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SOME SUFFICIENT CONDITIONS FOR ALPHA CONVEX FUNCTIONS OF ORDER BETA

ASKAR ALI AND JAN MUHAMMAD
[1]Department of Mathematics, Abdul Wali Khan University Mardan, Pakistan
[2]Department of Computer Science, Affiliation of second author
askarali@gmail.com

ABSTRACT. *The object of the present paper is to obtain new sufficiency criteria for a class of alpha convex functions and then discuss its applications to the generalized integral operator. Many known results appear as a special consequences of our work. Some applications of our work to the generalized integral operator is also given.*
Keywords: Analytic functions, alpha convex functions, integral operator

1. **Introduction.** Let $\mathcal{A}(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k \, z^k \quad (n \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We note that $\mathcal{A}_1(1) = \mathcal{A}$. Also let $\mathcal{S}^*(n, \beta)$, $\mathcal{C}(n, \beta)$ and $\mathcal{H}(\alpha, n, \beta)$, $0 \leq \alpha \leq 1$, $0 \leq \beta < 1$, $n \in \mathbb{N}$, denote the subclasses of $\mathcal{A}(n)$ consisting of all functions $f(z)$, of the form (1.1), which are defined, respectively, by

$$Re \frac{zf'(z)}{f(z)} > \beta, \ (z \in \mathbb{U}),$$

$$1 + Re \frac{zf''(z)}{f'(z)} > \beta, \ (z \in \mathbb{U}),$$

$$Re \left\{ (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{zf''(z)}{f'(z)} \right\} > \beta - \alpha, \ \ (z \in \mathbb{U}).$$

For $n = 1$, the above three classes reduce to the well known classes of starlike, convex and alpha convex functions of order β respectively. We also note that $\mathcal{H}(0, n, 0) \equiv \mathcal{S}^*(n, 0)$, $\mathcal{H}(1, n, 0) \equiv \mathcal{C}(n, 0)$.

Sufficient condition were studied by various authors for different subclasses of analytic functions, for some of the related work see [1, 2, 3, 4, 5, 6, 7]. The object of the present paper is to obtain sufficient conditions for the class $\mathcal{H}(\alpha, n, \beta)$ of alpha convex functions of order β. We also consider some special cases of our results which lead to various interesting corollaries and relevances of some of these with other known results are also mentioned.

We will assume throughout our discussion, unless otherwise stated, that $n \in \mathbb{N}$, $0 \leq \alpha \leq 1$, $0 \leq \beta < 1$.

To obtain our main results, we need the following Lemmas due to Mocanu [8].

Lemma 1.1. If $p(z) \in \mathcal{A}(n)$ satisfies the condition

$$|p'(z) - 1| < \frac{n+1}{\sqrt{(n+1)^2 + 1}} \quad (z \in \mathbb{U}),$$

then

$$p(z) \in S^*(n, 0).$$

Lemma 1.2. If $p(z) \in \mathcal{A}(n)$ satisfies the condition

$$|\arg p'(z)| < \frac{\pi}{2}\, \delta_n \quad (z \in \mathbb{U}),$$

where δ_n is the unique root of the equation

$$2\tan^{-1}[n(1-\delta_n)] + \pi(1 - 2\delta_n) = 0, \tag{1.2}$$

then

$$p(z) \in \mathcal{S}^*(n, 0).$$

2. **Sufficient conditions for the class.** $\mathcal{H}(\alpha, n, \beta)$.

Theorem 2.1. If $f(z) \in \mathcal{A}(n)$ satisfies

$$\left| \left(\frac{f(z)}{z^{1-\alpha}} \left(\frac{f'(z)}{f(z)} \right)^{\alpha} \right)^{\frac{1}{1-\beta}} \left\{ (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \frac{zf''(z)}{f'(z)} + \alpha - \beta \right\} - 1 + \beta \right|$$

$$< \frac{n+1}{\sqrt{(n+1)^2+1}}(1-\beta) \quad (z \in \mathbb{U}), \tag{2.1}$$

then $f(z) \in \mathcal{H}(\alpha, n, \beta)$.

Proof. Let us set a function $p(z)$ by

$$p(z) = z\left(\frac{f(z)}{z^{1-\alpha}} \left(\frac{f'(z)}{f(z)} \right)^{\alpha} \right)^{\frac{1}{1-\beta}} = z + \frac{\alpha(n+1)a_{n+1}}{(1-\beta)} z^{n+1} + \ldots \tag{2.2}$$

for $f(z) \in \mathcal{A}(n)$. Then clearly (2.2) shows that $p(z) \in \mathcal{A}(n)$.

Differentiating (2.2) logarithmically, we have

$$\frac{p'(z)}{p(z)} = \frac{1}{z} + \frac{1}{1-\beta}\left\{ (1-\alpha)\frac{f'(z)}{f(z)} + \alpha\frac{f''(z)}{f'(z)} - \frac{1-\alpha}{z} \right\} \tag{2.3}$$

which gives

$$|p'(z) - 1| = \left| \left(\frac{f(z)}{z^{1-\alpha}} \left(\frac{f'(z)}{f(z)} \right)^{\alpha} \right)^{\frac{1}{1-\beta}} \frac{1}{1-\beta} \right.$$

$$\left. \left\{ (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zf''(z)}{f'(z)} + \alpha - \beta \right\} - 1 \right|. \tag{2.4}$$

Thus using (2.1), we have

$$|p'(z) - 1| \leq \frac{n+1}{\sqrt{(n+1)^2+1}}, \quad (z \in \mathbb{U}).$$

Hence, using Lemma 1.1, we have $p(z) \in \mathcal{S}^*(n, 0)$.

From (2.3), we can write

$$\frac{zp'(z)}{p(z)} = 1 + \frac{1}{1-\beta}\left\{ (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zf''(z)}{f'(z)} + \alpha - 1 \right\}.$$

Since $p(z) \in \mathcal{S}^*(n, 0)$, it implies that $Re\frac{zp'(z)}{p(z)} > 0$. Therefore, we get

$$\frac{1}{1-\beta} Re\left\{ (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - \beta \right\} = Re\frac{zp'(z)}{p(z)} > 0$$

or

$$\frac{1}{1-\beta} Re\left\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-\beta\right\}>0.$$

and this implies that $f(z)\in\mathcal{H}(\alpha,n,\beta)$.

By taking $\alpha=0$ and $\alpha=1$ in Theorem 2.1, we obtain Corollary 2.2 and Corollary 2.3.

Corollary 2.2. If $f(z)\in\mathcal{A}(n)$ satisfies

$$\left|\left(\frac{f(z)}{z}\right)^{\frac{1}{1-\beta}}\left\{\frac{zf'(z)}{f(z)}-\beta\right\}-1+\beta\right|<\frac{n+1}{\sqrt{(n+1)^2+1}}(1-\beta)\quad(z\in\mathbb{U}),$$

for $0\le\beta<1$, then $f(z)\in\mathcal{S}^*(n,\beta)$.

Corollary 2.3. If $f(z)\in\mathcal{A}(n)$ satisfies

$$\left|\{f'(z)\}^{\frac{\beta}{1-\beta}}\{zf''(z)+(1-\beta)f'(z)\}-1+\beta\right|<\frac{n+1}{\sqrt{(n+1)^2+1}}(1-\beta)\quad(z\in\mathbb{U}),$$

for $0\le\beta<1$, then $f(z)\in\mathcal{C}(n,\beta)$.

Further If we take $n=1$ in Corollary 2.2 and Corollary 2.3, we get the following result proved by Uyanik et al [7].

Corollary 2.4. If $f(z)\in A$ satisfies

$$\left|\left(\frac{f(z)}{z}\right)^{\frac{1}{1-\beta}}\left\{\frac{zf'(z)}{f(z)}-\beta\right\}-1+\beta\right|<\frac{2}{\sqrt{5}}(1-\beta)\quad(z\in\mathbb{U}),$$

for $0\le\beta<1$, then $f(z)\in\mathcal{S}^*(\beta)$.

Corollary 2.5. If $f(z)\in\mathcal{A}$ satisfies

$$\left|(f'(z))^{\frac{\beta}{1-\beta}}\left\{f'(z)+\frac{1}{1-\beta}zf''(z)\right\}-1\right|<\frac{2}{\sqrt{5}}\quad(z\in\mathbb{U}),$$

for $0\le\beta<1$, then $f(z)\in\mathcal{C}(\beta)$.

Remark 2.1. If we put $\beta=0$ in Corollary 2.2 and Corollary 2.3, we get the result proved by Mocanu [5] and Nunokawa et al [6] respectively.

Theorem 2.6. If $f(z)\in\mathcal{A}(n)$ satisfies

$$\left|\arg\left\{\frac{f(z)}{z^{1-\alpha}}\left(\frac{f'(z)}{f(z)}\right)^{\alpha}\right\}+(1-\beta)\arg\left\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\frac{zf''(z)}{f'(z)}+\alpha-\beta\right\}\right|$$
$$<\frac{\pi}{2}\,\delta_n\,(1-\beta)\quad(z\in\mathbb{U}),$$

where δ_n is the unique root of (1.2), then $f(z)\in\mathcal{H}(\alpha,n,\beta)$.

Proof. Let us set

$$p(z)=z\left(\frac{f(z)}{z^{1-\alpha}}\left(\frac{f'(z)}{f(z)}\right)^{\alpha}\right)^{\frac{1}{1-\beta}}=z+\frac{\alpha(n+1)\ a_{n+1}}{(1-\beta)}z^{n+1}+\dots$$

for $f(z)\in\mathcal{A}(n)$. Then clearly (2.2) shows that $p(z)\in\mathcal{A}(n)$.

Differentiating (2.2), we have

$$p'(z)=\frac{p(z)}{z}\left\{\frac{1}{(1-\beta)}\left\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\frac{zf''(z)}{f'(z)}+\alpha-1\right\}+1\right\}$$

which gives

$$|\arg p'(z)|=\left|\arg\frac{p(z)}{z}+\arg\left\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\frac{zf''(z)}{f'(z)}+\alpha-\beta\right\}\right|.$$

Thus using (2.1), we have

$$|\arg p'(z)|\le\frac{\pi}{2}\,\delta_n\quad(z\in\mathbb{U}),$$

where δ_n is the root of (1.2). Hence, using Lemma 1.2, we have $p(z)\in\mathcal{S}^*(n,0)$.

From (2.3), we can write

$$\frac{zp'(z)}{p(z)} = \frac{1}{1-\beta}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zf''(z)}{f'(z)} + \alpha - \beta\right\}.$$

Since $p(z) \in \mathcal{S}^*(n,0)$, it implies that $Re\frac{zp'(z)}{p(z)} > 0$. Therefore, we get

$$\frac{1}{1-\beta}Re\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zf''(z)}{f'(z)} + \alpha - \beta\right\} = Re\frac{zp'(z)}{p(z)} > 0$$

or

$$\frac{1}{1-\beta}Re\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zf''(z)}{f'(z)} + \alpha - \beta\right\} > 0.$$

and this implies that $f(z) \in \mathcal{H}(\alpha, n, \beta)$.

Making $\alpha = 0$ in Theorem 2.6, we have

Corollary 2.7. If $f(z) \in \mathcal{A}(n)$ satisfies

$$\left|\arg\left(\frac{f(z)}{z}\right) + (1-\beta)\arg\left\{\frac{zf'(z)}{f(z)} - \beta\right\}\right| < \frac{\pi}{2}\,\delta_n\,(1-\beta)\quad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}^*(n,\beta)$.

Further if we take $n = 1$ in Corollary 2.7, we get the following result proved by Uyanik et al [7].

Corollary 2.8. If $f(z) \in \mathcal{A}$ satisfies

$$\left|\arg\left(\frac{f(z)}{z}\right) + (1-\beta)\arg\left\{\frac{zf'(z)}{f(z)} - \beta\right\}\right| < \frac{\pi}{2}\,\delta_1\,(1-\beta)\quad (z \in \mathbb{U}),$$

where δ_1 is the unique root of the equation

$$2\tan^{-1}[(1-\delta_1)] + \pi(1-2\delta_1) = 0,$$

then $f(z)$ belongs to the class of starlike functions of order β.

Taking $\alpha = 1$ in Theorem 2.6, we have

Corollary 2.9. If $f(z) \in \mathcal{A}(n)$ satisfies

$$\left|\arg(f'(z)) + (1-\beta)\arg\left\{\frac{zf''(z)}{f'(z)} + 1 - \beta\right\}\right| < \frac{\pi}{2}\delta_n(1-\beta),\quad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{C}(n,\beta)$.

Remark 2.2. If we take $n = 1$ in Corollary 2.9, we get the result proved in [7] and further for $\beta = 0$, we get the result proved by Mocanu [8].

3. Generalized Integral Operator

For $f(z) \in \mathcal{A}(n)$, we consider

$$G(z) = \left\{\delta_0^z t^{\delta-1}\left(\frac{f(t)}{t}\right)^{\gamma} dt\right\}^{\frac{1}{\delta}} = z + \frac{\gamma\ a_{n+1}}{(n+\delta)}\ z^{n+1} + \ldots \tag{3.1}$$

Clearly $G(z) \in \mathcal{A}(n)$ and when $\gamma = 1$ and $\delta = 1$, then (3.1) reduces to the well-known Alexander integral operator [9].

Theorem 3.1. If $\gamma \geq \delta$, $\delta > 0$ and $f(z) \in \mathcal{A}(n)$ satisfies

$$\left|\left(\frac{f(z)}{z}\right)^{\frac{\gamma}{\delta}}\left\{1 + \frac{\gamma}{\delta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} - 1\right| \leq \frac{(n+1)}{\sqrt{(n+1)^2+1}}, \tag{3.2}$$

then $f(z) \in \mathcal{S}^*(n,0)$.

Proof. From (3.1), we get

$$G^{\delta-1}(z)\,G'(z) = z^{\delta-1}\left(\frac{f(z)}{z}\right)^{\gamma}. \tag{3.3}$$

Differentiating (3.3), logarithmically, we get

$$(\delta-1)\frac{G'(z)}{G(z)} + \frac{G''(z)}{G'(z)} = (\delta-1)\frac{1}{z} + \gamma\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right). \tag{3.4}$$

Then by simple computation, we have,

$$\left|\frac{G(z)}{z}\left(\frac{zG'(z)}{G(z)}\right)^{\frac{1}{\delta}}\left\{\left(1-\frac{1}{\delta}\right)\frac{zG'(z)}{G(z)}+\frac{1}{\delta}\left(\frac{zG''(z)}{G'(z)}+1\right)\right\}-1\right|$$

$$=\left|\left(\frac{f(z)}{z}\right)^{\frac{\gamma}{\delta}}\left\{1+\frac{\gamma}{\delta}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}-1\right|\leq\frac{(n+1)}{\sqrt{(n+1)^2+1}},$$

where we have used (3.2). Therefore

$$\left|\frac{G(z)}{z}\left(\frac{zG'(z)}{G(z)}\right)^{\frac{1}{\delta}}\left\{\left(1-\frac{1}{\delta}\right)\frac{zG'(z)}{G(z)}+\frac{1}{\delta}\left(\frac{zG''(z)}{G'(z)}+1\right)\right\}-1\right|\leq\frac{(n+1)}{\sqrt{(n+1)^2+1}}.$$

By using Theorem 2.1 with $\beta=0$, $\alpha=\frac{1}{\delta}$, we have $G(z)\in\mathcal{H}\left(\frac{1}{\delta},n,0\right)$.

From (3.4), we can write

$$Re\left\{\left(1-\frac{1}{\delta}\right)\frac{zG'(z)}{G(z)}+\frac{1}{\delta}\left(\frac{zG''(z)}{G'(z)}+1\right)\right\}=\frac{\gamma}{\delta}Re\frac{zf'(z)}{f(z)}-\frac{\gamma}{\delta}+1,$$

or

$$Re\frac{zf'(z)}{f(z)}>\left(1-\frac{\delta}{\gamma}\right)\quad\left(\text{since } G(z)\in\mathcal{H}\left(\frac{1}{\delta},n,0\right)\right)$$

which shows that $f(z)\in\mathcal{S}^*(n,0)$, where $\gamma\geq\delta$.

Theorem 3.2. If $\gamma\geq\delta$, $\delta>0$ and $f(z)\in\mathcal{A}(n)$ satisfies

$$\left|\arg\left(\frac{f(z)}{z}\right)^{\frac{\gamma}{\delta}}+\arg\left\{\frac{\gamma}{\delta}\left(\frac{zf'(z)}{f(z)}-1\right)+1\right\}\right|<\frac{\pi}{2}\delta_n,$$

where δ_n is the unique root of (1.2), then $f(z)\in\mathcal{S}^*(n,0)$.

Proof. The result follows on similar lines as in the last theorem using Theorem 2.6 with $\beta=0$ and $\alpha=\frac{1}{\delta}$.

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