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A Study of Third Hankel Determinant for Certain Subclasses of Analytic Function

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Abstract

Recently the Hankel determinant problems got attractions of many well-known authors. Third Hankel determinant problems were determined for different subclasses of analytic functions. Here in our present investigation, we define certain new subclasses of analytic functions and then we obtain the upper bonds for the third Hankel determinant.

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1 Introduction and Definitions

Let the family of all functions that are holomorphic (or analytic) in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ be repesented by A, and having the following Taylor–Maclaurin series form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in E) \tag{1.1}$$

Further, subfamily of A, which contains functions that are univalent in E be represented by Bieberbach [8] in 1916 presented the familiar coefficient conjecture for the function $f \in S$ of the form (1.1) and proven by de-Branges [11] in1985. In the period 1916-1985,many researchers tried to prove or disprove this conjecture. As a result, several subfamilies of S connected with different image domains were defined by them. Among these, the families S^*, C and K of starlike functions, convex functions, and close-to-convex functions, respectively, are the most fundamental subfamilies of S and have a good geometric interpretation. These families are defined as:

$$S^* = \left\{ f \in S : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, (z \in E) \right\},$$

$$C = \left\{ f \in S : \frac{(zf'(z))'}{f'(z)} \prec \frac{1+z}{1-z}, (z \in E) \right\},$$

$$R = \left\{ f \in S : \frac{zf'(z)}{g(z)} \prec \frac{1+z}{1-z}, \text{ for } g(z) \in S^* \ (z \in E) \right\}$$

where the symbol " $\prec$ " denotes the familiar subordination between analytic functions and is defined as: the function h_1 is subordinated to the function h_2, symbolically written as $h_1 \prec h_2$ or $h_1(z) \prec h_2(z)$, if we can find a function w, called the Schwarz function, that is holomorphic in E with $w(0) = 0$ and $|w(z)| < 1$ such that $h_1(z) = h_2(w(z))\,(z \in E)$. In the case of univalency of h_2 in E, then the following relation holds:

$$h_1(z) \prec h_2(z) \qquad (z \in E) \iff \qquad h_1(0) = h_2(0) \qquad \text{and} \quad h_1(E) \subset h_2(E).$$

In 1985 [35] ,, a unified family of starlike and convex functions using familiar convolution with the function $z/(1-z)^a$ for $a \in \mathbb{R}$ defined by Padmanabhan and Parvatham. Later on Shanmugam [43] generalized this idea by introducing the family :

$$S_h^*(\phi) = \left\{ f \in A : \frac{z(f*h)'}{(f*h)} \prec \phi(z), \quad (z \in E) \right\},$$

where " $*$ " stands for the familiar convolution, ϕ is a convex, and h is a fixed function in A. Furthermore, if we replace h in $S_h^*(\phi)$ by $z/(1-z)$ and $z/(1-z)^2$, we obtain the families S_h^* and $C(\phi)$ respectively. In 1992, Ma and Minda [27] reduced the restiction to a weaker supposition that ϕ is a function, with Re $\phi(z) > 0$ in E, whose image domain is symmetric about the real axis and starlike with respect to $\phi(0) = 1$ with $\phi'(0) > 0$ and some properties including distortion, growth and covering theorems. The family $S^*(\phi)$ generalizes various subfamilies of the family A, for example:

1. If $\phi(z) = \frac{1+Az}{1+Bz}$ with $-1 \leq B < A \leq 1$, then $S^*[A,B] := S^*\left(\frac{1+Az}{1+Bz}\right)$ is the family of Janowski starlike functions; see [14, 31, 49]. Further, if $A = 1-2\alpha$ and $B = -1$ with $0 \leq \alpha < 1$, then we get the family $S^*(\alpha)$ of starlike functions of order α.

2. The family $S_L^* := S^*\left(\sqrt{1+z}\right)$ was introduced by Sokol abd Stankiewics [48, 18] consisting of functions $f \in A$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $\left|w^2 - 1\right| < 1$.

3. For $\phi(z) = 1 + \sin z$, the family $S^*(\phi)$ leads to the family $S^*_{\sin}$ introduced in [10, 13, 3].

4. When we take $\phi(z) = e^z$, then we have $S^*_e := S^*(e^z)$ [32, 44]

5. The family $S^*_R := S^*\ \phi(z)$ with $\phi(z) = 1 + \frac{z}{k}\frac{k+z}{k-z}, k = \sqrt{2}+1$ was studied in [20].

6. By setting $\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, the family $S^*\ (\phi)$ reduces to S^*_{car}, introduced by Sharma and his coauthors [46, 45], consisting of functions $f \in A$ such that $zf'(z)/f(z)$ lies in the region bounded by the cardioid given by:

$$\left(9x^2 + 9y^2 - 18x + 5\right)^2 - 16\left(9x^2 + 9y^2 - 6x + 1\right) = 0,$$

and also by the Alexandar-type relation, the authors in [46] defined the family C_{car} by:

$$C_{car} = \left\{f \in A : zf'(z) \in S^*_C \quad (z \in E)\right\}; \tag{1.2}$$

see also [41, 47]. For more special cases of the family $S^*(\phi)$, see [17, 40]. We now consider the following family connected with the cardioid domain:

$$R_{car} = \left\{f \in A : f'(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, (z \in E)\right\}. \tag{1.3}$$

For the given operators $q, n \in \mathbb{N} = \{1, 2, \cdots\}$, the Hankel determinant $H_{q,n}(f)$ was defined by Pommerenke [36, 37] for a function $f \in S$ of the form(1.1) given by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

For different subfamilies of univalent functions,the growth of $H_{q,n}(f)$ has been investigated. Specially, the absolute sharp bounds of the functional $H_{2,2}(f) = a_2a_4 - a_3^2$ where found in [15, 16] for each of the families C, S^* and R where the family R is a family of functions of bounded turnings. However, the exact estimate of this determinant of the family of close to convex functions is still undetermined [42]. Recently, Srivastava and his coauthors [50] found the estimate of the second Hankel detrminant for bi-valent functions involving the symmetric q-derivative operator, while in [51], Hankel and Toeplitz determinants for subfamilies of q-starlike functions connected with the conic domain has been studied by the authors. For more literature, see [9, 7, 12, 23, 2, 25, 33, 34].

The Hankel detrminant of third order is given as:

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = -a_5a_2^2 + 2a_2a_3a_4 - a_3^2 + a_3a_5 - a_4^2. \tag{1.4}$$

The estimation of the determinant $|H_{3,1}(f)|$ is very hard as compared to deriving the bound of $|H_{2,2}(f)|$. Babola [4] in 2010 gave the very first paper on $H_{3,1}(f)$, in which he obtained the upper bound of $H_{3,1}(f)$ for the families of S^*, C and R. Later on, many authors published their work regarding $|H_{3,1}(f)|$ for different subfamilies of univalent functions ; see [1, 6, 19, 38, 52]. In 2017, Zaprawa [53] improved the results of Babola as under:

$$|H_{3,1}(f)| \leq \begin{cases} 1 & \text{for } f \in S^*, \\ \frac{49}{540} & \text{for } f \in C, \\ \frac{41}{60} & \text{for } f \in R \quad . \end{cases}$$

and claimed that these bounds are still not the best possible. Further, for the sharpness, he examined the subfamilies of S^*, C and R consisting of functions with $m-$fold symmetry and obtained the sharp bounds. Moreover, Kwon et al. [22] modified the bounds of Zaprawa for $f \in S^*$ in 2018, and proved that $|H_{3,1}(f)| \leq \frac{8}{9}$, but it is not yet the best possible. The authors in [28, 29, 26] contributed in a similar direction by generalizing different families of univalent functions with respect to symmetric points. In 2018, Kowalczyk et al. [21] and Lecko et al. [24] obtained the sharp inequalities:

$$|H_{3,1}(f)| \leq \frac{4}{135} \quad \text{and} \quad |H_{3,1}(f)| \leq \frac{1}{9},$$

for the recognizable families K and $S^*(1/2)$, respectively, where the symbol $\mathrm{S}^*(1/2)$ stands for the family of starlike functions of order 1/2. Furthermore, we would like to cite the work done by Mahmood et al. [30] in which the third Hankel determinant for a subfamily of starlike functions in the q-analogue was studied by them. Additionally, Zhang et al. [54] studied this determinant for the family S_e^* and obtained the bound $|H_{3,1}(f)| \leq 0.565$. See for further studied [5, 45, 45, 39]

In the present article, our aim is to investigate the estimate of $|H_{3,1}(f)|$ for the subfamilies S_{car}^*, C_{car}, and R_{car} of analytic functions connected with the cardioid domain. Moreover, we also study this problem for families of m-fold symmetric function connected with the cardioid domain.

2 Class Of Lemmas

Lemma 2.1. *Let $w(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function. Then for any real numbers μ and υ such that*

$$(\mu\ \upsilon) \in \left\{ (\mu, \upsilon) : \frac{1}{2} \leq |\mu| \leq 2,\ \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \leq \upsilon \leq 1 \right\}. \tag{2.1}$$

the following sharp estimates holds

$$\left| c_3 + \mu c_1 c_2 + \upsilon c_1^3 \right| \leq 1.$$

Lemma 2.2. *Let $w(z) = c_1 z + c_2 z^2 + \cdots$, then*

$$|c_2| \leq 1 - |c_1|^2, \quad |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|} \quad \text{and} \quad |c_4| \leq 1 - |c_1|^2 - |c_2|^2. \tag{2.2}$$

3 Main Results

Theorem 3.1. *Let $f \in R_{car}$ be of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, then*

$$\left| a_2 a_4 - a_3^2 \right| \leq \frac{2}{9}, \quad |a_2 a_3 - a_4| \leq \frac{1}{3}, \quad \left| a_5 - a_3^2 \right| \leq \frac{137}{405}.$$

Proof. Let $f \in R_{car}$ be of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, then we can write in form of Schwarz function, as

$$f'(z) = 1 + \frac{4}{3} w(z) + \frac{2}{3} (w(z))^2.$$

Furthermore, we easily get

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots. \tag{3.1}$$

Also by expanding the series of w with simple calculations, we can write

$$\begin{aligned} 1 + \frac{4}{3} w(z) + \frac{2}{3} (w(z))^2 &= 1 + \frac{4}{3} c_1 z + \left(\frac{2}{3} c_1^2 + \frac{4}{3} c_2 \right) z^2 + \left(\frac{4}{3} c_1 c_2 + \frac{4}{3} c_3 \right) z^3 \\ &\quad + \left(\frac{4}{3} c_1 c_3 + \frac{44}{3} c_4 + \frac{2}{3} c_2^2 \right) z^4 + \cdots. \end{aligned} \tag{3.2}$$

By comparing (3.1) and (3.2), we get

$$a_2 = \frac{2}{3} c_1, \tag{3.3}$$

$$a_3 = \frac{1}{9} \left(2c_1^2 + 4c_2 \right), \tag{3.4}$$

$$a_4 = \frac{1}{3} \left(c_1 c_2 + c_3 \right), \tag{3.5}$$

$$a_5 = \frac{1}{15} \left(4c_1 c_3 + 4c_4 + 2c_2^2 \right). \tag{3.6}$$

Firsly we will determine the $a_2 a_4 - a_3^2$

$$\begin{aligned} a_2 a_4 - a_3^2 &= \frac{1}{81} \left(2c_1^2 c_2 - 4c_1^4 + 18 c_1 c_2 - 16 c_2^2 \right), \\ &= \frac{1}{81} \left[-2c_1^2 \left(c_1^2 - c_2 \right) - 2c_1^4 + 18 c_1 c_3 - 16 c_2^2 \right]. \end{aligned}$$

Applying triangle inequality and (2.2), we get

$$\begin{aligned} \left| a_2 a_4 - a_3^2 \right| &\leq \frac{1}{81} \left[2 |c_1|^2 + |2c_1|^4 + 18 |c_1| |c_3| + 16 |c_2|^2 \right], \\ &\leq \frac{1}{18} \left[2 |c_1|^2 + |2c_1|^4 + 18 |c_1| \left(1 - |c_1|^2 - \frac{|c_2^2|}{1 + |c_1|} \right) + 16 |c_2|^2 \right], \\ &= \frac{1}{18} h(x, y), \end{aligned} \tag{3.7}$$

By maximizing $h(x, y)$ in the set

$$\Omega = \left\{ (x, y) : x \geq 0, y \geq 0, y \leq 1 - x^2 \right\}.$$

We get

$$h(x, y) \leq 17.3788 < 18.$$

We may write (3.7) as

$$\left| a_2 a_4 - a_3^2 \right| \leq \frac{18}{81} = \frac{2}{9}.$$

Next we will determine the $a_2a_3 - a_4$

$$a_2a_3 - a_4 = \frac{1}{27}\left[4c_1^3 - c_1c_2 - 9c_3\right].$$

Application of triangle inequality and (2.2) leads us to

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{1}{27}\left[4|c_1|^3 + |c_1||c_2| + 9\left(1 - |c_1|^2 - \frac{|c_2^2|}{1+|c_1|}\right)\right], \qquad (3.8) \\ &= \frac{1}{27}h(x,y). \end{aligned}$$

By maximizing $h(x,y)$ in the set

$$\Omega = \left\{(x,y) : x \geq 0, y \geq 0, y \leq 1 - x^2\right\}.$$

We get

$$h(x,y) \leq 9.$$

We can write (3.8) as

$$|a_2a_3 - a_4| \leq \frac{9}{27} = \frac{1}{3}.$$

We will now determine $a_5 - a_3^2$

$$\begin{aligned} a_5 - a_3^2 &= \frac{1}{405}\left[-20c_1^4 - 80c_1^2c_2 + 108c_1c_3 - 26c_2^2 + 108c_4\right], \\ &= \frac{1}{405}\left[107c_1\left(c_3 - \frac{80}{107}c_1c_2\right) + c_1c_3 - 20c_1^4 - 26c_2^2 + 108c_4\right]. \end{aligned}$$

Applying triangle inequality and (2.2) along with $\left|c_3 - \frac{80}{107}c_1c_2\right| \leq 1$ we obtain

$$\begin{aligned} \left|a_5 - a_3^2\right| &\leq \frac{1}{405}\left[107|c_1| + |c_1||c_3| + 20|c_1|^4 + 26|c_2|^2 + 108|c_4|\right], \\ &\leq \frac{1}{405}\left[107|c_1| + |c_1|\left(1 - |c_1|^2 - \frac{|c_2^2|}{1+|c_1|}\right) + 20|c_1|^4 + 108\left(1 - |c_1|^2 - \left|c_2^2\right|\right)\right], \\ &= \frac{1}{405}h(x,y). \qquad (3.9) \end{aligned}$$

By maximizing $h(x,y)$ in the set

$$\Omega = \left\{(x,y) : x \geq 0, y \geq 0, y \leq 1 - x^2\right\}.$$

We get

$$h(x,y) \leq 136.4 < 137.$$

We can write(3.9) as

$$\left|a_5 - a_3^2\right| \leq \frac{137}{405}.$$

□

Theorem 3.2. *Let $f \in S^*_{car}$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{44}{81}, \quad |a_2a_3 - a_4| \leq \frac{64}{81}, \quad \left|a_5 - a_3^2\right| \leq \frac{470}{243}.$$

Proof. Let $f \in S^*_{car}$. Then in the form of the Schwarz functi on, we have

$$\frac{zf'(z)}{f(z)} = 1 + \frac{4}{3}w(z) + \frac{2}{3}w(z)^2.$$

Furthermore, we easily get

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= 1 + a_2z + \left(2a_3 - a_2^2\right)z^2 + \left(3a_4 - 3a_2a_3 + a_2^3\right)z^3 \\ &\quad + \left(4a_5 - 2a_3^2 - 4a_2a_4 + 4a_2^2a_3 - a_2^4\right)z^4 + \cdots \end{aligned} \tag{3.10}$$

By comparing (3.2) and (3.10) we get

$$a_2 = \frac{4}{3}c_1, \tag{3.11}$$

$$a_3 = \frac{1}{9}\left(11c_1^2 + 6c_2\right), \tag{3.12}$$

$$a_4 = \frac{1}{81}\left(108c_1c_2 + 36c_3 + 68c_1^3\right), \tag{3.13}$$

$$a_5 = \frac{1}{486}\left(235c_1^4 + 486c_1^2c_2 + 189c_2^2 + 450c_1c_3 + 162c_4\right). \tag{3.14}$$

Firsly we will determine the $a_2a_4 - a_3^2$

$$\begin{aligned} a_2a_4 - a_3^2 &= \frac{1}{243}\left[-91c_1^4 + 36c_1^2c_2 + 144c_1c_3 - 108c_2^2\right], \\ &= \frac{1}{243}\left[-36c_1^2\left(c_1^2 - c_2\right) - 55c_1^4 + 144c_1c_3 - 108c_2^2\right]. \end{aligned}$$

Applying triangle inequality and (2.2) we get

$$\begin{aligned} a_2a_4 - a_3^2 &\leq \frac{1}{243}\left[36|c_1|^2 + 55|c_1|^4 + 144|c_1|\left(1 - |c_1|^2 - \frac{\left|c_2^2\right|}{1 + |c_1|}\right) + 108|c_2|^2\right], \\ &= \frac{1}{243}h(x, y). \end{aligned} \tag{3.15}$$

By maximizing $h(x, y)$ in the set

$$\Omega = \left\{(x, y) : x \geq 0, y \geq 0, y \leq 1 - x^2\right\}.$$

We get

$$h(x, y) \leq 131.190 < 132.$$

We may write (3.15) as

$$\left|a_2a_4 - a_3^2\right| \leq \frac{132}{243} = \frac{44}{81}.$$

Next we will determine the $a_2a_3 - a_4$

$$a_2a_3 - a_4 = \frac{1}{81}\left[64c_1^3 - 36c_1c_2 - 36c_3\right].$$

Applying the triangle inequality and (2.2),we getSince the above function is decreasing w.r.t c_2 and we get maximum value at $c_2 = 0$,

$$\begin{aligned} |a_2 a_3 - a_4| &\leq 64\,|c_1|^3 + 36\,|c_1|\left(1 - |c_1|^2\right) + 36 - 36\,|c_1|^2\,, \\ &= \frac{1}{81} h\,(x)\,. \end{aligned} \tag{3.16}$$

By maximizing $h\,(x)$ over the interval $[0,1]$ we get

$$h\,(x) \leq 64.$$

We may write (3.16) as

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{64}{81}.$$

Lastly we will determine the $a_5 - a_3^2$

$$\begin{aligned} a_5 - a_3^2 &= \frac{1}{486}\left[-491c_1^4 - 108c_1^2c_2 + 450c_1c_3 - 27c_2^2 + 162c_4\right], \\ &= \frac{1}{486}\left[449c_1\left(c_3 - \frac{108}{449}c_1c_2\right) + c_1c_3 - 491c_1^4 - 27c_2^2 + 162c_4\right]. \end{aligned}$$

Applying the triangle inequality and (2.2) , along with$\left|c_3 - \frac{108}{449}c_1c_2\right| \leq 1$ we obtain

$$\begin{aligned} \left|a_5 - a_3^2\right| &\leq \frac{1}{486}\left[\begin{array}{c} 449\,|c_1| + |c_1|\left(1 - |c_1|^2 - \frac{|c_2^2|}{1+|c_1|}\right) \\ +491\,|c_1|^4 + 27\,|c_2|^2 + 162\left(1 - |c_1|^2 - |c_2|^2\right) \end{array}\right] \\ &= \frac{1}{486} h\,(x,y)\,. \end{aligned} \tag{3.17}$$

By maximizing $h\,(x,y)$ in the set

$$\Omega = \left\{(x,y) : x \geq 0, y \geq 0, y \leq 1 - x^2\right\}.$$

We get

$$h\,(x,y) \leq 940.$$

We may write (3.17) as

$$\left|a_5 - a_3^2\right| \leq \frac{940}{486} = \frac{470}{243}.$$

□

Theorem 3.3. *Let* $f \in \dot{C}_{car}$ *be of the form* $f\,(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ *then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{16}{243}, \quad |a_2 a_3 - a_4| \leq \frac{16}{81}, \quad \left|a_5 - a_3^2\right| \leq \frac{277}{3645}.$$

Proof. Let $f \in \dot{C}_{car}$ be of the form $f\,(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ then

$$1 + \frac{f''\,(z)}{f'\,(z)} = 1 + \frac{4}{3} w\,(z) + \frac{2}{3}\,(w\,(z))^2\,.$$

Furthermore, we easily get

$$
\begin{aligned}
1+\frac{f''(z)}{f'(z)} &= 1+2a_2z+\left(6a_3-4a_2^2\right)z^2+\left(12a_4-18a_2a_3+8a_2^3\right)z^3 \\
&\quad +\left(20a_5-32a_2a_4-18a_3^2+48a_2^2a_3-16a_2^4\right)z^4+\cdots . \qquad (3.18)
\end{aligned}
$$

By comparing (3.2) and(3.18) we get

$$a_2=\frac{2}{3}c_1, \qquad (3.19)$$

$$a_3=\frac{1}{27}\left(11c_1^2+6c_2\right), \qquad (3.20)$$

$$a_4=\frac{1}{81}\left(17c_1^3+27c_1c_2+9c_3\right), \qquad (3.21)$$

$$a_5=\frac{1}{2430}\left(235c_1^4+684c_1^2c_2+450c_1c_3+189c_2^2+162c_4\right). \qquad (3.22)$$

Firsly we will determine the $a_2a_4-a_3^2$

$$
\begin{aligned}
a_2a_4-a_3^2 &= \frac{1}{729}\left[-19c_1^4+30c_1^2c_2+54c_1c_3-36c_2^2\right], \\
&= \frac{1}{729}\left[-19c_1^2\left(c_1^2-c_2\right)+11c_1^2c_2+54c_1c_3-36c_2^2\right].
\end{aligned}
$$

Applying the triangle inequality and (2.2) , we get

$$
\begin{aligned}
\left|a_2a_4-a_3^2\right| &\le \frac{1}{729}\left[\begin{array}{c} 19\left|c_1\right|^2+11\left|c_1\right|^2\left|c_2\right|+54\left|c_1\right| \\ \left(1-\left|c_1\right|^2-\frac{\left|c_2^2\right|}{1+\left|c_1\right|}\right)+36\left|c_2\right|^2 \end{array}\right], \\
&= \frac{1}{729}h(x,y). \qquad (3.23)
\end{aligned}
$$

By maximizing $h(x,y)$ in the set

$$\Omega=\left\{(x,y):x\ge 0, y\ge 0, y\le 1-x^2\right\}.$$

We get

$$h(x,y)\le 47.7564<48.$$

We may write (3.23) as

$$\left|a_2a_4-a_3^2\right|\le\frac{48}{729}=\frac{16}{243}.$$

Next we will determine the $a_2a_3-a_4$

$$a_2a_3-a_4=\frac{1}{81}\left[5c_1^3-15c_1c_2-9c_3\right].$$

Applying triangle inequality and (2.2) we get

$$
\begin{aligned}
\left|a_2a_3-a_4\right| &\le \frac{1}{81}\left[5\left|c_1\right|^3+15\left|c_1\right|\left|c_2\right|+9\left(1-\left|c_1\right|^2-\frac{\left|c_2^2\right|}{1+\left|c_1\right|}\right)\right], \\
&= \frac{1}{81}h(x,y). \qquad (3.24)
\end{aligned}
$$

By maximizing $h(x,y)$ in the set

$$\Omega = \left\{(x,y) : x \geq 0, y \geq 0, y \leq 1 - x^2\right\}.$$

We get

$$h(x,y) \leq 15.5 < 16.$$

We may write (3.24) as

$$|a_2a_3 - a_4| \leq \frac{16}{81}.$$

Lastly we will determine the $a_5 - a_3^2$

$$\begin{aligned} a_5 - a_3^2 &= \frac{1}{7290}\left[-505c_1^4 + 732c_1^2c_2 + 135c_1c_3 + 207c_2^2 + 486c_4\right], \\ &= \frac{1}{7290}\left[-505c_1^2\left(c_1^2 - c_2\right) + 227c_1^2c_2 + 135c_1c_3 + 207c_2^2 + 486c_4\right]. \end{aligned}$$

Applying triangle inequality and (2.2), we get

$$\begin{aligned} \left|a_5 - a_3^2\right| &\leq \frac{1}{7290}\left[\begin{array}{c} 505\,|c_1|^2 + 227\,|c_1|^2\,|c_2| + 135\,|c_1|\left(1 - |c_1|^2 - \frac{|c_2^2|}{1+|c_1|}\right) \\ +207\,|c_2|^2 + 486\left(1 - |c_1|^2 - |c_2|^2\right) \end{array}\right], \\ &= \frac{1}{81}h(x,y). \end{aligned} \tag{3.25}$$

By maximizing $h(x,y)$ in the set

$$\Omega = \left\{(x,y) : x \geq 0, y \geq 0, y \leq 1 - x^2\right\}.$$

We get

$$h(x,y) \leq 553.087 < 554.$$

We may write (3.25) as

$$\left|a_5 - a_3^2\right| \leq \frac{554}{7290} = \frac{277}{3645}.$$

□

4 Third Hankel Determinant

We will now determine the third-order Hankel determinant $H_3(1)$

Theorem 4.1. *Let $f \in R_{car}$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ Then*

$$|H_3(1)| \leq \frac{11}{81}.$$

Proof. The third Hankel determinant can be written as;

$$H_3(1) = a_3\left(a_2a_4 - a_3^2\right) - a_4\left(a_4 - a_2a_3\right) + a_5\left(a_3 - a_2^2\right).$$

putting (3.3)-(3.6) we get

$$\begin{aligned} H_3(1) &= \frac{1}{3645}\begin{pmatrix} 120c_1^4c_2-40c_1^6+144c_1^3c_3-216c_1^2c_4 \\ -273c_1^2c_2^2+342c_1c_2c_3-104c_2^3+432c_2c_4-405c_3^2 \end{pmatrix}, \\ H_3(1) &= \frac{1}{3645}\left[-404c_3\left(c_3-\frac{342}{404}c_1c_2\right)-c_3^2+120c_1^2c_2\left(c_1^2-c_2\right)\right. \\ &\left. -153c_1^2c_2^2+216\left(c_2-c_1^2\right)c_4+216c_2c_4-40c_1^6+144c_1^3c_3-104c_2^3\right]. \end{aligned}$$

Applying triangle inequality along with $\left|c_3-\frac{342}{404}c_1c_2\right|\leq 1$ we get

$$\begin{aligned} |H_3(1)| &\leq \frac{1}{3645}\left[404|c_3|+|c_3|^2+120|c_1|^2|c_2|+153|c_1|^2|c_2|^2\right. \\ &\left.+216\left(|c_2|+|c_1|^2\right)|c_4|+216|c_2||c_4|+40|c_1|^6+144|c_1|^3|c_3|+104|c_2|^3\right], \\ |H_3(1)| &\leq \frac{1}{3645}\left[\begin{array}{c} \left(404+144|c_1|^3\right)\left(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|}\right)+ \\ \left(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|}\right)^2+120|c_1|^2|c_2|+153|c_1|^2|c_2|^2 \\ +216\left(|c_2|+|c_1|^2\right)\left(1-|c_1|^2-|c_2|^2\right)+216|c_2| \\ \left(1-|c_1|^2-|c_2|^2\right)+40|c_1|^6+104|c_2|^3 \end{array}\right], \\ |H_3(1)| &= \frac{1}{3645}h(x,y). \end{aligned} \tag{4.1}$$

By maximizing $h(x,y)$ in the set

$$\Omega=\left\{(x,y): x\geq 0, y\geq 0, y\leq 1-x^2\right\}.$$

We get

$$h(x,y)\leq 494.922 < 495.$$

We may write (4.1) as

$$|H_3(1)|\leq\frac{495}{3645}=\frac{11}{18}.$$

□

Theorem 4.2. *Let $f\in S^*_{car}$ be of the form $f(z)=z+a_2z^2+a_3z^3+\cdots$. Then*

$$|H_3(1)|\leq\frac{3271}{13122}.$$

Proof. The third Hankel determinant can be written as;

$$H_3(1)=a_3\left(a_2a_4-a_3^2\right)-a_4\left(a_4-a_2a_3\right)+a_5\left(a_3-a_2^2\right).$$

putting (3.11)-(3.14) we get

$$\begin{aligned} H_3(1) &= \frac{1}{13122}\left[-827c_1^6+1998c_1^4c_2+2466c_1^3c_3-4131c_1^2c_2^2\right. \\ &\left. -2430c_1^2c_4+2916c_1c_2c_3-486c_2^3+2916+2916c_2c_4-2592c_3^2\right], \\ &= \frac{1}{13122}\left[-2591c_3\left(c_3-\frac{2916}{2591}c_1c_2\right)-c_3^2+1998c_1^2c_2\left(c_1^2-cc_2\right)\right. \\ &-2133c_1^2c_2^2+2430c_4\left(c_2-c_1^2\right)+486c_2c_4 \\ &\left.+2466c_1^3c_3-486c_2^3-827c_1^6\right]. \end{aligned}$$

Applying triangle inequality and $\left|c_3 - \frac{2916}{2591}c_1c_2\right| \leq 1$ along with (2.2), we get

$$\begin{aligned}
|H_3(1)| &\leq \frac{1}{13122}\left[\begin{array}{c} 2591|c_3| + |c_3|^2 + 1998|c_1|^2|c_2| + 2133|c_1|^2|c_2|^2 \\ +2430|c_4|\left(|c_2| + |c_1|^2\right) + 486|c_2||c_4| \end{array}\right. \\
&\left. +2466|c_1|^3|c_3| + 486|c_2|^3 + 827|c_1|^6\right], \\
&\leq \frac{1}{13122}\left[\left(2591 + 2466|c_1|^3\right)\left(1 - |c_1|^2 - \frac{|c_2^2|}{1+|c_1|}\right)\right. \\
&+ \left(1 - |c_1|^2 - \frac{|c_2^2|}{1+|c_1|}\right)^2 + 1998|c_1|^2|c_2| \\
&+ 2133|c_1|^2|c_2|^2 + 2430\left(1 - |c_1|^2 - |c_2|^2\right)\left(|c_2| - |c_1|^2\right) \\
&\left. +486|c_2|\left(1 - |c_1|^2 - |c_2|^2\right) + 486|c_2|^3 + 827|c_1|^6\right], \\
&= \frac{1}{13122}h(x,y). \qquad (4.2)
\end{aligned}$$

By maximizing $h(x,y)$ in the set

$$\Omega = \left\{(x,y) : x \geq 0, y \geq 0, y \leq 1 - x^2\right\}.$$

We get

$$h(x,y) \leq 3270.84 < 3271.$$

We may write (4.2) as

$$|H_3(1)| \leq \frac{3271}{13122}.$$

□

Theorem 4.3. *Let $f \in \dot{C}_{car}$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ Then*

$$|H_3(1)| \leq 0.01463.$$

Proof. The third Hankel determinant can be written as;

$$H_3(1) = a_3\left(a_2a_4 - a_3^2\right) - a_4\left(a_4 - a_2a_3\right) + a_5\left(a_3 - a_2^2\right).$$

putting (3.19)-(3.22) we get

$$\begin{aligned}
H_3(1) &= \frac{1}{196830}[-245c_1^6 + 738c_1^4c_2 + 1350c_1^3c_3 - 2565c_1^2c_2^2 \\
&\quad -486c_1^2c_4 + 1242c_2^3 + 2916c_2c_4 - 2430c_3^2], \\
&= \frac{1}{196830}\left[738c_1^2c_2\left(c_1^2 - c_2\right) - 1827c_1^2c_2^2 + 486c_4\left(c_2 - c_1^2\right)\right. \\
&\quad \left.+2430c_2c_4 - 245c_1^6 + 1350c_1^3c_3 + 1242c_2^3 - 2430c_3^2\right].
\end{aligned}$$

Applying triangle inequality and (2.2) , we get

$$\begin{aligned}
|H_3(1)| &\leq \frac{1}{196830}\left[738|c_1|^2|c_2| + 1827|c_1|^2|c_2|^2 + 486|c_4|\left(|c_2| - |c_1|^2\right)\right. \\
&\left. +2430|c_2||c_4| + 1350|c_1|^3|c_3| + 1242|c_2|^3 + 2430|c_3|^2 + 245|c_1|^6\right], \\
&\leq \frac{1}{196830}\left[738|c_1|^2|c_2| + 1827|c_1|^2|c_2|^2 + 486\left(1 - |c_1|^2 - |c_2|^2\right)\left(|c_2| - |c_1|^2\right)\right. \\
&+2430|c_2|\left(1 - |c_1|^2 - |c_2|^2\right) + 1350|c_1|^3\left(1 - |c_1|^2 - \frac{|c_2^2|}{1+|c_1|}\right) \\
&\left. +1242|c_2|^3 + 2430\left(1 - |c_1|^2 - \frac{|c_2^2|}{1+|c_1|}\right)^2 + 245|c_1|^6\right], \\
|H_3(1)| &= \frac{1}{196830}h(x,y). \qquad (4.3)
\end{aligned}$$

By maximizing $h(x,y)$ in the set

$$\Omega = \left\{(x,y) : x \geq 0, y \geq 0, y \leq 1 - x^2\right\}.$$

We get

$$h(x,y) \leq 2880.63 < 2881.$$

We may write (4.3) as

$$|H_3(1)| \leq \frac{2880.63}{196830} = \frac{2881}{196830}.$$

□

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