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Upper Bound of the Third Hankel Determinant for a Subclass of Analytic Functions Subordinate to Cosine Function

Khurshid Ahmad[1], Serkan Araci[2,*], Mirajul Haq[3], Bilal Khan[4]

[1]Government Post Graduate Collage Dargai, Malakand, Pakistan
E-Mail: khurshidahmad410@gmail.com

[2]Department of Economics, Faculty of Economics Administrative and Social Sciences Hasan Kalyoncu University TR-27410 Gaziantep, Turkey
E-Mail: serkan.araci@hku.edu.tr
[*]**Corresponding Author**

[3]Department of Mathematics, Abdul Wali Khan University, Mardan 23200, Pakistan
E-Mail: merajkhan054@gmail.com

[4]School of Mathematical Sciences and Shanghai Key Laboratory of PMMP, East China Normal University, 500 Dongchuan Road, Shanghai 200241, Peoples Republic of China
E-Mail: bilalmaths789@gmail.com

Abstract

In this paper, we define a new subclass of analytic functions involving the cosine functions. For this function class we obtain upper bound of the third Hankel determinant.

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1 Introduction, Definitions and Motivation

Let by $\mathcal{H}(\mathbb{U})$ we denote the class of functions which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

where $\mathbb{C}$ is the set of complex numbers and let $\mathcal{A}$ be the class of analytic functions having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (\forall\ z \in \mathbb{U}), \tag{1.1}$$

in the open unit disk $\mathbb{U}$, centered at origin and normalized by the conditions. Also let $\mathcal{S}$ be a subclass of class $\mathcal{A}$, containing all univalent functions in $\mathbb{U}$ and are normalized by the conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

In 1916, working on the coefficients a_n of class $\mathcal{S}$, Bieberbach conjectured that

$$|a_n| \leq n \qquad (n = 2, 3, ...),$$

which was proved by Louis de Branges in 1984. During 1916-1984, researchers used different techniques and established a lot of coefficients results for some subfamilies of $\mathcal{S}$. Some of these subclasses are the class $\mathcal{S}^*$ of starlike functions, the class $\mathcal{K}$ of convex functions and the class $\mathcal{R}$ of bounded turning functions. These classes are defined as follows:

$$\mathcal{S}^* = \left\{ f \in \mathcal{S} : \mathfrak{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (\forall z \in \mathbb{U}) \right\},$$

$$\mathcal{K} = \left\{ f \in \mathcal{S} : \mathfrak{Re}\left(\frac{(zf'(z))'}{f'(z)}\right) > 0 \qquad (\forall z \in \mathbb{U}) \right\}$$

and

$$\mathcal{R} = \left\{ f \in \mathcal{S} : \mathfrak{Re}\left[f'(z)\right] > 0 \qquad (\forall z \in \mathbb{U}) \right\},$$

respectively.

We next denote by $\mathcal{P}$ the class of analytic functions p which are normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.2}$$

such that

$$\Re(p(z)) > 0 \qquad (\forall\ z \in \mathbb{U}).$$

Furthermore, we say that an analytic functions $f_1(z)$ is subordinated to $f_2(z)$ in $\mathbb{U}$ and is symbolically written as

$$f_1(z) \prec f_2(z) \qquad (\forall z \in \mathbb{U})$$

if there exists a schwarz function $u(z)$ with properties that

$$|u(z)| \leq 1 \quad \text{and} \quad u(0) = 1,$$

such that

$$f_1(z) = f_2(u(z)).$$

Moreover, if $f_2(z)$ is in the class $\mathcal{S}$, then we have the following equivalency,

$$f_1(0) = f_2(0) \ \text{and} \ f_1(\mathbb{U}) \subseteq f_2(\mathbb{U}).$$

Now by using the principle of subordination a generalized set of the classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}$ respectively are given as follows:

$$\mathcal{S}^*(\psi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \psi = \frac{1+z}{1-z} \qquad (\forall z \in \mathbb{U}) \right\}, \tag{1.3}$$

$$\mathcal{K}(\psi)=\left\{f\in\mathcal{S}:\frac{(zf'(z))'}{f'(z)}\prec\psi=\frac{1+z}{1-z}\qquad(\forall z\in\mathbb{U})\right\}\tag{1.4}$$

and

$$\mathcal{R}(\psi)=\left\{f\in\mathcal{S}:f'(z)\prec\psi=\frac{1+z}{1-z}\qquad(\forall z\in\mathbb{U})\right\}.\tag{1.5}$$

Also several other subclasses of starlike functions were introduced recently in [2, 7, 9, 19, 11, 17, 18, 1, 12] by choosing some particular functions for ψ such functions are associated with Bell numbers, shell-like curve connected with Fibonacci numbers and functions connected with the conic domains.

Lately, based on the techniques of Ma and Minda [21], Goel and Kumar in [10] defined the class $\mathcal{S}^*_{SG}$, based on subordination principle, as:

$$\frac{zf'(z)}{f(z)}\prec\frac{2}{1+e^{-z}}\qquad(\forall z\in\mathbb{U})$$

and studied its various important geometric properties.

Motivated by the above mentioned works, we now define the following.

Definition 1. *A function f of the form* (1.1) *is said to be in the class $\mathcal{SL}^*$ if and only if*

$$\frac{zp(z)}{f(z)}\prec\cos z.\tag{1.6}$$

Let $n\geq 0$ and $q\geq 1$. Then the q^{th} Hankel determinant is defined as:

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & . & . & . & a_{n+q-1}\\ a_{n+1} & . & & & & .\\ . & . & & & & .\\ . & . & & & & .\\ . & . & & & & .\\ a_{n+q-1} & . & . & . & . & a_{n+2(q-1)}\end{vmatrix}$$

The determinant has been studied by several authors. In particular, sharp upper bounds on $H_2(2)$ were obtained by the authors of articles [4, 13, 14, 27, 26, 22, 23] for various classes of functions. It is well-known that the Fekete-Szegö functional $\left|a_3-a_2^2\right|=H_2(1)$. This functional is further generalized as $\left|a_3-\mu a_2^2\right|$ for some μ real as well as complex. Fekete and Szegö gave sharp estimates of $\left|a_3-\mu a_2^2\right|$ for μ real and $f\in\mathcal{S}$, the class of univalent functions. It is also know that the functional $\left|a_2a_4-a_3^2\right|$ is equivalent to $H_2(2)$. Babalola [5] studied the Hankel determinant $H_3(1)$ for some subclasses of analytic functions. For more studied of third order Hankel determinat see, [6, 25, 24, 3]. In the present investigation, our focus is on the Hankel determinant $H_3(1)$ for the function class $\mathcal{SL}^*_q$.

2 Preliminaries

Lemma 1. (*see [20]*) *Let*

$$p(z)=1+c_1z+c_2z^2+\dots$$

is in the class $\mathcal{P}$ of functions positive real part in $\mathbb{U}$, then for any complex number υ

$$\left|c_2-\upsilon c_1^2\right| \leqq \begin{cases} -4\upsilon+2 & (\upsilon \leqq 0) \\ 2 & (0 \leqq \upsilon \leqq 1) \\ 4\upsilon-2 & (\upsilon \geqq 1)\,. \end{cases} \tag{2.1}$$

When $\upsilon < 0$ or $\upsilon > 1$, equality holds true in (2.1) if and only if

$$p(z)=\frac{1+z}{1-z}$$

or one of its rotations. If $0<\upsilon<1$, then equality holds true in (2.1) if and only if

$$p(z)=\frac{1+z^2}{1-z^2}$$

or one of its rotations. If $\upsilon = 0$, equality holds true in (2.1) if and only if

$$p(z)=\left(\frac{1+\rho}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\rho}{2}\right)\frac{1-z}{1+z} \qquad (0 \leqq \rho \leqq 1)$$

or one of its rotations. If $\upsilon = 1$,then the equality in (2.1) holds true if $p(z)$ is a reciprocal of one of the functions such that the equality holds true in the case when $\upsilon = 0$.

Lemma 2. *[15, 16] Let*

$$p(z)=1+c_1z+c_2z^2+\dots$$

is in the class $\mathcal{P}$ of functions positive real part in $\mathbb{U}$, then

$$2c_2=c_1^2+x\left(4-c_1^2\right)$$

for some x, $|x| \le 1$ and

$$4c_3=c_1^3+2\left(4-c_1^2\right)c_1x-\left(4-c_1^2\right)c_1x^2+2\left(4-c_1^2\right)\left(1-|x|^2\right)z$$

for some z, $|z| \le 1$.

Lemma 3. *[8] Let*

$$p(z)=1+c_1z+c_2z^2+\dots$$

is in the class $\mathcal{P}$ of functions positive real part in $\mathbb{U}$, then

$$|c_k| \le 2 \qquad (k \in \mathbb{N})$$

and the inequality is sharp.

3 Main Results

In this section, we will prove our main results.

Theorem 1. *Let $f \in \mathcal{SL}_q^*$ and be of the form (1.1), then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} -4\mu + \frac{5}{2} & \left(\mu < \frac{1}{8}\right) \\ 2 & \left(\frac{1}{8} \leq \mu \leq \frac{9}{8}\right) \\ 4\mu - \frac{5}{2} & \left(\mu > \frac{9}{8}\right). \end{cases}$$

These results are sharp.

Proof. If $f \in \mathcal{SL}_q^*$ then it follows from definition that

$$\frac{zp(z)}{f(z)} \prec \cos z. \tag{3.1}$$

Define a function

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + ...$$

it is clear that $p \in \mathcal{P}$. This implies that

$$w(z) = \frac{1 - p(z)}{1 + p(z)}.$$

From (3.1) we have

$$\frac{zp(z)}{f(z)} = \cos(w(z)),$$

or

$$f(z) = \frac{zp(z)}{\cos(w(z))}$$

Now

$$\frac{zp(z)}{\cos(w(z))} = z + c_1 z^2 + \left[c_2 + \frac{1}{8}c_1^2\right] z^3 + [c_3 + c_1 c_2] z^4 + ...$$

Similarly

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Therefore

$$a_2 = c_1 \tag{3.2}$$

$$a_3 = c_2 + \frac{1}{8}c_1^2 \tag{3.3}$$

$$a_4 = c_3 + \frac{c_1 c_2}{4} \tag{3.4}$$

and

$$a_5 = c_4 - \frac{7}{384}c_1^4 + \frac{1}{4}c_1c_3 + \frac{1}{8}c_2^2.$$

Thus

$$\left|a_3 - \mu a_2^2\right| = \left|c_2 - \left(\mu - \frac{1}{8}\right)c_1^2\right|, \tag{3.5}$$

Using Lemma (1) on (3.5), we obtain the required result. □

Theorem 2. *Let* $f \in \mathcal{SL}_q^*$ *and of the form* (1.1), *then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{70}{11}. \tag{3.6}$$

Proof. Making use of (3.2), (3.3) and (3.4)

$$a_2a_4 - a_3^2 = -\frac{1}{64}c_1^4 - c_2^2 + c_1c_3$$

Putting the value of c_2 and c_3 from Lemma 2, using triangular inequality and replacing $|x| < 1$ by ρ and c_1 by c, we have

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| \leq & \left[\frac{15c^4}{64} + \frac{c^2}{2} + \frac{\left(4-c^2\right)\rho}{2} + \frac{c^2\left(4-c^2\right)\rho}{2}\right. \\ & \left. + \frac{c^2\left(4-c^2\right)\rho^2}{4} + \frac{\left(4-c^2\right)\left(1-\rho^2\right)c}{2}\right] \\ & = F\left(c,\rho\right) \end{aligned} \tag{3.7}$$

Differentiating (3.7) with respect to ρ, we have

$$\frac{\partial F}{\partial \rho} = \frac{\left(4-c^2\right)}{2} + \frac{c^2\left(4-c^2\right)}{2} + \frac{c^2\left(4-c^2\right)\rho}{2} - \left(4-c^2\right)c\rho$$

It is clear that

$$\frac{\partial F(c,\rho)}{\partial \rho} > 0.$$

Which show that $F(c,\rho)$ is an increasing function on the closed interval $[0,1]$. This implies that maximum value occurs at $\rho = 1$. That is for maximum of

$$F(c,\rho) = F(c,1) = G(c) \qquad \text{(say)}.$$

Now

$$G(c) = \left[\frac{15c^4}{64} + \frac{c^2}{2} + \frac{\left(4-c^2\right)}{2} + \frac{c^2\left(4-c^2\right)}{2} + \frac{c^2\left(4-c^2\right)}{4}\right] \tag{3.8}$$

Differentiating (3.8) with respect to c we have

$$G'(c) = 6c - \frac{33c^3}{16}$$

Differentiating again above equation with respect to c we have

$$G''(c) = 6 - \frac{99c^2}{16}$$

For $c = \frac{4}{11}\sqrt{22}$ this shows that the max $G(c)$ occurs at $c = 0$. Hence we obtain

$$\left|a_2a_4 - a_3^2\right| \leq \frac{70}{11}.$$

□

Theorem 3. *Let $f \in \mathcal{SL}_q^*$ and of the form* (1.1), *then*

$$|a_2a_3 - a_4| \leq 10.$$

Proof. Using the values of (3.2), (3.3) and (3.4)

$$a_2a_3 - a_4 = c_1c_2^2 + \frac{c_1^3}{8} - c_3 - \frac{c_1c_2}{4}$$

Using Lemma 2 and since $c_1 \leq 2$ by Lemma 3, let $c_1 = c$ and assume without restriction that $c \in [0, 2]$. Taking absolute and applying the triangle inequality with $\rho = |x|$ we obtain

$$\begin{aligned} |a_2a_3 - a_4| \leq & \frac{c_1^5}{5} + \frac{\rho^2 c\left(4-c^2\right)^2}{4} + \frac{\rho c^3\left(4-c^2\right)}{2} + \frac{c^3}{4} \\ & + \frac{5\rho c\left(4-c^2\right)}{8} + \frac{\rho^2 c\left(4-c^2\right)}{4} + \frac{\left(1-\rho^2\right)\left(4-c^2\right)^2}{2} \\ & = F(\rho) \text{ say.} \end{aligned}$$

Differentiating $F(\rho)$ with respect to ρ we have

$$\begin{aligned} F'(\rho) = & \frac{\rho c\left(4-c^2\right)^2}{2} + \frac{c^3\left(4-c^2\right)}{2} + \frac{5c\left(4-c^2\right)}{8} \\ & + \frac{\rho c\left(4-c^2\right)}{2} + \frac{(c-1)\left(4-c^2\right)\rho}{2} \\ & > 0. \end{aligned}$$

This implies that $F(\rho)$ is an increasing function of ρ on the closed interval $[0, 1]$.

Hence $F(\rho) \leq F(1)$ for all $\rho \in [0, 1]$ that is

$$\begin{aligned} F(\rho) & \leq \frac{15c}{2} - \frac{5c^3}{8} \\ & = G(c) \text{ say.} \end{aligned}$$

Differentiating $G(c)$ with respect to c we have

$$G'(c) = \frac{15}{2} - \frac{15c^2}{8}.$$

Again differentiating the above equation with respect to c we have

$$G''(c) = -\frac{15c}{4}.$$

Since $c \in [0,2]$, by the assumption, it follows that $G(c)$ attains maximum at $c = 2$,which corresponds to $\rho = 1$, and it is the desired upper bound. □

Theorem 4. *Let* $f \in \mathcal{SL}_q^*$ *and of the form* (1.1)*, then*

$$H_3(1) \leq \frac{6907}{132}.$$

Proof. Since

$$H_3(1) \leq |a_3| \left|\left(a_2a_4 - a_3^2\right)\right| + |a_4| \left|(a_2a_3 - a_4)\right| + |a_5| \left|\left(a_1a_3 - a_2^2\right)\right|.$$

Using the fact that $a_1 = 1$, with Theorem 1, Theorem 2, Theorem 3 and Lemma 3, we have the required result. □

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