VFAST Transactions on Mathematics
http://vfast.org/journals/index.php/VTM@ 2016 ISSN(e): 2309-0022; ISSN(p): 2411-6343
Volume 4, Number 1, January-December, 2016
pp. 15-20

FUNDAMENTAL THEIOREM OF FUNCTIONS

GEZAHAGNE MULAT ADDIS*[1],
[1] Department of Mathematics, Dilla University, Dilla, Ethiopia.
Email: buttu412@yahoo.com

Revised October 2016

ABSTRACT. *From the fundamental theorem of homomorphism, it is well known that any homomorphism of groups (or rings or modules or vector spaces and of general universal algebras) can be decomposed as a composition of a monomorphism and an epimorphism. This result can also be extended to general functions defined on abstract sets; that is, any function can be expressed as a composition of an injection and a surjection. The main theorem in this paper called 'Fundamental Theorem of Functions' provides the uniqueness of such a decomposition of functions as a composition of an injection and a surjection. The uniqueness in this theorem is proved up to the level of associates by introducing the notion of an associate of functions*

Keywords: Injections, surjections, composition of functions, associate of a function, uniqueness up to associate.

1. **Introduction.** It is well known that, if $f: G \to G'$ is a homomorphism of groups (or rings or modules) the quotient group $G/_{\ker f}$ is isomorphic to the image of f which is a subgroup of the codomain group G'. This isomorphism g is simply induced by f, in the sense that, g can be defined by $g(a + \ker f) = f(a)$ for any coset $a + \ker f, a \in G$ Also, we have the natural epimorphism $h: G \to G/_{\ker f}$, defined by $h(a) = a + \ker f$ for all $a \in G$. In other words, we have decomposition $f = g \circ h$

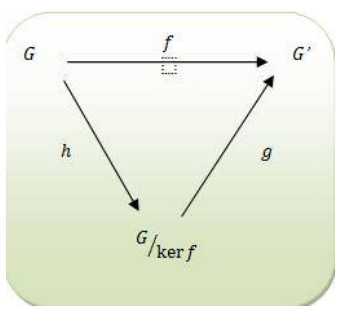

Figure1: Decomposition of a homomorphism of Groups

where g is a monomorphism and h is an epimorphism. This is known as the Fundamental Theorem of Homomorphisms [1- 4]. This result can be extended to any homomorphism $f: A \to A'$ of universal algebras of the same type by considering the binary relation θ on the domain algebra A defined by,

$$\theta = \{(a, b) \in A \times A : f(a) = f(b)\}$$

and the quotient algebra $A/_{\theta}$. In the general case of universal algebras, this θ is defined as the kernel of f and is actually a congruence relation on the domain algebra A and $A/_{\theta}$ is the set of all congruence classes of elements of A corresponding to θ. In the familiar cases of homomorphisms of groups (or rings or modules) the kernel of f is a normal subgroup (or an ideal or a submodule respectively) of the domain and the congruence classes are precisely the cosets of the conventional kernels [2, 5, 8]. In all these cases there is an order isomorphism of the set of normal subgroups (or ideals or submodules) onto the set of congruence relations on the domain algebra. In this scenario, we can say that any homomorphism f of algebras of any type can be decomposed as a composition of a monomorphism and an epimorphism.

It is a natural question whether this can be extended to general functions between abstract sets; that is, for a given function f of a set A into another set B, can we express f as $g \circ h$, where g is an injection and h is a surjection? If so, what can we say about the uniqueness of such a decomposition of f?

2. Method/Approach

For any sets A and B and a function $f: A \rightarrow B$, let us consider the binary relation θ defined by:

$$\theta = \{(x, y) \in A \times A : f(x) = f(y)\}$$

Then θ becomes an equivalence relation on A. Also, the natural map $h: A \rightarrow A/_{\theta}$, defined by $h(a) = \theta(a)$, the equivalence class of a corresponding to θ is a surjection and the function $g: A/_{\theta} \rightarrow B$, defined by $g(\theta(a)) = f(a)$ for all $a \in A$, is an injection. Now, we have $f = g \circ h$. [3, 6, 7]

To discuss about the uniqueness of the functions g and h in this decomposition, we first define the notion of an associate of a function and we will prove the uniqueness of such a decomposition up to the level of associate; that is, if a function f can be decomposed in two ways as $g \circ h$ as $g' \circ h'$ where g and g' are injections and h and h' are surjections, then we will prove that g and g' (respectively h and h') are associate to each other. This says that, the decomposition of a function as a composition of an injection and a surjection is unique upto associate. In this vein we can generalize and unify all the known fundamental theorems of homomorphisms and those isomorphism theorems.

3. Fundamental Theorem of Functions

As explained earlier, since the uniqueness of the decomposition of functions as a composition of an injection and a surjection is taken up to the level of associates we should first define the notion of an associate of a function in a formal way.

Definition 3.1: Let A, B, C and D be any nonempty sets and $f: A \rightarrow B$ and $g: C \rightarrow D$ be any functions. Then f is defined to be an associate of g, if there exist two bijections $\alpha: C \rightarrow A$ and $\beta: D \rightarrow B$ such that, the diagram

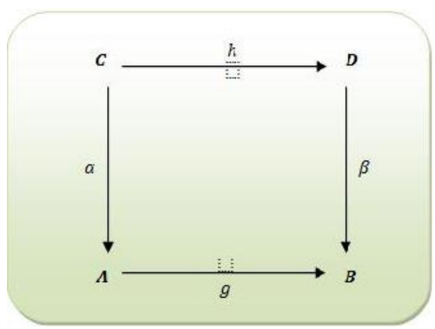

Figure 2: Two associate functions

is commutative; in the sense that, $f \circ \alpha = \beta \circ g$, and we write $f \sim g$ to say that f is an associate of g.

Following the above definition we can observe that the binary relation is an equivalence relation on the class of all functions. Furthermore, if any two functions are given to be associate to each other, then we can draw the properties of functions like injectivity, surjectivity and bijectivity which is held in one of the two associate functions to the other. The theorem is given below

Theorem 3.1 (Fundamental Theorem of Functions)

Let A and B be any nonempty sets and $f: A \rightarrow B$ be any function. Then

1. There exist a nonempty set X, a surjective function $h: A \rightarrow X$ and an injective function $g: X \rightarrow$B such that;

$$f = g \circ h \quad (i.e.\ f(a) = g \circ h(a) \text{ for all } a \in A)$$

   This property is known as decomposition of functions and

2. This decomposition is unique up to associate, in the sense that, if f can be decomposed in one way as $f = g \circ h$; where $h: A \to X$ is a surjection and $g: X \to B$ for some nonempty set X, and if f can also be decomposed in another way as $f = g' \circ h'$ where $h': A \to Y$ is a surjection and $g': Y \to B$ for some nonempty set Y, then g and g' (respectively h and h') are associate to each other (or $g \sim g'$ and $h \sim h'$).

Proof:

1. Define a binary relation θ on the domain set A by:

$$\theta = \{(a, b) \in A \times A: f(a) = f(b)\}$$

Then it is clear that θ is an equivalence relation on A.

Let $x = A/_{\theta} = \{\theta(a): a \in A\}$; be the set of all equivalence classes of A corresponding to θ, where $\theta(a) = \{b \in A: (a, b) \in \theta\}$. Now consider a natural map $h: A \to X$ defined by $h(a) = \theta(a)$ for all $a \in A$ which is a surjection. Also it is trivial to see that a mapping $g: X \to B$ defined by $g(\theta(a)) = f(a)$ for all $\theta(a) \in A/_{\theta}$, $a \in A$ is an injection.

Now for any $a \in A$, consider;

$$\begin{aligned} g \circ h(a) &= g(h(a)) \\ &= g(\theta(a)) \\ &= f(a) \end{aligned}$$

Therefore $g \circ h = f$, and this is the required decomposition of f.

2. Next we prove the uniqueness (up to associates)

Suppose that $f = g \circ h$ be a decomposition of f, where $h: A \longrightarrow X$ is a surjection and $g: X \longrightarrow B$ is an injection for some nonempty set X, and let $f = g' \circ h'$ be another decomposition of f, where $h': A \longrightarrow Y$ is a surjection and $g': Y \longrightarrow B$ is an injection for some set Y.

Claim 1: $h \sim h'$ (or h and h' are associates)

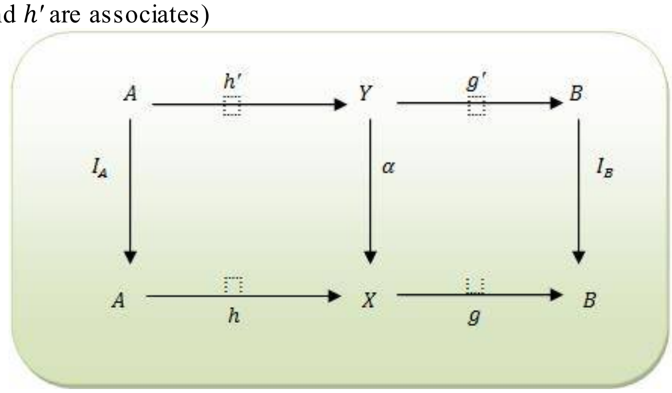

Figure 3: Two decompositions of a function f

Consider the identity map I_A on A, which is a bijection. On the other hand, since $h': A \longrightarrow Y$ is a surjection, every element of Y can be expressed as $y = h'(a)$ for some $a \in A$, define a function $\alpha: Y \to X$ by

$$\alpha\left(h'(a)\right) = h(a) \quad for\ all\ \ h'(a) \in Y \text{ where } a \in A.$$

Then we prove that,

a) α is well defined

For any $y_1, y_2 \in Y$ there exist a_1 and $a_2 \in A$ such that $y_1 = h'(a_1)$ and $y_2 = h'(a_2)$

Therefore,

$$\begin{aligned} y_1 = y_2 &\Rightarrow h'(a_1) = h'(a_2) \text{ in Y} \\ &\Rightarrow g'\left(h'(a_1)\right) = g'(\left(h'(a_2)\right) \text{ in B} \\ &\Rightarrow g' \circ h'(a_1) = g' \circ h'(a_2) \text{ in B} \\ &\Rightarrow f(a_1) = f(a_2) \text{ in B } (\because f = g' \circ h') \\ &\Rightarrow g \circ h(a_1) = g \circ h(a_2) \text{ in B } (\because f = g \circ h) \\ &\Rightarrow g(h(a_1)) = g(h(a_2)) \text{ in B} \\ &\Rightarrow h(a_1) = h(a_2) \ \ in\ X \quad (\because g \text{ is an injection}) \end{aligned}$$

Therefore, α is well defined.

b) α is an injection.

Let $y_1 = h'(a_1)$ and $y_2 = h'(a_2) \in Y$ for some a_1 and $a_1 \in A$. Then

$\alpha(y_1) = \alpha(y_2)$ in X $\Longrightarrow \alpha\left(h'(a_1)\right) = \alpha\left(h'(a_2)\right)$ in X

$\Longrightarrow h(a_1) = h(a_2)$ in X (∵by the definition of α)

$\Longrightarrow g(h(a_1)) = g(h(a_2))$ in B

$\Longrightarrow g \circ h(a_1) = g \circ h(a_2)$ in B

$\Longrightarrow f(a_1) = f(a_2)$ in B (∵ $f = g \circ h$)

$\Longrightarrow g' \circ h'(a_1) = g' \circ h'(a_2)$ in B (∵ $f = g' \circ h'$)

$\Longrightarrow g'\left(h'(a_1)\right) = g'\left(h'(a_2)\right)$ in B

$\Longrightarrow h'(a_1) = h'(a_2)$ in Y (∵ g' is an injection)

$\Longrightarrow y_1 = y_2$ in Y

Therefore, α is an injection.

c) α is a surjection.

$x \in X \Longrightarrow x = h(a)$ for some $a \in A$ (∵ h is a surjection)

$\Longrightarrow x = \alpha\left(h'(a)\right)$ for some $a \in A$, where $h'(a) \in Y$

$\Longrightarrow x = \alpha(y)$ for some $y \in Y$, where $y = h'(a)$

Therefore, α is a surjection.

Thus, by the results in (a), (b) and (c) we get that α is a bijection.

Now, for any $a \in A$, consider,

$\alpha \circ h'(a) = \alpha\left(h'(a)\right)$

$= h(a)$ (∵by the definition of α)

$= h\,(I_A(a))$ (∵ I_A is an identity function on A)

$= h \circ I_A\,(a)$

Therefore $\alpha \circ h' = h \circ I_A$ *and hence* $h \sim h'$ ($or\ h\ and\ h'\ are\ associates$).

Claim 2: $g \sim g'$ (or g and g' are associates)

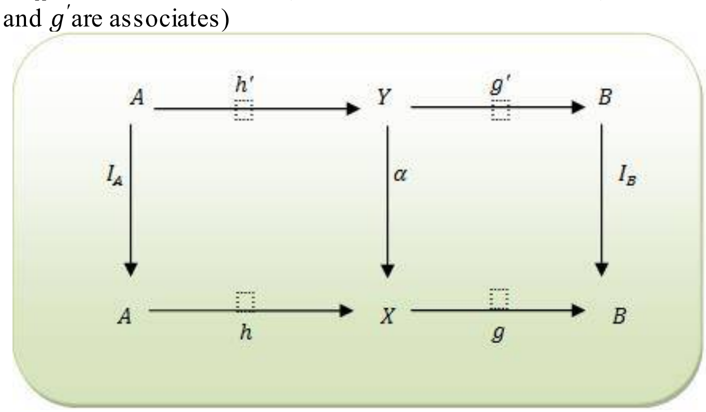

Figure 4: Two decompositions of a function f

From claim (1) we have, the identity map I_A on A and a function $\alpha: Y \to X$, which is defined by $\alpha\left(h'(a)\right) = h(a)$ for all $h'(a) \in Y$ where a∈A, are bijections. Also, consider the identity map I_B on B which is also a bijection. Since $h': A \to Y$ is a surjection, for any $y \in Y$, there exist $a \in A$ such that $y = h'(a)$.

Now consider,

$I_B \circ g'(y) = I_B \circ g'(h'(a))$ in B

$= I_B(g'\left(h'(a)\right)$ in B

$= g'\left(h'(a)\right)$ in B

$= g' \circ h'(a)$ in B

$= f(a)$ in B

$= g \circ h(a)$ in B (∵ $f = g \circ h$)

$= g(h(a))$ in B

$= g(\alpha(h'(a)))$ in B (by the definition of α)

$= g \circ \alpha(h'(a))$ in B

$= g \circ \alpha\,(y)$ in B $(\because y = h^{'}(a))$

Therefore, $I_B \circ g^{'} = g \circ \alpha$.

Thus, we have two bjections, $I_B: B \to B$ and $\alpha: Y \to X$ satisfying that, the diagram;

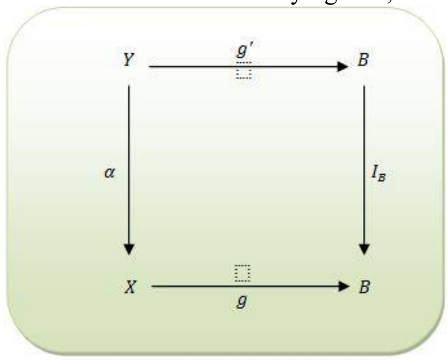

Figure 5: Associate functions g and g'

is commutative; that is, $I_B \circ g^{'} = g \circ \alpha$. Therefore $g{\sim}g^{'}$ (or g and g' are associates). This says that the decomposition of functions as a composition of an injection and a surjection is unique up to associate.

4. **Applications of Results**

The Fundamental Theorem of Functions can be applied to derive the known Fundamental Theorem of Homomorphisms of groups (or rings or modules or vector spaces and of general universal algebras). Also, various isomorphism theorems can also be deduced from this theorem. Furthermore, as a consequence of the Fundamental Theorem of Functions, we can formulate and prove the first, the Second and the third bijection theorems which are a simple copy of those known Isomorphism theorems.

In addition, one of the advantages of this research is, it introduces the notion of an associate of a function, which mainly uses to prove the uniqueness of the decomposition of functions Also, it is mostly important to draw a property of functions like surjectivity, injectivity and bijectivty, which is held in one of the two associate functions to the other.

5. **Conclusion and Recommendations**

On the basis of the Fundamental Theorem of Functions which is formulated and proved in this paper, we can conclude that any function defined on general abstract sets can be decomposed as a composition of an injection and a surjection and this decomposition is unique upto associate; in the sense that, if a given function f defined from a given nonempty set A into another nonempty set B, can be decomposed in one way as $f = g \circ h$ where g is an injection and h is a surjection, and if f can also be decomposed in another way as $f = g^{'} \circ h^{'}$ where g' is an injection and h' is a surjection then we get that $g{\sim}g^{'}$ (or g and g' are associate to each other) and $h{\sim}h'$ (or h and h' are associate to each other). We say that two functions g and h defined on general abstract sets are associate to each other, if there exists two bijections α and β such that; $g \circ \alpha = \beta \circ h$. Since α and β are bijections and hence invertible it is equivalently saying that one can be expressed as a composition of the other and two bijections; that is, $g \circ \alpha = \beta \circ h \Leftrightarrow g = \beta \circ h \circ \alpha^{-1} \Leftrightarrow h = \beta^{-1} \circ g \circ \alpha$. It follows from this definition that, two associate functions have several properties in common such as: one of the two associate functions is an injection (respectively a surjection and a bijection) if and only if the other is an injection (respectively a surjection and a bijection). Moreover, their domain (respectively codomain) are cardinal to each other.

It is finally recommended that it needs additional researches on the class of associate functions of general abstract sets to identify and characterize them in a more general way.

Acknowledgments

Above all, I would like to thank the almighty God for his mercy that endowed upon me. Next, I would like to offer my sincerest and deepest gratitude to my advisor Prof.U.M. Swamy, who has supported me throughout this research work with his patency and knowledge whilst allowing me the room to work in my own way. One can simply could not wish for a better or friendly advisor. I am also indebted to my friends helping me throughout my work in different ways. Finally, I would like to express my heartfelt thanks to my parents, mom, dad and my brothers. In my daily work I have been blessed with their unconditional love, kindness, support and encouragement. From the start until the accomplishment of this manuscript, they have been my

source of strength and courage. I would not let this opportunity to escape without me saying "mom, I love you and always treasure you”.

REFERENCES

[1] Artin, M., (1991). Algebra, Prentice-Hall, Englewood cliffs, New Jersey.
[2] Birkhoff, G., (1967). Lattice Theory, Amer. Malt. Soc. Coll. Publications, Providence.
[3] Burris, S.and Sankappanavor, H.P., (1981). A Course in Universal Algebra, Springer-Verlag, GTM78.
[4] Cohn, P.M., (1965). Universal Algebra, Harper and Row, New York.
[5] Gratzer, G., (1979). Universal Algebra, Springer-Verlag, New York.
[6] Halmos, P.R. , (1960). Naive Set Theory, Van Nostrand, New York.
[7] Jacobson, N., (1980). Basic Algebra-Vol I, W. H. Freeman, San Francisco,
[8] Pierce, R.S., (1968). Introduction to the theory of Abstract Algebra, Halt, Rinehart and winston, New York.