The methods investigated in this paper include Forward Difference (FD), Backward Difference (BD), Crank Nicolson (CN), Alternate Directions Scheme (ADI) and Dufort-Frankel Scheme (DF). Here we present a short description of all these methods used in this paper for performing computational analysis.

A. Forward Difference (FD)

The finite difference approximation to the above partial differential equation known as forward difference formula is given below [6].

$$\frac{w_i^{k+1} - w_i^k}{\Delta t} = c\,\frac{w_{i+1}^k - 2w_i^k + w_{i-1}^k}{\Delta x^2} \tag{2}$$

OR

$$w_i^{k+1} = w_i^k + \lambda\left(w_{i+1}^k - 2w_i^k + w_{i-1}^k\right)$$

$$wehere \quad \lambda = c\,\frac{\Delta t}{\Delta x^2}$$

The scheme can easily be extended to 2-D as:

$$\begin{aligned} w_{i,j}^{k+1} = {} & w_{i,j}^k + \lambda(w_{i+1,j}^k + w_{i-1,j}^k + \\ & w_{i,j+1}^k + w_{i,j-1}^k - 4w_{i,j}^k) \end{aligned} \tag{3}$$

$$wehere \quad \lambda = c\,\frac{\Delta t}{h^2};\ h = \Delta x = \Delta y$$

Also this method can be extended to 3-D as given below:

$$\begin{aligned} w_{i,j,k}^{k+1} = {} & w_{i,j,k}^k + \lambda(w_{i+1,j,k}^k + w_{i-1,j,k}^k + w_{i,j+1,k}^k + \\ & w_{i,j-1,k}^k + w_{i,j,k+1}^k + w_{i,j,k-1}^k - 6w_{i,j,k}^k) \end{aligned} \tag{4}$$

$$wehere \quad \lambda = c\,\frac{\Delta t}{h^2};\ h = \Delta x = \Delta y = \Delta z$$

B. Backward Difference (BD)

The finite difference approximation to the above partial differential equation known as backward difference formula is given below [6].

$$\frac{w_i^{k+1} - w_i^k}{\Delta t} = c\,\frac{w_{i+1}^{k+1} - 2w_i^{k+1} + w_{i-1}^{k+1}}{\Delta x^2} \tag{5}$$

OR

$$w_i^{k+1} = w_i^k + \lambda\left(w_{i+1}^{k+1} - 2w_i^{k+1} + w_{i-1}^{k+1}\right)$$

$$wehere \quad \lambda = c\,\frac{\Delta t}{\Delta x^2}$$

The scheme can easily be extended to 2-D as:

$$w_{i,j}^{k+1} = w_{i,j}^{k} + \lambda \begin{pmatrix} w_{i+1,j}^{k+1} + w_{i-1,j}^{k+1} + \\ w_{i,j+1}^{k+1} + w_{i,j-1}^{k+1} - 4w_{i,j}^{k+1} \end{pmatrix} \tag{6}$$

$$wehere \quad \lambda = c\frac{\Delta t}{h^2};\ h = \Delta x = \Delta y$$

Also 3-D extension of Backward Difference Method is given as given below:

$$w_{i,j,k}^{k+1} = w_{i,j,k}^{k} + \lambda \begin{pmatrix} w_{i+1,j,k}^{k+1} + w_{i-1,j,k}^{k+1} + \\ w_{i,j+1,k}^{k+1} + w_{i,j-1,k}^{k+1} + \\ w_{i,j,k+1}^{k+1} + w_{i,j,k-1}^{k+1} - 6w_{i,j,k}^{k+1} \end{pmatrix} \tag{7}$$

$$wehere \quad \lambda = c\frac{\Delta t}{h^2};\ h = \Delta x = \Delta y = \Delta z$$

C. Crank-Nicolson (CN)

Another methods known Crank–Nicolson method is centered in both space and time suggested in [6] is given below:

$$\frac{w_i^{k+1} - w_i^{k}}{\Delta t} = \frac{c}{2}\begin{pmatrix} \frac{w_{i+1}^{k+1} - 2w_i^{k+1} + w_{i-1}^{k+1}}{\Delta x^2} + \\ \frac{w_{i+1}^{k} - 2w_i^{k} + w_{i-1}^{k}}{\Delta x^2} \end{pmatrix} \tag{8}$$

OR

$$w_i^{k+1} = w_i^{k} + \lambda \begin{pmatrix} w_{i+1}^{k+1} - 2w_i^{k+1} + w_{i-1}^{k+1} + \\ w_{i+1}^{k} - 2w_i^{k} + w_{i-1}^{k} \end{pmatrix}$$

$$wehere \quad \lambda = c\frac{\Delta t}{2\Delta x^2}$$

The scheme can easily be extended to 2-D as:

$$w_{i,j}^{k+1} = w_{i,j}^{k} + \lambda \begin{pmatrix} w_{i+1,j}^{k+1} + w_{i-1,j}^{k+1} + w_{i,j+1}^{k+1} + \\ w_{i,j-1}^{k+1} - 4w_{i,j}^{k+1} + w_{i+1,j}^{k} + \\ w_{i-1,j}^{k} + w_{i,j+1}^{k} + w_{i,j-1}^{k} - 4w_{i,j}^{k} \end{pmatrix} \tag{9}$$

$$wehere \quad \lambda = c\frac{\Delta t}{2h^2};\ h = \Delta x = \Delta y$$

Also this method can be extended to 3-D as given below:

$$
w_{i,j,k}^{k+1} = w_{i,j,k}^{k} + \lambda \begin{pmatrix} w_{i+1,j,k}^{k+1} + w_{i-1,j,k}^{k+1} + \\ w_{i,j+1,k}^{k+1} + w_{i,j-1,k}^{k+1} + \\ w_{i,j,k+1}^{k+1} + w_{i,j,k-1}^{k+1} - 6w_{i,j,k}^{k+1} + \\ w_{i+1,j,k}^{k} + w_{i-1,j,k}^{k} + \\ w_{i,j+1,k}^{k} + w_{i,j-1,k}^{k} + \\ w_{i,j,k+1}^{k} + w_{i,j,k-1}^{k} - 6w_{i,j,k}^{k} \end{pmatrix} \quad (10)
$$

$$
wehere \quad \lambda = c\frac{\Delta t}{2h^2};\ h = \Delta x = \Delta y = \Delta z
$$

D. Alternate Directions Scheme (ADI)

Another method to solve this heat equation is known as alternating direction (ADI) algorithm; which has only 2-D and 3-D extensions. The 2D formulation of this method is give as [8]:

$$
\begin{aligned} \frac{w_{ij}^{n+1/2} - w_{ij}^{n}}{\Delta t / 2} &= \left(\delta_x^2 w_{ij}^{n+1/2} + \delta_y^2 w_{ij}^{n}\right) \\ \frac{w_{ij}^{n+1} - w_{ij}^{n+1/2}}{\Delta t / 2} &= \left(\delta_x^2 w_{ij}^{n+1/2} + \delta_y^2 w_{ij}^{n+1}\right) \end{aligned} \quad (11)
$$

Also the 3D extension is give as [8]:

$$
\begin{aligned} \frac{w_{ijk}^{n+1/3} - w_{ijk}^{n}}{\Delta t / 3} &= \left(\delta_x^2 w_{ijk}^{n+1/3} + \delta_y^2 w_{ijk}^{n} + \delta_z^2 w_{ijk}^{n}\right) \\ \frac{w_{ij}^{n+2/3} - w_{ij}^{n+1/3}}{\Delta t / 3} &= \begin{pmatrix} \delta_x^2 w_{ij}^{n+1/3} + \delta_y^2 w_{ij}^{n+2/3} + \\ \delta_z^2 w_{ijk}^{n+1/3} \end{pmatrix} \\ \frac{w_{ij}^{n+1} - w_{ij}^{n+2/3}}{\Delta t / 3} &= \begin{pmatrix} \delta_x^2 w_{ij}^{n+2/3} + \delta_y^2 w_{ij}^{n+2/3} + \\ \delta_z^2 w_{ijk}^{n+1} \end{pmatrix} \end{aligned} \quad (12)
$$

E. DuFort- Frankel Scheme (DF)

Another method known as Dufort–Frankel scheme for the one-dimensional heat equation is give as in [7] :

$$
\frac{w_i^{k+1} - w_i^{k-1}}{2\Delta t} = c\left(\frac{w_{i+1}^{k} + w_{i-1}^{k}}{\Delta x^2} - \frac{w_i^{k+1} + w_i^{k-1}}{\Delta x^2}\right) \quad (13)
$$

OR

$$
\begin{aligned} w_i^{k+1} &= w_i^{k-1} + \lambda\left(w_{i+1}^{k+1} + w_{i-1}^{k} - w_i^{k+1} - w_i^{k-1}\right) \\ wehere \quad \lambda &= c\frac{2\Delta t}{\Delta x^2} \end{aligned}
$$

This scheme is easily extended in 2D as follows:

$$w_{i,j}^{k+1} = w_{i,j}^{k-1} + \lambda \begin{pmatrix} w_{i+1,j}^{k} + w_{i-1,j}^{k} + w_{i,j+1}^{k} + \\ w_{i,j-1}^{k} - 2\left(w_{i,j}^{k+1} + w_{i,j}^{k-1}\right) \end{pmatrix} \quad (14)$$

$$wehere \quad \lambda = c\frac{2\Delta t}{h^2};\ h = \Delta x = \Delta y$$

Also the 3D extension of method is as:

$$w_{i,j,k}^{k+1} = w_{i,j,k}^{k-1} + \lambda \begin{pmatrix} w_{i+1,j,k}^{k} + w_{i-1,j,k}^{k} + \\ w_{i,j+1,k}^{k} + w_{i,j-1,k}^{k} + w_{i,j,k+1}^{k} + \\ w_{i,j,k-1}^{k} - 3\left(w_{i,j,k}^{k+1} + w_{i,j,k}^{k-1}\right) \end{pmatrix} \quad (15)$$

$$wehere \quad \lambda = c\frac{2\Delta t}{h^2};\ h = \Delta x = \Delta y = \Delta z$$

3. Problems Solved. Different types of problems were considered to solve standard heat equation. These problems exploit various computational aspects of these methods to elaborate analysis.

F. 1-D Applications

Three 1-D problems considered as exponential, sinusoidal and linear with sinusoidal initial conditions. These three problems have been solved for both Neuman and Dirichlet boundary conditions.

1. Exponential initial conditions

General 1-D Heat Equation (1) is Considered with exponential initial conditions as first case.

The Dirichlet boundary conditions for problem (A1-D) are given as:

$u(x,0) = e^{-x}$, $u(0,t) = e^{t}$, $u(1,t) = e^{t-1}$,

$0 \le x \le 1$, $0 \le t \le 1$

Also the Neuman boundary conditions for problem (A1-N) are given as:

$u(x,0) = e^{-x}$, $u_x(0,t) = -e^{t}$, $u_x(1,t) = -e^{t-1}$,

$0 \le x \le 1$, $0 \le t \le 1$

This problem has Analytical solution as:

$$u(x,t) = e^{t-x} \quad (16)$$

2. Sinusoidal initial conditions

General 1-D Heat Equation (1) is Considered with sinusoidal initial conditions as second case.

The Dirichlet boundary conditions for problem (1D-b (D)) are given as:

$u(x,0) = \sin(\pi x)$, $u(0,t) = u(1,t) = 0$,

$0 \le x \le 1$, $0 \le t \le 1$

Also the Neuman boundary conditions for this problem (1D-b (N)) are given as

$u(x,0) = \sin(\pi x), u_x(0,t) = \pi e^{-\pi t}$,

$u_x(1,t) = -\pi e^{-\pi t}$, $0 \le x \le 1$, $0 \le t \le 1$

This problem has the analytical solution as:

$$u(x,t) = e^{-\pi t} \sin(\pi x)$$

3. Linear combined with Sinusoidal initial conditions

General 1-D Heat Equation (1) is Considered with sinusoidal initial conditions as third case.

The Dirichlet boundary conditions for problem (1D-c (D)) are given as:

$$u(x,0) = x + \sin(\pi x),\ u(0,t) = u(1,t) = 0,$$
$$0 \le x \le 1,\ 0 \le t \le 1$$

Also the Neuman boundary conditions for this problem (1D-c (N)) are given as

$$u(x,0) = x + \sin(\pi x), u_x(0,t) = \pi e^{-\pi t},$$
$$u_x(1,t) = -\pi e^{-\pi t},\ 0 \le x \le 1,\ 0 \le t \le 1$$

G. 2-D Applications

General 2-D Heat Equation Considered with exponential initial conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{17}$$

Subject to Drichlet boundary conditions (2D (D)) as:

$$u(x,y,0) = e^{-x} + e^{-y},\ u(x,0,t) = e^{t-x} + e^{t},$$
$$u(x,1,t) = e^{t-x} + e^{t-1},\ u(0,y,t) = e^{t-y} + e^{t},$$
$$u(1,y,t) = e^{t-y} + e^{t-1},\ 0 \le x, y \le 1,\ 0 \le t \le 1$$

The problem also solved for the Neuman Boundary conditions (2D (N)) as:

$$u(x,y,0) = e^{-x} + e^{-y}, u(x,0,t) = e^{t-x} + e^{t},$$
$$u(x,1,t) = e^{t-x} + e^{t-1}, u(0,y,t) = e^{t-y} + e^{t},$$
$$u(1,y,t) = e^{t-y} + e^{t-1},\ 0 \le x, y \le 1,\ 0 \le t \le 1$$

This problem has following Analytical solution

$$u(x,y,t) = e^{t}\left(e^{-x} + e^{-y}\right) \tag{18}$$

H. 3-D Applications

General 3-D Heat Equation Considered with exponential initial conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{19}$$

Subject to Drichlet boundary conditions (3D (D)) as:

$$u(x,y,z,0)=e^{-x}+e^{-y}+e^{-z},$$

$$u(x,y,0,t)=e^{t-x}+e^{t-y}+e^{t},$$

$$u(x,y,1,t)=e^{t-x}+e^{t-y}+e^{t-1},$$

$$u(x,0,z,t)=e^{t-x}+e^{t-z}+e^{t},$$

$$u(x,1,z,t)=e^{t-x}+e^{t-z}+e^{t-1},$$

$$u(0,y,z,t)=e^{t-y}+e^{t-z}+e^{t},$$

$$u(1,y,z,t)=e^{t-y}+e^{t-z}+e^{t-1},$$

$$0\le x,y,z\le 1, 0\le t\le 1$$

The problem also solved for the Neuman Boundary conditions (3D (N)) as:

$$u(x,y,z,0)=e^{-x}+e^{-y}+e^{-z},$$

$$u(x,y,0,t)=e^{t-x}+e^{t-y}+e^{t},$$

$$u(x,y,1,t)=e^{t-x}+e^{t-y}+e^{t-1},$$

$$u(x,0,z,t)=e^{t-x}+e^{t-z}+e^{t},$$

$$u(x,1,z,t)=e^{t-x}+e^{t-z}+e^{t-1},$$

$$u(0,y,z,t)=e^{t-y}+e^{t-z}+e^{t},$$

$$u(1,y,z,t)=e^{t-y}+e^{t-z}+e^{t-1},$$

$$0\le x,y,z\le 1,\ 0\le t\le 1$$

This problem has following Analytical solution

$$u(x,y,z,t)=e^{t}\left(e^{-x}+e^{-y}+e^{-z}\right) \tag{20}$$

4. Results and Analysis. A unified computation error and timing analysis for said conventional numerical methods has been performed for five heat equation problems stated earlier. Results have been obtained by incrementing the time and space steps for each case. These results are analyzed for both accuracy and computation time, are summarized in this section.

Figure 1 presents a uniform error analysis for small time step i.e. $\Delta t = 1/1000$ whereas space step is fixed at 1/4 in this case. It can be observed that problems with Dirichlet boundary conditions have reduced error than their Neuman counterpart. We deem this is due to derivative estimation on boundary points.

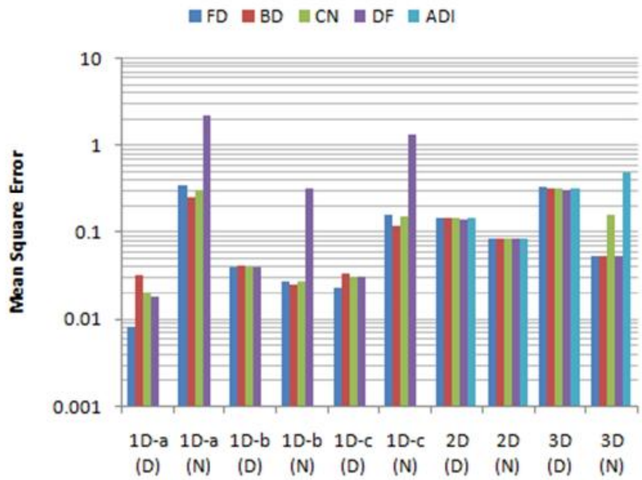

Figure 1: Uniform error analysis for small time step

Uniform error analysis for large time step is presented in Figure **2**. In this case the time step is taken as 1/40 while still the space step is fixed at 1/4. This is evident that CN performs best in case of exponential 1D problems while FD and DF performs best for 1D sinusoidal problem. Similarly BD is best for 3D Neuman problem and ADI performs worse in case of 3D problems.

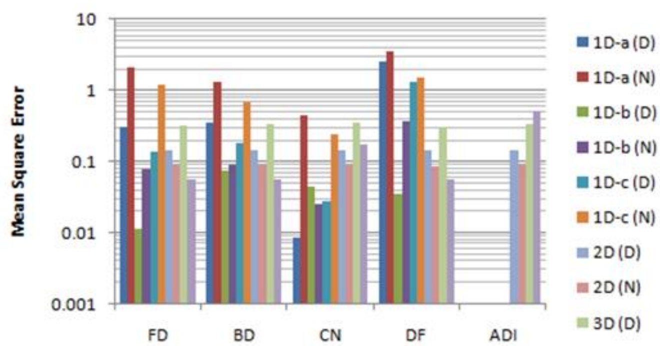

Figure 2: Uniform error analysis for large time step

Error analysis in case of fine and course grid is presented in Figure 3 and Figure 4. The time step is fixed at 1/1000 for both the cases while space step is 1/100 for fine grid and it is set at 1/10 for course grid solution. A close observation indicates that almost every method performs its best for linear and sinusoidal combined problem. It is also evident is DF is not a good choice in coarse grid for any type of 1D problem.

Uniform computational time analysis is given in **Figure 5**. FD takes the minimum time to solve a problem and DF is second most efficient to solve the problem. At the same time ADI is the most time taking scheme and the 2nd worse is CN for the same problem. The latter two method take more time as they are the most computationally intensive than all methods.

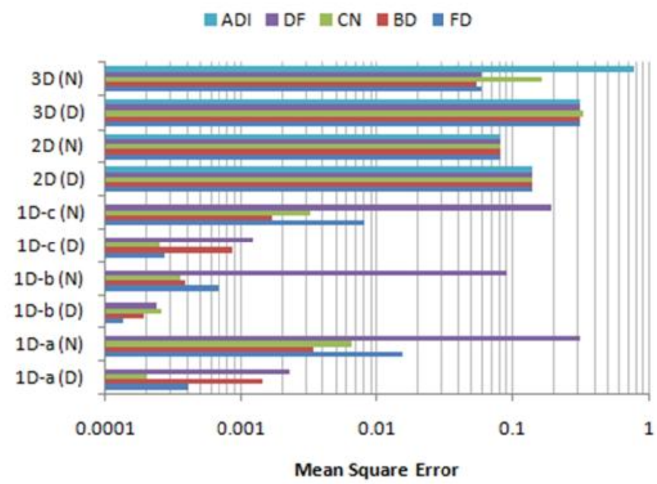

Figure 3: Uniform error analysis for fine grid

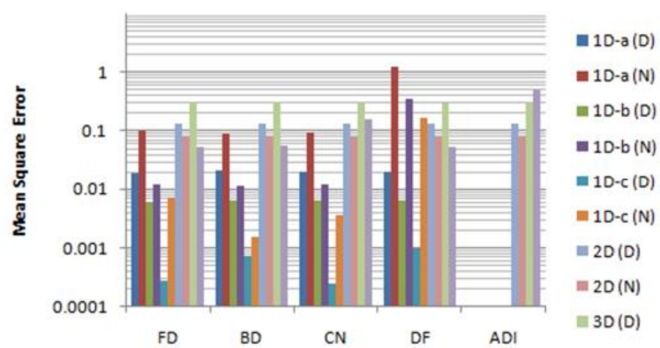

Figure 4: Uniform error analysis for coarse grid

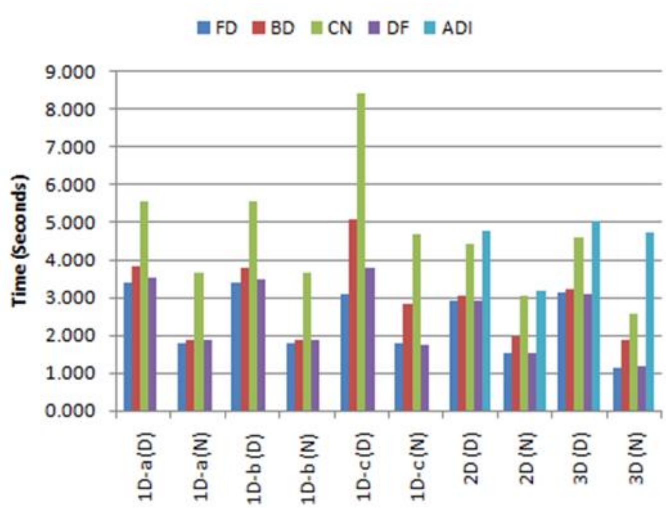

Figure 5: Uniform computation time analysis

5. Summary. In this paper we have presented a uniform analysis of some conventional numerical method to solve five different problems of heat equation. All these problems and methods have been analyzed for accuracy and computation time. The summary of obtained results is furnished below.

I. For Drichlet boundary conditions

- FD method provides better accuracy for 1D problem.
- DF provides more accuracy for both 2D and 3D problems.
- BD is least accurate to solve 1D and 2D problems.
- CN should be avoided for 3D problem as this is also least accurate for the case.

J. For Neuman boundary conditions

- BD method provides better accuracy for 1D problem.
- DF provides more accuracy for 2D problems but at the same time it is least accurate for 1D problem.
- FD is most accurate for 3D problems but meanwhile it is least accurate for 2D problems.

ADI should be avoided for 3D problem as this is also least accurate for the case.

References

[1] Heath, Michael T. (2001), Scientific computing. McGraw-Hill,.

[2] Pinchover, Y., & Rubinstein, J. (2005). *An introduction to partial differential equations*. Cambridge university press.

[3] Romão, E. C., Campos-Silva, J. B., & De Moura, L. F. M. (2009). Error analysis in the numerical solution of 3D convection-diffusion equation by finite difference methods. *Revista de Engenharia Térmica*, *8*(1), 12-17.

[4] Babuška, I., & Szymczak, W. G. (1982). An error analysis for the finite element method applied to convection diffusion problems. *Computer Methods in Applied Mechanics and Engineering*, *31*(1), 19-42.

[5] Ascher, U. M., Ruuth, S. J., & Wetton, B. T. (1995). Implicit-explicit methods for time-dependent partial differential equations. *SIAM Journal on Numerical Analysis*, *32*(3), 797-823.

[6] Lambert, J. D. (1991). *Numerical methods for ordinary differential systems: the initial value problem*. John Wiley & Sons, Inc..

[7] Zwillinger, D. (1998). *Handbook of differential equations* (Vol. 1). Gulf Professional Publishing.

[8] Morton, K. W., & Mayers, D. F. (2005). *Numerical solution of partial differential equations: an introduction*. Cambridge university press.

[9] LeVeque, R. J. (2007). *Finite difference methods for ordinary and partial differential equations: steady-state and time-dependent problems*. Society for Industrial and Applied Mathematics.