle e^a, \Delta^{01} P_{g_2h_2} \rangle + \right. \\
&\langle \Delta^{11} P_{rl}, e^a \times \Delta^{01} P_{g_1h_1} \rangle \langle \Delta^{10} P_{jk}, \Delta^{01} P_{g_2h_2} \rangle) + \alpha^{m,p}_{g,r,j_1,j_2} \beta^{n,q}_{h,l,k_1,k_2} (\langle \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh}, \Delta^{11} P_{rl} \rangle \langle \Delta^{10} P_{j_2k_2}, e^a \rangle + \\
&\langle \Delta^{11} P_{rl}, \Delta^{10} P_{j_1k_1} \times e^a \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{01} P_{gh} \rangle) + \beta^{m,p}_{j_1,j_2,g_1,g_2} \beta^{n,q}_{h_1,h_2,k_1,k_2} \langle e^a, \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{g_1h_1} \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{01} P_{g_2h_2} \rangle) + \\
&\frac{m^3 n^2 (n-1)}{(5n-2)(5m-2)} \sum_{j_1,k_1=0}^{m-1,n} \sum_{j_2,k_2=0}^{m-1,n} \sum_{j_3,k_3=0}^{m-1,n} \sum_{g,h=0}^{m,n-1} \sum_{r,l=0}^{m,n-2} (\alpha^{m,p}_{j_1,j_2,j_3,g,p} \eta^{n,q}_{k_1,k_2,k_3,h} \langle e^a, \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{10} P_{j_3k_3} \rangle + \\
&\lambda^{n,q}_{l,h,k_1,k_2} \beta^{m,p}_{j_1,j_2,g,r} (\langle \Delta^{02} P_{rl}, e^a \times \Delta^{01} P_{gh} \rangle \langle \Delta^{10} P_{j_1k_1}, \Delta^{10} P_{j_2k_2} \rangle + 2 \times \langle \Delta^{10} P_{j_1k_1} \times \Delta^{01} P_{gh}, \Delta^{02} P_{rl} \rangle \langle e^a, \Delta^{10} P_{j_2k_2} \rangle) + \\
&\alpha^{m,p}_{j_1,j_2,j_3,r,p} \eta^{n,q}_{l,k_1,k_2,k_3} \langle \Delta^{02} P_{rl}, \Delta^{10} P_{j_1k_1} \times e^a \rangle \langle \Delta^{10} P_{j_2k_2}, \Delta^{10} P_{j_3k_3} \rangle)
\end{aligned} \tag{4.41}
$$

$$\frac{\partial \mathcal{J}(\mathcal{P})}{\partial x^{a}_{pq}} =$$
$$\sum_{j,k=0}^{m-1,n}\sum_{g_1,h_1=0}^{m,n-1}\sum_{g_2,h_2=0}^{m,n-1}\sum_{g_3,h_3=0}^{m,n-1}\sum_{j_1,k_1=0}^{m-1,n}\sum_{j_2,k_2=0}^{m-1,n}\sum_{j_3,k_3=0}^{m-1,n}\sum_{g,h=0}^{m,n-1}\left(\frac{n^3(m-1)m^2}{(5m-2)(5n-2)}\sum_{r,l=0}^{m-2,n}\left(\lambda^{m,p}_{p,j,g_1,g_2,r}\beta^{n,q}_{h_1,h_2,k,l}2\langle e^a,\Delta^{01}P_{g_2h_2}\rangle\times\right.\right.$$
$$\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1},\Delta^{20}P_{rl}\rangle+\alpha^{n,q}_{k,h_1,h_2,h_3}\eta^{m,p}_{g_1,g_2,g_3,j}\langle e^a,\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle+\gamma^{m,p}_{r,g_1,g_2,g_3}\alpha^{n,q}_{l,h_1,h_2,h_3}$$
$$\langle\Delta^{20}P_{rl},e^a\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle+\lambda^{m,p}_{p,j,g_1,g_2,r}\beta^{n,q}_{h_1,h_2,k,l}\langle\Delta^{20}P_{rl},\Delta^{10}P_{jk}\times e^a\rangle\langle\Delta^{01}P_{g_1h_1},\Delta^{01}P_{g_2h_2}\rangle\Big)-$$
$$2\frac{m^3n^3}{(5m-2)(5n-2)}\sum_{r,l=0}^{m-1,n-1}\Big(\alpha^{n,q}_{k,l,h_1,h_2}\beta^{m,p}_{j,r,g_1,g_2}\left(\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1},\Delta^{11}P_{rl}\rangle\langle e^a,\Delta^{01}P_{g_2h_2}\rangle+\langle\Delta^{11}P_{rl},e^a\times\Delta^{01}P_{g_1h_1}\rangle\times\right.$$
$$\langle\Delta^{10}P_{jk},\Delta^{01}P_{g_2h_2}\rangle\Big)+\alpha^{m,p}_{g,r,j_1,j_2}\beta^{n,q}_{h,l,k_1,k_2}\left(\langle\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{gh},\Delta^{11}P_{rl}\rangle\langle\Delta^{10}P_{j_2k_2},e^a\rangle+\langle\Delta^{11}P_{rl},\Delta^{10}P_{j_1k_1}\times e^a\rangle\times\right.$$
$$\langle\Delta^{10}P_{j_2k_2},\Delta^{01}P_{gh}\rangle\Big)+\beta^{m,p}_{j_1,j_2,g_1,g_2}\beta^{n,q}_{h_1,h_2,k_1,k_2}\langle e^a,\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{10}P_{j_2k_2},\Delta^{01}P_{g_2h_2}\rangle\Big)+\frac{m^3n^2(n-1)}{(5n-2)(5m-2)}\times$$
$$\sum_{r,l=0}^{m,n-2}\Big(\alpha^{m,p}_{j_1,j_2,j_3,g,p}\eta^{n,q}_{k_1,k_2,k_3,h}\langle e^a,\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{gh}\rangle\langle\Delta^{10}P_{j_2k_2},\Delta^{10}P_{j_3k_3}\rangle+\lambda^{n,q}_{l,h,k_1,k_2}\beta^{m,p}_{j_1,j_2,g,r}\left(\langle\Delta^{02}P_{rl},e^a\times\Delta^{01}P_{gh}\rangle\times\right.$$
$$\langle\Delta^{10}P_{j_1k_1},\Delta^{10}P_{j_2k_2}\rangle+2\langle\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{gh},\Delta^{02}P_{rl}\rangle\langle e^a,\Delta^{10}P_{j_2k_2}\rangle\Big)+\alpha^{m,p}_{j_1,j_2,j_3,r,p}\gamma^{n,q}_{l,k_1,k_2,k_3}\langle\Delta^{02}P_{rl},\Delta^{10}P_{j_1k_1}\times e^a\rangle\times$$
$$\langle\Delta^{10}P_{j_2k_2},\Delta^{10}P_{j_3k_3}\rangle\Big)\tag{4.42}$$

$$\frac{\partial \mathcal{J}(\mathcal{P})}{\partial x^{a}_{pq}} = \frac{m^2n^2}{(5m-2)(5n-2)}\times$$
$$(n(m-1)\sum_{j,k=0}^{m-1,n}\sum_{g_1,h_1=0}^{m,n-1}\sum_{g_2,h_2=0}^{m,n-1}\sum_{g_3,h_3=0}^{m,n-1}\sum_{j_1,k_1=0}^{m-1,n}\sum_{j_2,k_2=0}^{m-1,n}\sum_{j_3,k_3=0}^{m-1,n}\sum_{g,h=0}^{m,n-1}\left(\sum_{r,l=0}^{m-2,n}\left(\lambda^{m,p}_{p,j,g_1,g_2,r}\beta^{n,q}_{h_1,h_2,k,l}2\langle e^a,\Delta^{01}P_{g_2h_2}\rangle\times\right.\right.$$
$$\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1},\Delta^{20}P_{rl}\rangle+\alpha^{n,q}_{k,h_1,h_2,h_3}\eta^{m,p}_{g_1,g_2,g_3,j}\langle e^a,\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle+\gamma^{m,p}_{r,g_1,g_2,g_3}\alpha^{n,q}_{l,h_1,h_2,h_3}\times$$
$$\langle\Delta^{20}P_{rl},e^a\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle+\lambda^{m,p}_{p,j,g_1,g_2,r}\beta^{n,q}_{h_1,h_2,k,l}\langle\Delta^{20}P_{rl},\Delta^{10}P_{jk}\times e^a\rangle\langle\Delta^{01}P_{g_1h_1},\Delta^{01}P_{g_2h_2}\rangle\Big)$$
$$-2mn\sum_{r,l=0}^{m-1,n-1}\Big(\alpha^{n,q}_{k,l,h_1,h_2}\beta^{m,p}_{j,r,g_1,g_2}\left(\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1},\Delta^{11}P_{rl}\rangle\langle e^a,\Delta^{01}P_{g_2h_2}\rangle+\langle\Delta^{11}P_{rl},e^a\times\Delta^{01}P_{g_1h_1}\rangle\times\right.$$
$$\langle\Delta^{10}P_{jk},\Delta^{01}P_{g_2h_2}\rangle\Big)+\alpha^{m,p}_{g,r,j_1,j_2}\beta^{n,q}_{h,l,k_1,k_2}\left(\langle\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{gh},\Delta^{11}P_{rl}\rangle\langle\Delta^{10}P_{j_2k_2},e^a\rangle+\langle\Delta^{11}P_{rl},\Delta^{10}P_{j_1k_1}\times e^a\rangle\times\right.$$
$$\langle\Delta^{10}P_{j_2k_2},\Delta^{01}P_{gh}\rangle\Big)+\beta^{m,p}_{j_1,j_2,g_1,g_2}\beta^{n,q}_{h_1,h_2,k_1,k_2}\langle e^a,\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{10}P_{j_2k_2},\Delta^{01}P_{g_2h_2}\rangle\Big)+m(n-1)\times$$
$$\sum_{r,l=0}^{m,n-2}\Big(\alpha^{m,p}_{j_1,j_2,j_3,g,p}\eta^{n,q}_{k_1,k_2,k_3,h}\langle e^a,\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{gh}\rangle\langle\Delta^{10}P_{j_2k_2},\Delta^{10}P_{j_3k_3}\rangle+\lambda^{n,q}_{l,h,k_1,k_2}\beta^{m,p}_{j_1,j_2,g,r}\left(\langle\Delta^{02}P_{rl},e^a\times\Delta^{01}P_{gh}\rangle\times\right.$$
$$\langle\Delta^{10}P_{j_1k_1},\Delta^{10}P_{j_2k_2}\rangle+2\langle\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{gh},\Delta^{02}P_{rl}\rangle\langle e^a,\Delta^{10}P_{j_2k_2}\rangle\Big)+\alpha^{m,p}_{j_1,j_2,j_3,r,p}\gamma^{n,q}_{l,k_1,k_2,k_3}\langle\Delta^{02}P_{rl},\Delta^{10}P_{j_1k_1}\times e^a\rangle\times$$
$$\langle\Delta^{10}P_{j_2k_2},\Delta^{10}P_{j_3k_3}\rangle\Big)\Big)\tag{4.43}$$

$$\frac{\partial \mathcal{J}(\mathcal{P})}{\partial x^{a}_{pq}} =$$
$$\frac{m^2n^2}{(5m-2)(5n-2)}(n(m-1)\sum_{j,k=0}^{m-1,n}\sum_{g_1,h_1=0}^{m,n-1}\sum_{g_2,h_2=0}^{m,n-1}\sum_{g_3,h_3=0}^{m,n-1}\sum_{r,l=0}^{m-2,n}\Big(\lambda^{m,p}_{p,j,g_1,g_2,r}\beta^{n,q}_{h_1,h_2,k,l}\left(2\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1},\Delta^{20}P_{rl}\rangle\times\right.$$
$$\langle e^a,\Delta^{01}P_{g_2h_2}\rangle+\langle\Delta^{20}P_{rl},\Delta^{10}P_{jk}\times e^a\rangle\langle\Delta^{01}P_{g_1h_1},\Delta^{01}P_{g_2h_2}\rangle\Big)+\alpha^{n,q}_{k,h_1,h_2,h_3}\eta^{m,p}_{g_1,g_2,g_3,j}\langle e^a,\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1}\rangle\times$$
$$\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle+\gamma^{m,p}_{r,g_1,g_2,g_3}\alpha^{n,q}_{l,h_1,h_2,h_3}\langle\Delta^{20}P_{rl},e^a\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle\Big)-$$
$$2mn\sum_{j,k=0}^{m-1,n}\sum_{r,l=0}^{m-1,n-1}\sum_{g,h=0}^{m,n-1}\sum_{g_1,h_1=0}^{m,n-1}\sum_{g_2,h_2=0}^{m,n-1}\sum_{j_1,k_1=0}^{m-1,n}\sum_{j_2,k_2=0}^{m-1,n}\Big(\alpha^{n,q}_{k,l,h_1,h_2}\beta^{m,p}_{j,r,g_1,g_2}\left(\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1},\Delta^{11}P_{rl}\rangle\langle e^a,\Delta^{01}P_{g_2h_2}\rangle+\right.$$
$$\langle\Delta^{11}P_{rl},e^a\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{10}P_{jk},\Delta^{01}P_{g_2h_2}\rangle\Big)+\alpha^{m,p}_{g,r,j_1,j_2}\beta^{n,q}_{h,l,k_1,k_2}\left(\langle\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{gh},\Delta^{11}P_{rl}\rangle\langle\Delta^{10}P_{j_2k_2},e^a\rangle+\right.$$
$$\langle\Delta^{11}P_{rl},\Delta^{10}P_{j_1k_1}\times e^a\rangle\langle\Delta^{10}P_{j_2k_2},\Delta^{01}P_{gh}\rangle\Big)+\beta^{m,p}_{j_1,j_2,g_1,g_2}\beta^{n,q}_{h_1,h_2,k_1,k_2}\langle e^a,\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{10}P_{j_2k_2},\Delta^{01}P_{g_2h_2}\rangle\Big)+$$
$$m(n-1)\sum_{j_1,k_1=0}^{m-1,n}\sum_{j_2,k_2=0}^{m-1,n}\sum_{j_3,k_3=0}^{m-1,n}\sum_{g,h=0}^{m,n-1}\sum_{r,l=0}^{m,n-2}\Big(\alpha^{m,p}_{j_1,j_2,j_3,g,p}\eta^{n,q}_{k_1,k_2,k_3,h}\langle e^a,\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{gh}\rangle\langle\Delta^{10}P_{j_2k_2},\Delta^{10}P_{j_3k_3}\rangle+$$
$$\lambda^{n,q}_{l,h,k_1,k_2}\beta^{m,p}_{j_1,j_2,g,r}\left(\langle\Delta^{02}P_{rl},e^a\times\Delta^{01}P_{gh}\rangle\langle\Delta^{10}P_{j_1k_1},\Delta^{10}P_{j_2k_2}\rangle+2\times\langle\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{gh},\Delta^{02}P_{rl}\rangle\langle e^a,\Delta^{10}P_{j_2k_2}\rangle\right)+$$
$$\alpha^{m,p}_{j_1,j_2,j_3,r,p}\gamma^{n,q}_{l,k_1,k_2,k_3}\langle\Delta^{02}P_{rl},\Delta^{10}P_{j_1k_1}\times e^a\rangle\langle\Delta^{10}P_{j_2k_2},\Delta^{10}P_{j_3k_3}\rangle\Big)\Big)\tag{4.44}$$

$$\frac{\partial \mathcal{J}(\mathcal{P})}{\partial x^{a}_{pq}} =$$
$$\frac{m^2n^2}{(5m-2)(5n-2)}(n(m-1)\sum_{\substack{j,j_1=0\\ j_2,j_3=0}}^{m-1}\sum_{\substack{k,k_1=0\\ k_2,k_3=0}}^{n}\sum_{\substack{g,g_1=0\\ g_2,g_3=0}}^{m}\sum_{\substack{h,h_1=0\\ h_2,h_3=0}}^{n-1}\left(\sum_{r,l=0}^{m-2,n}\left(\lambda^{m,p}_{j,r,g_1,g_2}\beta^{n,q}_{k,l,h_1,h_2}2\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1},\Delta^{20}P_{rl}\rangle\times\right.\right.$$
$$\langle e^a,\Delta^{01}P_{g_2h_2}\rangle+\alpha^{n,q}_{k,h_1,h_2,h_3}\eta^{m,p}_{j,g_1,g_2,g_3}\langle e^a,\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle+\gamma^{m,p}_{r,g_1,g_2,g_3}\alpha^{n,q}_{l,h_1,h_2,h_3}\times$$
$$\langle\Delta^{20}P_{rl},e^a\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{01}P_{g_2h_2},\Delta^{01}P_{g_3h_3}\rangle+\lambda^{m,p}_{j,r,g_1,g_2}\beta^{n,q}_{k,l,h_1,h_2}\langle\Delta^{20}P_{rl},\Delta^{10}P_{jk}\times e^a\rangle\langle\Delta^{01}P_{g_1h_1},\Delta^{01}P_{g_2h_2}\rangle\Big)$$
$$-2mn\sum_{r,l=0}^{m-1,n-1}\Big(\alpha^{n,q}_{k,l,h_1,h_2}\beta^{m,p}_{j,r,g_1,g_2}\left(\langle\Delta^{10}P_{jk}\times\Delta^{01}P_{g_1h_1},\Delta^{11}P_{rl}\rangle\langle e^a,\Delta^{01}P_{g_2h_2}\rangle+\langle\Delta^{11}P_{rl},e^a\times\Delta^{01}P_{g_1h_1}\rangle\times\right.$$
$$\langle\Delta^{10}P_{jk},\Delta^{01}P_{g_2h_2}\rangle\Big)+\alpha^{m,p}_{g,r,j_1,j_2}\beta^{n,q}_{h,l,k_1,k_2}\left(\langle\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{gh},\Delta^{11}P_{rl}\rangle\langle\Delta^{10}P_{j_2k_2},e^a\rangle+\langle\Delta^{11}P_{rl},\Delta^{10}P_{j_1k_1}\times e^a\rangle\times\right.$$
$$\langle\Delta^{10}P_{j_2k_2},\Delta^{01}P_{gh}\rangle\Big)+\beta^{m,p}_{j_1,j_2,g_1,g_2}\beta^{n,q}_{h_1,h_2,k_1,k_2}\langle e^a,\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{g_1h_1}\rangle\langle\Delta^{10}P_{j_2k_2},\Delta^{01}P_{g_2h_2}\rangle\Big)+m(n-1)\times$$
$$\sum_{r,l=0}^{m,n-2}\Big(\alpha^{m,p}_{g,j_1,j_2,j_3}\eta^{n,q}_{h,k_1,k_2,k_3}\langle e^a,\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{gh}\rangle\langle\Delta^{10}P_{j_2k_2},\Delta^{10}P_{j_3k_3}\rangle+\lambda^{n,q}_{l,h,k_1,k_2}\beta^{m,p}_{g,r,j_1,j_2}\left(\langle\Delta^{02}P_{rl},e^a\times\Delta^{01}P_{gh}\rangle\times\right.$$
$$\langle\Delta^{10}P_{j_1k_1},\Delta^{10}P_{j_2k_2}\rangle+2\langle\Delta^{10}P_{j_1k_1}\times\Delta^{01}P_{gh},\Delta^{02}P_{rl}\rangle\langle e^a,\Delta^{10}P_{j_2k_2}\rangle\Big)+\alpha^{m,p}_{r,j_1,j_2,j_3}\gamma^{n,q}_{l,k_1,k_2,k_3}\langle\Delta^{02}P_{rl},\Delta^{10}P_{j_1k_1}\times e^a\rangle\times$$
$$\langle\Delta^{10}P_{j_2k_2},\Delta^{10}P_{j_3k_3}\rangle\Big)\Big)\tag{4.45}$$

The mean curvature functional (3.7) has an extremal if and only if $\frac{\partial M(P)}{\partial x^{a}_{ij}}=0$, $a\in\{1,2,3\}$ for a particular set of control net of points $P=\{P_{ij}\}_{i,j=0}^{n,m}$ of the corresponding B\'{e}zier surface, for which it is a quasi-minimal, gives us the constraint eq. (4.6).

5. CONCLUSION

The mean curvature functional (3.10) is exploited to find the quasi-minimal B´ezier surface x(u, v) as variational improvement in it by finding the vanishing condition for the gradient of mean curvature functional by finding the interior control points as algebraic constraints on the boundary control points. For a quasi-minimal B´ezier patch, the control net of the patch must satisfy the constraint (4.6). It is obtained by finding the gradient of mean curvature functional for the patch with respect to inner unknown control points and setting it to zero, given in theorem 4.1. The algorithm presented is rather computational in its nature and suitable programming in computer algebra system can find the minimal surfaces of desired degree and accuracy.

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