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ANALYTICAL SOLUTION OF AN SEIV EPIDEMIC MODEL BY HOMOTOPY PERTURBATION METHOD

SAEED ISLAM[1], SYED FARASAT SADDIQ[2], GUL ZAMAN[3], MUHAMMAD ALTAF KHAN*[1]
FAROOQ AHMAD [3] AND MURAD ULLAH[2]

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[1]Department of Mathematics, Abdul Wali Khan, University Mardan, Khyber Pakhtunkhwa, Pakistan
[2]Department of Mathematics, Islamia College University, Peshawar, Khyber Pakhtunkhwa, Pakistan
[3]Department of Mathematics, University of Malakand, Chakdara Dir(L), Khyber Pakhtunkhwa, Pakistan
Email: altafdir@gmail.com

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ABSTRACT. *In this paper, we consider an SEIV epidemic model which represents the interection of infected and susceptible individuals in the population through horizontal transmission. We find the analytical solution of the proposed model by Homotopy perturbation method which is one of the best method for finding the solution of the nonlinear problem. By using this techniques, first, we solve the problem analytically and then compare the numerical results with other standards methods. We also justfy the numerical simulation and their results. Mostly nonlinear problem have upon some difficulties, and their solution is some time difficult to obtain. However, this techniques help us to obtain their approximate as well as analytical solution just after few perturbation terms.*

Keywords*:* Epidemic models; Homotopy Perturbation Method; Numerical simulations.

1. **Introduction.** Mathematical modeling has become important tools in analyzing the spread and control of infectious diseases. These models help us to understand different factors like the transmission and recovery rates and predict how the diseases will spread over a period of time [1, 2, 3]. In recent years, many attempts have been made to develop realistic mathematical models for investigating the transmission dynamics of infectious diseases and different possible equilibria see for example [4]. To understand the behavior of epidemic model, we need to know the analysis of steady states and their stability [5]. To understand the dynamical interaction of epidemic in population usually ordinary differential equations (ODEs) system namely an SIR (susceptible, infectious, recovered) model is used [6,16,17]. In the SIR epidemic model the disease incubation is negligible such that once infected, each susceptible individual becomes infectious instantaneously and later recovers with a temporary acquired immunity.

Most of the biological problems in the form of epidemic models are inherently nonlinear. Therefore it's not only difficult but always impossible to find the exact solutions that represent the complete biological phenomena. So, the scientists are in search to find such numerical methods or perturbations method to find the exact solution and approximate solution to these non-linear problems. In the numerical methods, stability and convergence should be considered so as to avoid divergence or inappropriate results. While, in the analytical perturbation method, we need to exert the small parameter in the equation. Therefore, finding the small parameter and exerting it into the equation are difficulties of this method. However, there are some limitations with the common perturbation method, like the common perturbation method is based upon the existence of a small parameter, which is difficult to apply to real world problems. Therefore, many different powerful

mathematical methods have been recently introduced to vanish the small parameters, such as artificial parameter method [7, 8].

The Homotopy Analysis Method (HAM) is one of the wellknown methods to solve the nonlinear equations. In the last decade, the idea of homotopy was combined with perturbation. The fundamental work was done by Liao and He. This method involves a free parameter, whose suitable choice results into fast convergence. First time He [9], introduced Homotopy Perturbation Method (HPM) and its application in several problems see for example and the references there in [10, 11]. Ali et al [12], presented the solution of multi points boundary values by using Optimal Homotopy Analysis Method (OHAM). These methods are independent of the assumption of small parameter as well as they covered all the advantages of the perturbation method.

In this paper, we consider the model presented in [13] by applying the Homotopy perturbation method, to find the approximate solution. First, we formulate our problem and then apply the HPM to find the analytical as well as numerical solutions.

The paper is organized as follows. Section 2 is devoted to the basic idea of HPM and the mathematical formulation of the model. In Section 3 the model is solved by HPM. We present the numerical solution and discussion in section 4.

Basic Idea of HPM: In this section, we explain the Homotopy perturbation method in detail and then we apply the technique of HPM to our proposed epidemic model. HPM was first time introduced by He [7, 14] for solving the nonlinear differential equations.

$$B(m) = f(d), \qquad d \in \Lambda \qquad (1)$$

with the boundary conditions

$$\psi\left(m, \frac{\partial m}{\partial n}\right) = 0, \qquad d \in \Omega \qquad (2)$$

Here B represents the general differential operator, ψ is the boundary operator, $f(d)$ is the analytic function, Ω is the boundary of the domain Λ, $\frac{\partial}{\partial n}$ represents the differentiation along the normal vector drawn outward from Λ. The operator B is divided in two parts, H is linear and K is nonlinear. So we get equation (3) in the following form:

$$H(m)+K(m)=f(d). \qquad (3)$$

Define the homotopy $v(r, p) : \Lambda \times [0,1]$™ □ , that satisfies

$$F(v, p) = (1-p)\left[H(v) - H(m_o)\right]$$
$$+p\left[B(v) - f(d)\right] = 0 \quad (4),$$

also we can simplified form:

$$F(v,p)=H(v)+pH(m_o)+p[K(m)-f(d)]=0, \qquad (5)$$

where m_o shows the initial approximation of (5) and p is the embedding parameter, $p \in [0,1]$. We see that

$$F(v,0) = \left[H(v) - H(m_o)\right] = 0,$$
$$F(v,1) = \left[B(v) - f(d)\right] = 0. \qquad (6)$$

For p=0 we get,

$$\left[H(v)-H(m_o)\right].$$

While for p=1, we get

$$F(v,1)=\left[B(v)-f(d)\right].$$

To applying the perturbation technique, we consider p is the small parameter then the solution of equation (4) can be considered as series in p, which is given by

$$v = v_o + pv_1 + p^2v_2 + p^3v_3 + ..., \quad (7)$$

when p approaches 1 the equation (4) becomes the original equation (3) and (7) becomes the approximate solution of (3) given by

$$m = \lim_{p™ 1} v = v_o + pv_1 + p^2v_2 + p^3v_3 + ..., \quad (8)$$

The series (8) is convergent for most of the cases, for reader sees [6,8].

2. Formulation of the Problem. In this section, we formulate our problem here, let S(t), E(t), I(t) and V(t) respectively represents the susceptible, exposed, Infected and vaccinated individuals.

$$\begin{aligned}
\frac{dS(t)}{dt} &= (1-p)\Lambda - \frac{\beta S(t)I(t)}{\psi(I)} - \gamma S(t) + \alpha V(t),\\
\frac{dE(t)}{dt} &= \frac{\beta S(t)I(t)}{\psi(I)} - (\gamma+n)E(t), \qquad (9)\\
\frac{dI(t)}{dt} &= nE(t) - (\gamma+\delta)I(t),\\
\frac{dV(t)}{dt} &= p\Lambda - \gamma V(t) + \delta I(t) - \alpha V(t),
\end{aligned}$$

Subject to the initial conditions

$$S(0)\geq 0, \quad E(0)\geq 0, \quad I(0)\geq 0, \quad V(0)\geq 0. \quad (10)$$

Here p represents the fraction of recovered individuals, β represents the transmission rate, the recruitment rate for human individuals is Λ, α is the rate of vaccine wanes, the natural death rate of the population is shown by γ, at the rate of n the exposed individuals become infected and the rate of recovery of treated for the infected individuals is denoted by δ.

Solution of Model by HPM. Now we apply the homotopy perturbation method to our proposed model (9), to get the following form.

$$\begin{aligned}
\mathrm{L}S(t) - \mathrm{L}S^o(t) &= q\Big((1-p)\Lambda - \frac{\beta S(t)I(t)}{\psi(I)} - \gamma S(t) + \alpha V(t) - \mathrm{L}S^o(t)\Big),\\
\mathrm{L}E(t) - \mathrm{L}E^o(t) &= q\Big(\frac{\beta S(t)I(t)}{\psi(I)} - (\gamma+n)E(t) - \mathrm{L}E^o(t)\Big),\\
\mathrm{L}I(t) - \mathrm{L}I^o(t) &= q\Big(nE(t) - (\gamma+\delta)I(t) - \mathrm{L}I^o(t)\Big), \qquad (11)\\
\mathrm{L}V(t) - \mathrm{L}V^o(t) &= q\Big(p\Lambda - \gamma V(t) + \delta I(t) - \alpha V(t) - \mathrm{L}V^o(t)\Big)
\end{aligned}$$

Here we define the operator $\mathcal{L} = \frac{d}{dt}$. The initial condition is

$$S_o(t) = S(0), \quad E_o(t) = E(0), \quad I_o(t) = I(0), \quad V_o(t) = V(0). \quad (12)$$

In the following we assume the solution for system (11) in the form,

$$S^*(t) = S_o^*(t) + qS_1^*(t) + q^2S_2^*(t) + ...$$
$$E^*(t) = E_o^*(t) + qE_1^*(t) + q^2E_2^*(t) + ...$$
$$I^*(t) = I_0^*(t) + qI_1^*(t) + q^2I_2^*(t) + ... \qquad (13)$$
$$V^*(t) = V_0^*(t) + qV_1^*(t) + q^2V_2^*(t) + ...$$

Making use of (13) in (11) and comparing the coefficient of the same power, we get

$$\mathsf{L}S(t) - \mathsf{L}S^o(t) = 0,$$
$$\mathsf{L}E(t) - \mathsf{L}E^o(t) = 0,$$
$$\mathsf{L}I(t) - \mathsf{L}I^o(t) = 0, \qquad (14)$$
$$\mathsf{L}V(t) - \mathsf{L}V^o(t) = 0,$$

And

$$\mathsf{L}S_1^*(t) = \left((1-p)\Lambda - \frac{\beta S_o^*(t)I_o^*(t)}{\psi(I_o^*)} - \gamma S_o^*(t) + \alpha V_o^*(t) - \mathsf{L}S_o^*(t)\right),$$
$$\mathsf{L}E_1^*(t) = \left(\frac{\beta S_o^*(t)I_o^*(t)}{\psi(I_o^*)} - (\gamma+n)E_o^*(t) - \mathsf{L}E_o^*(t)\right), \qquad (15)$$
$$\mathsf{L}I_1^*(t) = \left(nE_o^*(t) - (\gamma+\delta)I_o^*(t) - \mathsf{L}I_o^*(t)\right),$$
$$\mathsf{L}V_1^*(t) = \left(p\Lambda - \gamma V_o^*(t) + \delta I_o^*(t) - \alpha V_o^*(t) - \mathsf{L}V_o^*(t)\right),$$

with the conditions

$$S_1^*(t) = 0, \qquad E_1^*(t) = 0, \qquad I_1^*(t) = 0, \qquad V_1^*(t) = 0. \qquad (16)$$

And

$$\mathsf{L}S_2^*(t) = \frac{\beta(S_o^*(t)I_1^*(t) + S_1^*(t)I_o^*(t))}{\psi(I_1^*)} - \gamma S_1^*(t) + \alpha V_1^*(t),$$
$$\mathsf{L}E_2^*(t) = \frac{\beta(S_o^*(t)I_1^*(t) + S_1^*(t)I_o^*(t))}{\psi(I_1^*)} - (\gamma+n)I_1^*(t),$$
$$\mathsf{L}I_2^*(t) = \left(nE_1^*(t) - (\gamma+\delta)I_1^*(t)\right), \qquad (17)$$
$$\mathsf{L}V_2^*(t) = -\gamma V_1^*(t) + \delta I_1^*(t) - \alpha V_1^*(t),$$

with the conditions

$$S_2^*(t) = 0, \qquad E_2^*(t) = 0, \qquad I_2^*(t) = 0, \qquad V_2^*(t) = 0. \qquad (18)$$

In similar fashion, we obtain

$$\mathsf{L}S_3^*(t) = \frac{\beta(S_o^*(t)I_2^*(t) + S_1^*(t)I_1^*(t) + S_2^*(t)I_o^*(t))}{\psi(I_2^*)} - \gamma S_2^*(t) + \alpha V_2^*(t),$$
$$\mathsf{L}E_3^*(t) = \frac{\beta(S_o^*(t)I_2^*(t) + S_1^*(t)I_2^*(t) + S_2^*(t)I_o^*(t))}{\psi(I_2^*)} - (\gamma+n)I_2^*(t),$$
$$\mathsf{L}I_3^*(t) = \left(nE_2^*(t) - (\gamma+\delta)I_2^*(t)\right), \qquad (19)$$
$$\mathsf{L}V_3^*(t) = -\gamma V_2^*(t) + \delta I_2^*(t) - \alpha V_2^*(t),$$

To find the solution, we put p=1 in the system (13), we get

$$
\begin{aligned}
S^*(t) &= S_o^*(t) + S_1^*(t) + S_2^*(t) + \ldots \\
E^*(t) &= E_o^*(t) + E_1^*(t) + E_2^*(t) + \ldots \\
I^*(t) &= I_0^*(t) + I_1^*(t) + I_2^*(t) + \ldots \qquad (20) \\
V^*(t) &= V_0^*(t) + V_1^*(t) + V_2^*(t) + \ldots
\end{aligned}
$$

The convergence of HPM is rapid, for few iterations of both linear and non-linear.

Zeroth order solution or P^0. $S_0^*(t) = 100, \quad E_0^*(t) = 8, \quad I_0^*(t) = 10, \quad V_0^*(t) = 20$

First order solution or P^1

$$
\begin{aligned}
\mathsf{L}S_1^*(t) &= \left((1-p)\Lambda - \frac{\beta g_1 g_3}{\psi(g_3)} - \gamma g_1 + \alpha g_4\right)t, \\
\mathsf{L}E_1^*(t) &= \left(\frac{\beta g_1 g_3}{\psi(g_3)} - (\gamma + n)g_2\right)t, \\
\mathsf{L}I_1^*(t) &= \left(ng_2 - (\gamma + \delta)g_3\right)t, \\
\mathsf{L}V_1^*(t) &= \left(p\Lambda - \gamma g_4 + \delta g_3 - \alpha g_4\right)t,
\end{aligned}
$$

$$
S_0^*(t) = 100 = g_1, \quad E_0^*(t) = 8 = g_2, \quad I_0^*(t) = 10 = g_3, \quad V_0^*(t) = 20 = g_4.
$$

Second order solution or P^2

$$
\begin{aligned}
S_2^*(t) = &-\frac{\beta g_3\left((1-p)\Lambda - \frac{\beta g_1 g_3}{\psi(g_3)} - \gamma g_1 + \alpha g_4\right) + \beta g_1\left(ng_2 - (\gamma+\delta)g_3\right)}{\psi(g_2)}\frac{t^2}{2} \\
&-\gamma\left((1-p)\Lambda - \frac{\beta g_1 g_3}{\psi(g_3)} - \gamma g_1 + \alpha g_4\right)\frac{t^2}{2} + \alpha\left(p\Lambda - \gamma g_4 + \delta g_3 - \alpha g_4\right)\frac{t^2}{2},
\end{aligned}
$$

$$
\begin{aligned}
E_2^*(t) = &\frac{\beta g_3\left((1-p)\Lambda - \frac{\beta g_1 g_3}{\psi(g_3)} - \gamma g_1 + \alpha g_4\right)\frac{t^2}{2} + \beta g_1\left(ng_2 - (\gamma+\delta)g_3\right)}{\psi(g_2)}\frac{t^2}{2} \\
&-(\gamma + n)\left(ng_2 - (\gamma+\delta)g_3\right)\frac{t^2}{2},
\end{aligned}
$$

$$
\begin{aligned}
I_2^*(t) &= n\left(\frac{\beta g_1 g_3}{\psi(g_3)} - (\gamma+n)g_2\right)\frac{t^2}{2} - (\gamma+\delta)\left(ng_2 - (\gamma+\delta)g_3\right)\frac{t^2}{2}, \\
V_2^*(t) &= -\gamma\left(p\Lambda - \gamma g_4 + \delta g_3 - \alpha g_4\right)\frac{t^2}{2} + \delta\left(ng_2 - (\gamma+\delta)g_3\right)\frac{t^2}{2} - \\
&\alpha\left(p\Lambda - \gamma g_4 + \delta g_3 - \alpha g_4\right)\frac{t^2}{2}.
\end{aligned}
$$

3. Numerical results. In this section, we solve our proposed model numerically by using HPM method, the Runge-Kutta order 4 and the non standard finite difference method (NSFD). The parameter values used for numerical simulation is given in Table1. Our numerical simulations results represented in **Figure -1 to 4**, represents, susceptible, exposed, infected and vaccinated individuals, respectively. The solution obtained from HPM have a good agreement with RK4 and NSFD.

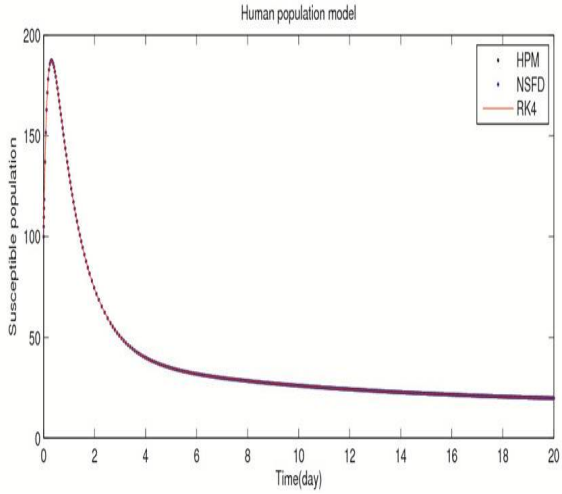

Figure-1: The plot represents the comparison of HPM with others standards methods of susceptible individuals.

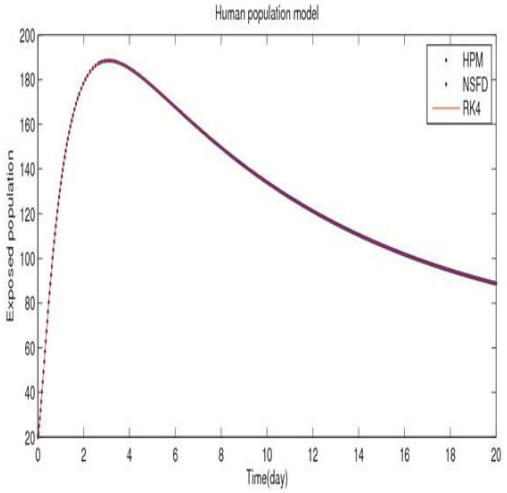

Figure-2: The plot represents the comparison of HPM with others standards methods of exposed individuals.

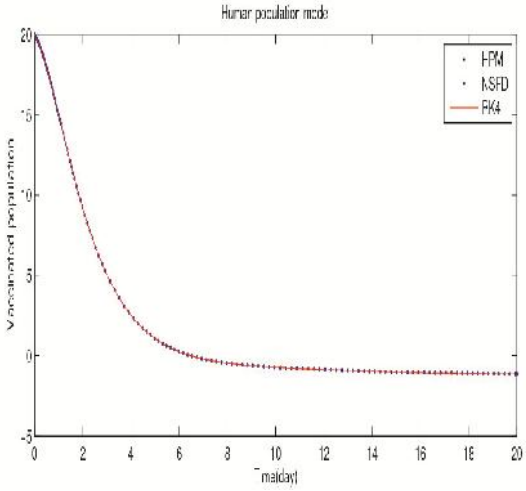

Figure-3: The plot represents the comparison of HPM with others standards methods of vaccinated individuals.

Table 1. Values used in the numerical simulation

p	The fraction of recovered individuals	0.4
β	The rate at which the susceptible individuals become infected	0.078
Λ	Represent the birth rate	2
α	Represent the rate at which vaccine wanes	0.03
γ	The natural mortality rate	0.0052
n	The rate at which the exposed individuals become infected	0.0087
δ	The recovery rate of individuals	9.3×10^{-3}

Conclusion.In this paper, we considered an SEIV epidemic model and applied the homotopy perturbation techniques. By applying this technique, we obtained the solution of zeroth, first and second order. Our analytical solution shown that for non-linear ODEs just a few iterations of HPM gives a good results. We also solved numerically the epidemic model and compared our numerical results with NSFD and RK4 methods. The numerical results shown good agreement with others numerical method.

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