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SOLUTION OF AN SEIR EPIDEMIC MODEL IN FRACTIONAL ORDER

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ABSTRACT. *In this paper, we consider the SEIR (Susceptible-Exposed-Infected-Recovered) epidemic model (with out of bilinear incidence rates) in fractional order. First the non-negative solution of the SEIR model in fractional order is discussed. Then calculate an approximate solution of the proposed model. The obtained results are compaired with those obtained by forth order Runge-Kutta method and nonstandard numerical method in the integer case. Finally, we present some numerical results.*

***Keywords**:* Fractional differential equations; Mathematical model; Epidemic model; Non-standard scheme; Differential transform method.

1. **Introduction.** For analyzing the spread and control of different diseases mathematical models have used as important tools. So for this a lot of mathematical models for different infectious diseases were proposed by several researchers for the purpose to overcome on the different infectious diseases. Hethcote [1], presented the interaction of susceptible S(t), infected I(t) and recovered R(t) individuals is given by:

$$\frac{dS(t)}{dt} = N(t) - \mu S(t) - \beta S(t)I(t),$$

$$\frac{dI(t)}{dt} = \beta S(t)I(t) - (\gamma + \mu)I(t),$$

$$\frac{dR(t)}{dt} = \gamma I(t) - \mu R(t).$$

Here $N(t)$ is the total population, β is the interaction rate of infection, μ is the death rate and γ is the recovery rate. Also Shulgin et al. [2] considered this model with pulse vaccination. Zaleta and Henandez [3] considered a simple two dimensional SIS model with vaccination showing backward bifurcation. There is a lot of work presented by many authors about the vaccination to control the diseases [1-11]. Here we consider the model of Zaman et al. [6] that presented the vaccination of SEIR model. In their work they shown that by introducing vaccination stratigy it is possible to eradicate or minimize the disease. But all these work has been done in the integer order differential equations.
Because of best presentaion of many phenmena the fractional calculus become more important. So every mathemetician try to use the fractional calculus in different fields of sciences. This is also the generalization of ordinary differential equations.
In this paper, we consider an SEIR model presented in Zaman et al. [6] in fractional order. First, we show the nonnegative solution of this model. Then we use the multi-step generalized differential transform method to approximate the numerical solution. Finally, we compare our numerical results with nonstandard numerical method and forth order Runge-Kutta method. This paper is organized as follows.

In Section 2, we present the model with some basic definitions and notations related to this work. In Section 3, we show the non-negative solution and uniqueness of the model. In Section 4, the multi-step generalized differential transform method (MSGDTM) is applied to the model. In Section 5, the numerical simulations are presented graphically. Finally we give the conclusion.

2. Formulation of Model with Preliminaries. The model presented in [6], is given by

$$\begin{aligned}
\frac{dS(t)}{dt} &= \lambda - \alpha E(t)S(t) - \beta_1 I(t)S(t) - \mu S(t), \\
\frac{dE(t)}{dt} &= \alpha E(t)S(t) - (\beta_2 + \gamma_1 + \mu)E(t), \\
\frac{dI(t)}{dt} &= \beta_1 I(t)S(t) + \beta_2 E(t) - (\gamma_2 + \mu)I(t), \\
\frac{dR(t)}{dt} &= \gamma_1 E(t) + \gamma_2 I(t) - \mu R(t),
\end{aligned} \tag{1}$$

with $S(0) = S_0,\ E(0) = E_0,\ I(0) = I_0,\ R(0) = R_0.$

The total population size is $N(t) = S(t) + E(t) + I(t) + R(t).$

So we obtain by adding all equations of the system (1)

$$\frac{dN(t)}{dt} = \lambda - \mu N(t). \tag{2}$$

Here λ is the recruitment rate of susceptible class, α is the rate of reduction percapita in the susceptible due to exposed class, β_1 is the transmission rate from susceptible to exposed class due to infected class, β_2 is the rate moving from exposed class to the infected class, γ_1 and γ_2 is the recovery rates of exposed class and infected class respectively and μ is the natural death rate.

Now we introduced fractional order to the system (1) which is consisting of ordinary differential equations. The new system is described by the following set of fractional order differential equations:

$$\begin{aligned}
D_t^{\alpha} S(t) &= \lambda - \alpha E(t)S(t) - \beta_1 S(t)I(t) - \mu S(t), \\
D_t^{\alpha} E(t) &= \alpha E(t)S(t) - (\beta_2 + \gamma_1 + \mu)E(t), \\
D_t^{\alpha} I(t) &= \beta_1 S(t)I(t) + \beta_2 E(t) - (\gamma_2 + \mu)I(t), \\
D_t^{\alpha} R(t) &= \gamma_1 E(t) + \gamma_2 I(t) - \mu R(t), \\
D_t^{\alpha} N(t) &= \lambda - \mu N(t).
\end{aligned} \tag{3}$$

Here we consider the Caputo sense fractional derivatives.

For basic defintions of fractional calculus see the appendix A.

3. Non-negative solutions

Let $R_+^5 = \{X \in R^5 : X \geq 0\}$ and $X(t) = (S(t), E(t), I(t), R(t), N(t))^T$. For the proof of the theorem about non-negative solutions we shall need the following Lemma [12].

Lemma 3.1. (Generalized Mean Value Theorem) Let $f(x) \in C[a,b]$ and $D^{\alpha} f(x) \in C[a,b]$ for $0 < \alpha \leq 1$. Then we have

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} D^{\alpha} f(\xi)(x-a)^{\xi},$$

with $0 \le \xi \le x$, for all $x \in (a,b]$.

Remark 3.2. Suppose $f(x) \in C[0,b]$ and $D^{\alpha} f(x) \in C[0,b]$ for $0 < \alpha \le 1$. It is clear from Lemma 3.1 that if $D^{\alpha} f(x) \ge 0$ for all $x \in (0,b)$, then the function *f* is non-decreasing, and if $D^{\alpha} f(x) \le 0$ for all $x \in (0,b)$, then the function *f* is non-increasing.

Theorem 3.3. There is unique solution for the initial value problem given in system (3), and the solution remains in R_+^5.

Proof. The existence and uniqueness of the solution of (3), in $(0,\infty)$ can be obtained from [13, Theorem 3.1 and Remark 3.2]. We need to show that the domain R_+^5 is positively invariant. Since

$$D_t^{\alpha} S(t)\Big|_{S=0} = \lambda \ge 0, \;\; D_t^{\alpha} E(t)\Big|_{E=0} = 0, \;\; D_t^{\alpha} I(t)\Big|_{I=0} = \beta_2 E(t) \ge 0,$$

$$D_t^{\alpha} R(t)\Big|_{R=0} = \gamma_1 E(t) + \gamma_2 I(t) \ge 0, \;\; D_t^{\alpha} N(t)\Big|_{N=0} = \lambda \ge 0.$$

On each hyperplane bounding the non-negative orthant, the vector field points into R_+^5.

4. Multi-step generalized differential transform method. We applying the multi-step generalized differential transform method to fine the approximate solution of system (3), which gives an accurate solution over a longer time frame as compaired to the standard generalized differential transform method. Taking the differential transform of system (3) with respect to time we obtain,

$$
\begin{aligned}
S(k+1) &= \frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}\Big(\lambda - \alpha \sum_{s=0}^{k} E(k-s)S(s) - \beta_1 \sum_{s=0}^{k} I(k-s)S(s) - \mu S(k)\Big),\\
E(k+1) &= \frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}\Big(\alpha \sum_{s=0}^{k} E(k-s)S(s) - (\beta_2 + \gamma_1 + \mu)E(k)\Big),\\
I(k+1) &= \frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}\Big(\beta_1 \sum_{s=0}^{k} I(k-s)S(s) + \beta_2 E(k) - (\gamma_2 + \mu)I(k)\Big),\\
R(k+1) &= \frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}\Big(\gamma_1 E(k) + \gamma_2 I(k) - \mu R(k)\Big),\\
N(k+1) &= \frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}\Big(\lambda - \mu N(k)\Big).
\end{aligned}
\tag{4}
$$

Here $S(k), E(k), I(k), R(k)$ and $N(k)$ are the differential transformation of $S(t), E(t), I(t), R(t)$ and $N(t)$. The differential transform of the initial conditions are

$S(0) = S_0,\ E(0) = E_0,\ I(0) = I_0,\ R(0) = R_0$ and $N(0) = N_0$.

In view of the differential inverse transform, the differential transform series solution for the system can be obtained as

$$
\begin{cases}
S(t) = \sum_{k=0}^{K} S(k)t^{\alpha k},\\
E(t) = \sum_{k=0}^{K} E(k)t^{\alpha k},\\
I(t) = \sum_{k=0}^{K} I(k)t^{\alpha k},\\
R(t) = \sum_{k=0}^{K} R(k)t^{\alpha k},\\
N(t) = \sum_{k=0}^{K} N(k)t^{\alpha k}.
\end{cases}
\tag{5}
$$

Now according to the multi-step generalized differential transform method the series solution for the system of equations (3) is suggested by

$$S(t)=\begin{cases}\sum_{k=0}^{K} S_1(k)t^{\alpha k}, & t\in[0,t_1]\\ \sum_{k=0}^{K} S_2(k)(t-t_1)^{\alpha k}, & t\in[t_1,t_2]\\ \cdot & \\ \cdot & \\ \cdot & \\ \sum_{k=0}^{K} S_M(k)(t-t_{M-1})^{\alpha k}, & t\in[t_{M-1},t_M]\end{cases} \tag{6}$$

Similar equations can be constructed for other individuals. The multi-step approach introduces a new idea for finding the approximate solution. Assume that the interval [*0,T*] is divided into *M* subintervals $[t_{i-1},t_i]$, for $i=1,2,...,M$ of equal step size $h=T/M$ by the nodes $t=ih$.

Here $S_i(k), E_i(k), I_i(k), R_i(k)$ and $N_i(k)$ for $i=1,2,...,M$ satisfy the following recurrence relations

$$\begin{aligned}
S_i(k+1)&=\frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}\left(\lambda-\alpha\sum_{s=0}^{k}E_i(k-s)_i S_i(s)-\beta_1\sum_{s=0}^{k}I_i(k-s)_i S_i(s)-\mu S_i(k)\right),\\
E_i(k+1)&=\frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}\left(\alpha\sum_{s=0}^{k}E_i(k-s)_i S_i(s)-(\beta_2+\gamma_1+\mu)E_i(k)\right),\\
I_i(k+1)&=\frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}\left(\beta_1\sum_{s=0}^{k}I_i(k-s)_i S_i(s)+\beta_2 E_i(k)-(\gamma_2+\mu)I_i(k)\right),\\
R_i(k+1)&=\frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}\left(\gamma_1 E_i(k)+\gamma_2 I_i(k)-\mu R_i(k)\right),\\
N_i(k+1)&=\frac{\Gamma(\alpha k+1)}{\Gamma((\alpha k+1)+1)}\left(\lambda-\mu N_i(k)\right).
\end{aligned} \tag{7}$$

With the initial conditions

$$S_i(0)=S_{i-1}(0),\quad E_i(0)=E_{i-1}(0),\quad I_i(0)=I_{i-1}(0),\quad R_i(0)=R_{i-1}(0),\quad \text{and}\quad N_i(0)=N_{i-1}(0).$$

Finally start with initial conditions $S(0)=S_0,\ E(0)=E_0,\ I(0)=I_0,\ R(0)=R_0\ \ and\ N(0)=N_0$, and use the recurrence relation given in the above system, we can obtained the multi-step generalized differential transform solution given for susceptible individual in (6) and for other individuals on the same way.

5. Numerical Method and Simulation. We solve analytically the system (3) with transform initial conditions by using the multi-step generalized differential transform method (MSGDTM). We also use nonstandard numerical method and forth-order Runge-Kutta method for numerical results. For numerical simulation we use a set of parameters given in Table 1. To demonstrate the effectiveness of proposed algorithm as an approximate tool for solving the nonlinear system (3) for large time *t*, we apply this algorithm on the interval [0-30].

Table 1: Parameter values for the numerical simulation

Notation	Parameter description	Value
μ	Natural death rate	0.007
β_1	Transmission rate from susceptible to exposed class due to infected class	0.0098
β_2	Transmission rate from susceptible to exposed class	0.007
γ_1	Recovery rate of exposed class	0.0091
γ_2	Recovery rate of infected class	0.007
α	The rate of reduction percapita due to exposed class	0.005

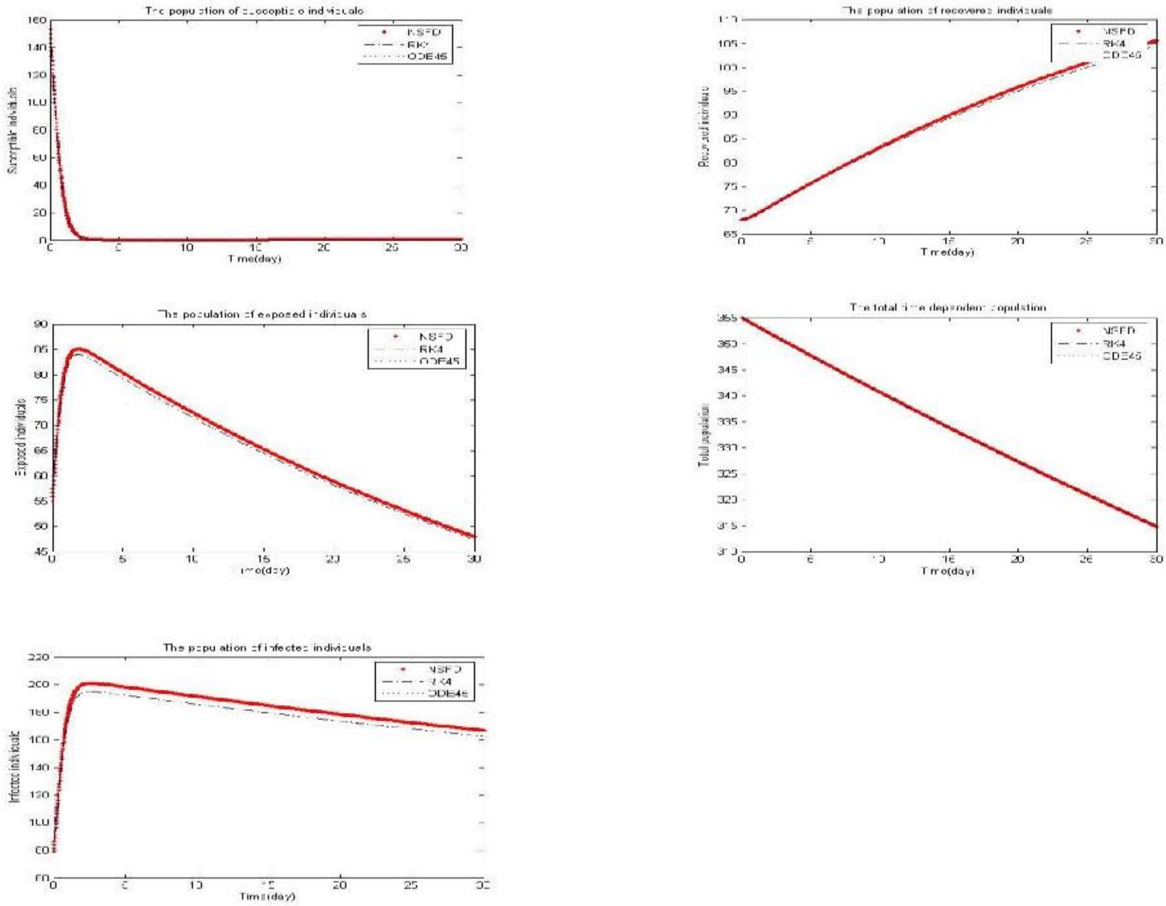


Fig 1. Shows the Susceptible, Exposed, Infected, Recovered and total time dependent population

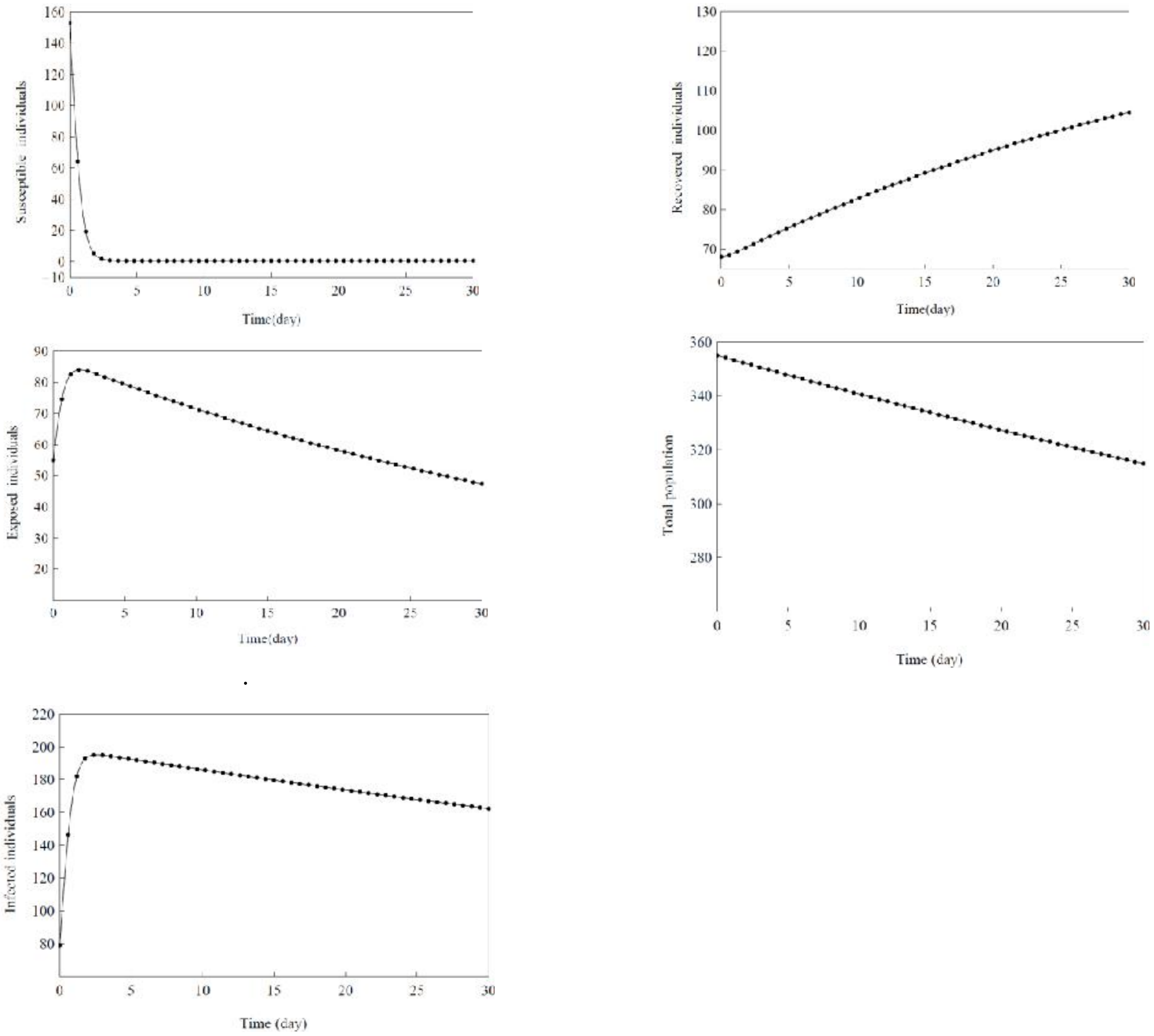
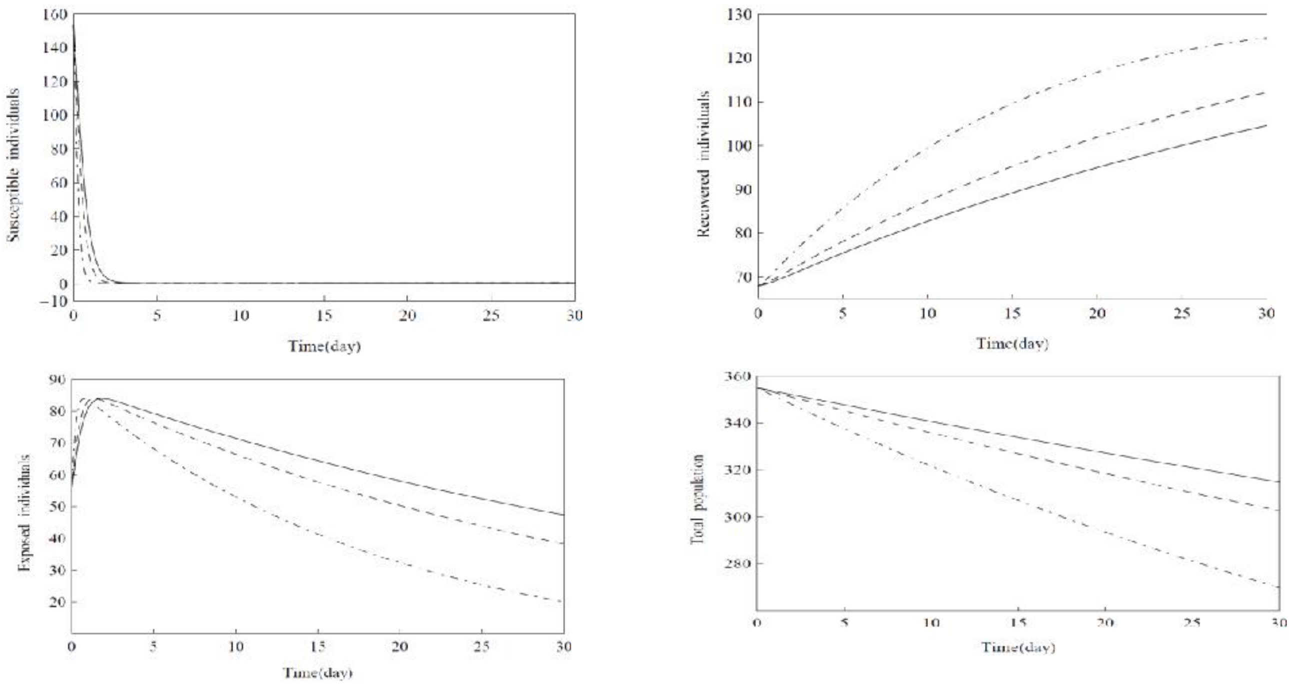


Fig. 2. $S(t), E(t), I(t), R(t), N(t)$ versus *t*: (solid line) MSGDTM, (dotted line) Runge-Kutta method.



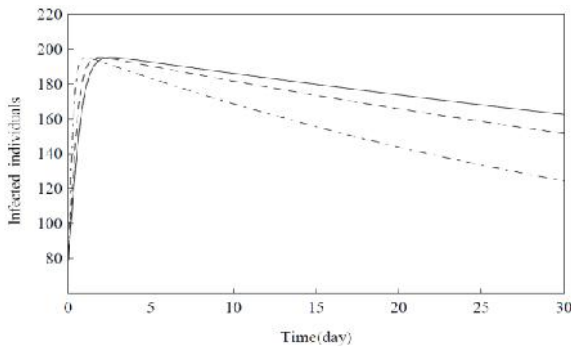


Fig. 3. $S(t), E(t), I(t), R(t), N(t)$ versus t: (solid line) $\alpha = 1.0$ (dashed line) $\alpha = 0.95$, (dot-dashed line) $\alpha = 0.85$

6. Conclusion. In this paper, a fractional order system for SEIR (Susceptible-Exposed-Infected-Recovered) epidemic model (with out of bilinear incidence rates) is studied and its approximate solution is presented using the multi-step generalized differential transform method (MSGDTM).

The approximate solution obtained by multi-step generalized differential transform method are highly accurate and valid for a long time in the integer case. This method is very applicable and also this is a good approach for the solutions of differential equations of such order. This tool is the best one for modeling in mathematics and other fields also.

Now we give some basic definitions and properties of the fractional calculus theory which are used further in this paper [13-16].

7. Appendix A

Definition 1. A function $f(x)(x>0)$ is said to be in the space $C_\alpha\,(\alpha \in R)$ if it can be written as $f(x) = x^p f_1(x)$ for some $p > \alpha$ where $f_1(x)$ is continuous in $[0,\infty)$, and it is said to be in the space $C_\alpha{}^m$ if $f^{(m)} \in C_\alpha,\ m \in \boldsymbol{N}$.

Definition 2. The Riemann–Liouville integral operator of order $\alpha > 0$ with $a \ge 0$ is defined as

$$(J_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \qquad x > a,$$

$$(J_a^0 f)(x) = f(x).$$

Properties of the operator can be found in [16]. We only need here the following:

For $f \in C_\alpha, \alpha, \beta > 0, a \ge 0, c \in R$ and $\gamma > -1,$ we have

$$(J_a^\alpha J_a^\beta f)(x) = (J_a^\beta J_a^\alpha f)(x) = (J_a^{\alpha+\beta} f)(x),$$

$$J_a^\alpha x^\gamma = \frac{x^{\gamma+\alpha}}{\Gamma(\alpha)} B_{\frac{x-a}{x}}(\alpha, \gamma+1),$$

where $B_\tau(\alpha, \gamma+1)$ is the incomplete beta function which is defined as

$$B_\tau(\alpha, \gamma+1) = \int_0^\tau t^{\alpha-1}(1-t)^\gamma\,dt,$$

$$J_a^{\alpha} e^{cx} = e^{ac}(x-a)^{\alpha} \sum_{k=0}^{\infty} \frac{[c(x-a)]^k}{\Gamma(\alpha+k+1)}.$$

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_a^{α} proposed by Caputo in his work on the theory of viscoelasticity.

Definition 3. The Caputo fractional derivative of $f(x)$ of order $\alpha > 0$ with $a \geq 0$ is defined as

$$(D_a^{\alpha} f)(x) = (J_a^{m-\alpha} f^{(m)})(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}} dt,$$

for $m-1 < \alpha \leq m, m \in \boldsymbol{N}, x \geq a, f(x) \in C_{-1}^{m}.$

The Caputo fractional derivative was investigated by many authors, for $m-1 < \alpha \leq m, f(x) \in C_{\alpha}^{m}$ and $\alpha \geq -1$, we have

$$(J_a^{\alpha} D_a^{\alpha} f)(x) = J^m D^m f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(a) \frac{(x-a)^k}{k!}.$$

For mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

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