VFAST Transactions on Mathematics
http://vfast.org/journals/index.php/VTM@ 2014 ISSN(e): 2309-0022; ISSN(p): 2411-6343
Volume 2, Number 1, January-December, 2014

pp. 16-24

ON BANACH LATTICE ALGEBRAS

Ruşen YILMAZ[1] and Yilmaz ALTUN[2]

[1] Department of Mathematics, Recep Tayyip Erdoğan University, 53100 Rize, TURKEY
rusen.yilmaz@erdogan.edu.tr, ryilmaz00@yahoo.com
[2] Department of Mathematics, Artvin Çoruh University, 08100 Artvin, TURKEY
yilmazaltun@artvin.edu.tr

Abstract. *In this paper we investigate a characterization of a Banach lattice algebra with unit to be represented as an AM-f-algebra. We also consider, for a locally compact Hausdorff topological space L and a Banach lattice space A, the identification of $C_b(L, Z(A))$ with the center $Z(C_0(L, A))$ of $C_0(L, A)$.*

Keywords: Lattice Ordered Algebra; ℓ-algebra; Riesz Algebra; f-algebra; Banach Lattice Algebra; Banach f-algebra; AM-space.

1. **Introduction.** The main result of this paper is the representation of a Banach lattice algebra with unit as an algebra of continuous functions on a compact Hausdorff topological space K, with the usual partial ordering, pointwise multiplication $\cdot$, and the supremum norm; that is, $(A, *) \cong (C(K), \cdot)$, where $* : A \times A \to A$ is the algebraic multiplicative operator in a Banach lattice algebra A. The results in this direction were given by Martignon in [15]. In this paper we improve her results and prove that, under certain conditions, $(A, *)$ and $(C(K), \cdot)$ are isometrically and algebraically ℓ-isomorphic as AM-f-algebras. We also attempt to clarify some results in the same direction due to Ercan and Wickstead [7]. In particular, we prove that, if A is a Banach lattice space and L a locally compact Hausdorff space, the center of the ℓ-space $C_0(L, A)$ of all continuous A-valued functions on L is isometrically isomorphic to the space $C_b(L, Z(A))$ of all norm bounded $Z(A)$-valued functions on L, where $Z(A)$ is the center of A endowed with the strong operator topology.

2. **Preleminaries.**

Definition 2.1. A real lattice ordered linear space (ℓ-space) A is said to be a *lattice ordered algebra* (an *ℓ-algebra* or a *Riesz algebra*) if it is a linear algebra (not necessarily associative) such that if $a, b \in A^+$, then $ab \in A^+$. An ℓ-algebra A is said to be

(i) an *f-algebra (function algebra)* if $a \wedge b = 0$ implies $ac \wedge b = ca \wedge b = 0$ for all $c \in A^+$, (although the following classes of ℓ-algebras are not explicitly used, we include their definitions for the sake of completeness)

(ii) an *almost f-algebra* if $a \wedge b = 0$ implies $ab = 0$,

(iii) a *d-algebra* if $c(a \vee b) = ca \vee cb$ and $(a \vee b)c = ac \vee bc$ for all $a, b \in A$ and $c \in A^+$.

The notion of an f-algebra, as given in the above definition, first appeared in a paper by Birkhoff and Pierce [5] in 1956 to be followed a decade later by the class of almost f-algebras introduced by Birkhoff in [4]. The notion of a d-algebra was introduced by Kudláček [13] in 1962. In general, these classes of algebras are distinct, but there are relations between them; for example, it is clear that every f-algebra is an almost f-algebra and a d-algebra. Every Archimedean (that is, for all $a, b \in A^+$ and $n = 1, 2, ...,$ $na \leq b$ implies $a = 0$) f-algebra is commutative and associative. It turns out that every Archimedean almost f-algebra is commutative but not necessarily associative. In any semi-prime associative ℓ-algebra the classes of f-algebras, almost f-algebras and d-algebras are equivalent. In particular, this holds for an associative ℓ-algebra with (algebraic) unit element $e > 0$. We refer the reader to [3] for details of these results as given by Bernau and Huijmans.

For the elementary theory of ℓ-space and terminology not explained here we refer to [2, 14, 16, 20].

Let E and F be ℓ-spaces. A linear operator $T : E \to F$ is said to be *order bounded* if the image under T of an order bounded set in E is again an order bounded set in F. The operator T is called *positive* $T(E^+) \subset F^+$. A linear operator $T : E \to F$ is called an *ℓ-homomorphism* (or a *Riesz homomorphism*) whenever $a \wedge b = 0$ in E implies $Ta \wedge Tb = 0$. Clearly, every ℓ-homomorphism is positive. An order bounded linear operator $\pi : E \to E$ is called an *orthomorphism* if and only if, for all $a, b \in E$, $a \perp b$ implies $\pi a \perp b$. The collection of all orthomorphisms on E is denoted by $Orth(E)$. Obviously every positive orthomorphism is an ℓ-homomorphism. It is well-known that the ordered vector space $\mathcal{L}_b(E, F)$ of all order bounded linear mappings of an ℓ-space E into a Dedekind complete ℓ-space F is a Dedekind complete ℓ-space (see, for example, [2, Theorem 1.13]). In the case that $E = F$, $\mathcal{L}_b(E, F)$ is denoted by $\mathcal{L}_b(E)$. It is also well-known that, if E is an (Archimedean) ℓ-space, then $Orth(E)$ is an (Archimedean) f-algebra under multiplication by composition, possessing the identity operator I on E as a multiplicative identity (see [2]), and we recall that $Orth(E)$ is an f-subalgebra of $\mathcal{L}_b(E)$.

The notion of an orthomorphism is related to that of a multiplier. An order bounded operator $T : E \to E$ is said to be a *multiplier* on an ℓ-algebra E if $T(ab) = (Ta)b = a(Tb)$ for all $a, b \in E$. The algebra of all multipliers on E is denoted by $M(A)$. If E is an Archimedean f-algebra, then every orthomorphism $\pi : E \to E$ is a multiplier; that is $Orth(E) \subseteq M(E)$. Indeed, for each $a \in E$, the operator π_a defined by $\pi_a(b) = ab$ ($b \in E$) is an orthomorphism, and so,

$$\pi(ab) = \pi(\pi_a b) = (\pi\pi_a)b = (\pi_a\pi)b = a\pi b.$$

If E is an Archimedean semi-prime f-algebra, then $Orth(E) = M(E)$ [18].

Definition 2.2. Let A be an ℓ-space. The solid subspace generated by the identity operator I in $\mathcal{L}_b(A)$ is said to be the *center* of A and is denoted by $Z(A)$; that is,

$$Z(A) = \{T \in \mathcal{L}_b(A) : |T| \leq nI \text{ for some } n \in I\!N\}.$$

The center $Z(A)$ of an ℓ-space A is an ℓ-subalgebra of $Orth(A)$. To see this, we first note that $Z(A)$ is itself an ℓ-space since $Z(A)$ is a solid subspace. Moreover, it can easily be seen that $0 \leq ST \in Z(A)$ for all $0 \leq S, T \in Z(A)$; that is, $Z(A)$ is itself an ℓ-algebra. So it is sufficient to show that $a \perp b$ in A implies that $Ta \perp b$ in A for all $T \in Z(A)$. Observe that $|Ta| \leq |T||a|$ for all $T \in \mathcal{L}_b(A)$ and $a \in A$. Hence, for $T \in Z(A)$,

$$\begin{aligned} |Ta| \wedge |b| &\leq |T||a| \leq nI|a| \wedge |b| \quad \text{(some } n \in IN) \\ &= n|a| \wedge |b| \leq n(|a| \wedge |b|) \quad \text{(all } n \in IN) \\ &= 0; \end{aligned}$$

that is, $Ta \perp b$. It is also obvious that T is order bounded, from which follows that $T \in Orth(A)$. Moreover, we see that, for all $T \in Orth(A)$ and $a \in A^+$, $|Ta| = |T|a \leq nIa = na$ for some $n \in IN$ (note that $|Ta| = |T||a|$ for all $T \in Orth(A)$ and $a \in A$; see, for instance, [2, Thoerem 8.6]). Therefore the above definition may be refined as follows,

$$Z(A) = \{T \in Orth(A) : |Ta| \leq na \text{ for some } n \in IN \text{ and all } a \in A^+\}.$$

Now it is obvious that $Z(A)$ of an Archimedean ℓ-space A is an Archimedean f-subalgebra of $Orth(A)$, and so is necessarily commutative and associative.

A norm $\|\cdot\|$ on an ℓ-space A is said to be a *lattice norm (ℓ-norm)* if $|a| \leq |b|$ in A implies $\|a\| \leq \|b\|$, and the pair $(A, \|\cdot\|)$ is called a *normed lattice space (normed ℓ-space)*. If a normed ℓ-space $(A, \|\cdot\|)$ is norm complete, then $(A, \|\cdot\|)$ is referred to as a *Banach lattice space (Banach ℓ-space)*. A Banach ℓ-space $(A, \|\cdot\|)$ is said to be an *AM-space (abstract M-space)* if $\|a \vee b\| = \|a\| \vee \|b\|$ holds for all $a, b \in A^+$.

We recall that every normed ℓ-space is Archimedean and an orthomorphism π on a normed ℓ-space A is norm bounded (i.e. $\|\pi a\| \leq \|\pi\| \|a\|$ for all $a \in A$) if and only if there exists a positive real number λ such that $|\pi| \leq \lambda I$; that is, $\pi \in Orth(A)$ is norm bounded if and only if $\pi \in Z(A)$ (for details see [19, §144]). Note that this statement indicates that $Z(A)$ consists precisely of all norm bounded orthomorphisms on A. Moreover, the operator norm $\|\cdot\|$ on $Z(A)$ is an ℓ-norm and coincides with the Minkowski functional of $[-I, I]$

$$\|T\|_\infty = \inf\{\lambda > 0 : |T| \leq \lambda I\}$$

for all $T \in Z(A)$; more explicitly, $\|T\| = \|T\|_\infty$ for all $T \in Z(A)$.

In the special case when A is a Banach ℓ-space every orthomorphism on A is norm bounded; that is, $Orth(A) = Z(A)$ ([17, Corollary 4.2]). The Banach ℓ-space $(Orth(A), \|\cdot\|)$ is indeed a Banach ℓ-algebra since the operator norm $\|\cdot\|$ on $\mathcal{L}_b(A)$ is an algebra norm, and it is an ℓ-norm as $\|T\| = \|T\|_\infty$. Wickstead proved in [17] that, if A is a Banach ℓ-space, then $Orth(A)$ is an AM-space with the identity operator I; in fact, $Orth(A)$ is a Banach f-algebra since $Orth(A)$ is an f-algebra itself.

Next we define a concept of an AM-f-algebra.

Definition 2.3. A Banach f-algebra $(A, \|\cdot\|)$ with the property $\|a \vee b\| = \|a\| \vee \|b\|$ for all $a, b \in A^+$ is called an *AM-f-algebra;* in other words, an AM-space which is also an f-algebra is called an *AM-f-algebra.*

The space $C_b(X)$ of continuous bounded functions on a topological space X, and its closed ℓ-subspaces, with the usual partial ordering, multiplication (pointwise) and the operator norm, are examples of AM-f-algebras.

We summarize our results in the following.

Theorem 2.4. *If A is a Banach ℓ-space, then*

$$Z(A) = Orth(A) = \{\pi \in \mathcal{L}_b(A) : |\pi| \leq \lambda I \text{ for some } \lambda > 0\},$$

and $\|\pi\| = \inf\{\lambda > 0 : |\pi| \leq \lambda I\}$ holds for all $\pi \in Z(A)$. In particular, $Z(A)$ under this norm is an AM-f-algebra with the identity operator I as its multiplicative identity.

A positive element u in a normed ℓ-space A is called a *norm order unit* if and only if $\|u\| = 1$ and, for any $a \in A^+$, with $\|a\| \leq 1$, $a \leq u$. A positive element u in an ℓ-space A is said to be an *order unit* if and only if, for every $a \in A$, there exists a positive integer n depending upon a such that $|a| \leq nu$ (or, equivalently, if and only if the solid subspace S_u generated by u is equal to A). Clearly, in a normed ℓ-space every norm order unit is an order unit, and in an ℓ-space every order unit is a weak order unit (A positive element u in an ℓ-space A is said to be an *weak order unit* if $a \wedge u = 0$ implies $a = 0$). Indeed, let $a \wedge u = 0$. Then $0 \leq a \wedge nu \leq n(a \wedge u) = 0$ for all $n \in IN$; that is, $a \wedge nu = 0$ for all $n \in IN$. If u is an order unit, then $0 \leq a \leq nu$ for some $n \in IN$, and so $a = a \wedge nu = 0$, as required.

Every weak order unit need not be an order unit. For example, the ℓ-space $C([0, \infty))$ has the weak order unit $u(x) = 1$ $(0 \leq x \leq \infty)$. However, $C([0, \infty))$ has no order unit; for, if $0 \leq f(x) = (u(x) + nx)^2$ in $C([0, \infty))$, then there exists no $n \in IN$ and $u(x) \geq 0$ such that $f(x) \leq nu(x)$ for all $0 \leq x < \infty$.

In general, if A is a normed ℓ-space with a norm order unit u, then $a \in A$ and $\|a\| \leq 1$ imply that $|a| \leq u$; for,

$$\|a^+ + a^-\| = \|\,|a|\,\| = \|a\| \leq 1,$$

and so $a^+ + a^- \leq u$ since $a^+ + a^- \in A^+$; that is, $|a| \leq u$.

Remark 2.5. We note that the identity operator I on a normed ℓ-space A is a norm order unit in $Z(A)$ (and hence in $Orth(A)$), which is also a multiplicative identity. In fact, in general if A is a normed ℓ-algebra with multiplicative modulus and a multiplicative identity $e \geq 0$, then e is the norm order unit of A by 2.1. Lemma of [15].

3. **A Representation Theorem for an AM-f-algebra.** In this section we discuss necessary and sufficient conditions for a Banach ℓ-algebra $(A, *)$ with unit to be represented as an algebra $(C(K), \cdot)$ of continuous real-valued functions on a compact Hausdorff topological space K, with the usual partial ordering, pointwise multiplication $\cdot$, and the supremum norm. Investigations in this direction have been carried out by Martignon [15]. Here we improve her results and prove that, under certain conditions, $(A, *)$ and $(C(K), \cdot)$ are isometrically and algebraically ℓ-isomorphic as AM-f-algebras.

It is well known that if A is an Archimedean ℓ-algebra with a multiplicative unit $e > 0$ which is an order unit (even a weak unit), then it is an f-algebra ([3, Corallary 1.10.]). Moreover, if A is equipped with two f-algebra multiplications with the same multiplicative unit, then these multiplications coincide on A. More precisely, if A is an Archimedean ℓ-space and $e > 0$ in A, then there exists at most one product on A

that makes A an f-algebra having e as its multiplicative unit ([2, Theorem 8.23.]). Now combining these facts we obtain the following result, due to Martignon [15, 1.4. Proposition].

Theorem 3.1. *Let K be a compact Hausdorff topological space and suppose that $(C(K), *)$ is an ℓ-algebra such that $\mathit{1}(x) = 1$ for all $x \in K$. Furthermore, if $\mathit{1}$ is the multiplicative identity of $(C(K), *)$, then $(C(K), *) = (C(K), \cdot)$; in other words, $*$ is the pointwise multiplication $\cdot$.*

Suppose that A is a Banach ℓ-space with an order unit u. Then, by the definition of an order unit, $S_u = A$ holds, and so it follows from Theorem 12.20 of [2] that A, endowed with the norm

$$\|a\|_\infty = \inf\{\lambda > 0 : |a| \leq \lambda u\},$$

is an AM-space having the order interval $[-u, u]$ as its closed unit ball. Moreover, $\|a\|_\infty$ is equivalent to the original norm on A since all ℓ-norms making an ℓ-space a Banach ℓ-space are equivalent (see [2, Corollary 12.4]. Summarizing these results, we have

Theorem 3.2. *If a Banach ℓ-space A has an order unit u, then A can be remormed such that A becomes an AM-space having $[-u, u]$ as its closed unit ball. In particular, the order unit u is a norm order unit.*

As observed before, every norm order unit in a normed ℓ-space is an order unit. Thus, the preceding theorem yields the following characterization.

Corollary 3.3. *If u is a positive element in a Banach ℓ-space, then u is an order unit if and only if it is a norm order unit.*

In the sequel, unless otherwise stated, we shall mean by the phrase "AM-space with unit" a Banach ℓ-space with an order unit, whose norm is the $\|\cdot\|_\infty$-norm.

The following is an extension of Kakutani's Representation Theorem to AM-algebras with unit. We first note that Kakutani's Representation Theorem deals with AM-spaces with unit. This result is due to Kakutani [9, Theorems 2 and 21], and was later extended by M. Krein and S. Krein [10, 11]. We also note that an injective ℓ-homomorphism T from an ℓ-space A into an ℓ-space B is referred to as an *ℓ-isomorphism*, and that A and B are said to be *ℓ-isomorphic* if T is also surjective.

Theorem 3.4. *A Banach ℓ-algebra $(A, *)$ with unit u is an AM-algebra if and only if $(A, *)$ is isometrically and algebraically ℓ-isomorphic to $(C(K), \cdot)$ for some (unique up to homeomorphism) compact Hausdorff topological space K. K can be chosen to be the set of all algebraic ℓ-functionals f from A into IR such that $f(u) = 1$, endowed with the weak* topology $\sigma(A', A)$.*

In general, a Banach ℓ-algebra $(A, *)$ is an AM-algebra if and only if $(A, *)$ is isometrically and algebraically ℓ-isomorphic to a closed ℓ-subalgebra of a $(C(K), \cdot)$ algebra.

Proof. A complete proof of the theorem can be found in [2, Theorem 12.28]. We give here a sketch of the proof. Suppose that $A = C(K)$ for a compact Hausdorff topological space K. Clearly, A is an AM-algebra with unit the constant function

1. We now discuss the description of K in connection with the algebraic lattice structure of A. Set

$$L = \{f \in (U')^+ : f \text{ is an extreme point of } (U')^+ \text{ with } f(u) = \|f\| = 1\}$$

and

$$K = \{g \in (U')^+ : g \text{ is an algebraic } \ell\text{-functional with } g(u) = \|g\| = 1\},$$

where $U' = \{h \in A' : \|h\| \leq 1\}$, the closed unit ball of A'. Now since U' is $\sigma(A', A)$-compact by Alaoglu's Theorem (see, for example, [6, §3 of Chapter V]), and so L being $\sigma(A', A)$-closed in U' is also $\sigma(A', A)$-compact. Since A is an AM-space, it follows from Theorem 12.27 of [2] that $K \subseteq L$. Therefore K is $\sigma(A', A)$-compact and Hausdorff.

On the other hand, suppose that A is an AM-algebra with unit u and define the mapping $T : A \to C(K)$ by $Ta(g) = g(a)$ for all $a \in A$ and $g \in K$. It follows from the Krein-Milman theorem (see [12] or [2, Theorem 9.14]) that T is a norm preserving ℓ-isomorphism. The fact that every element $g \in K$ is an algebraic homomorphism implies that T is also an algebraic homomorphism. Indeed,

$$T(a * b)(g) = g(a * b) = g(a)g(b) = (Ta(g))(Tb(g)) = (Ta \cdot Tb)(g)$$

holds for all $g \in K$, and so $T(a * b) = Ta \cdot Tb$ for all $a, b \in A$. Moreover, the algebraic ℓ-isomorphism T maps the unit element u of A onto the multiplicative identity $\mathtt{1}$ of $C(K)$ (the constant function $\mathtt{1}$ on K); for, $(Tu)(g) = g(u) = 1$ for all $g \in K$, and so $Tu = \mathtt{1}$. Furthermore, $T(A)$ separates the points of K. It follows from Stone-Weierstrass theorem (see [1, Theorem 8.3]) that $T(A)$ is (norm) dense in $C(K)$, and so $T(A) = \overline{T(A)} = C(K)$ since $T(A)$ is closed. This shows that T is surjective, as required.

Theorem 3.5. *If A is a Banach ℓ-space, then there exists a compact Hausdorff topological space K such that $(Z(A), \circ)$ and $(C(K), \cdot)$ are isometrically and algebraically ℓ-isomorphic as AM-f-algebras.*

Proof. By Theorem 2.4, $Z(A)$ is an AM-f-algebra with unit, the multiplicative identity I. Hence, by Theorem 3.4, there exists a compact Hausdorff topological space K such that $Z(A)$ and $C(K)$ are isometrically ℓ-isomorphic. As observed in the proof of Theorem 3.4, the ℓ-isomorphism maps the multiplicative identity I of $Z(A)$ onto the multiplicative identity $\mathtt{1}$ of $C(K)$. It follows from Corollary 5.5 of [8] that this ℓ-isomorphism is also algebraic. Therefore, we have $(Z(A), \circ) \cong (C(K), \cdot)$, as required.

Definition 3.6. An algebra A is called *faithful* if, for each $a \in A$, $Aa = aA = \{0\}$ implies that $a = 0$.

Before presenting the main result of this section, we recall that $Z(A) \subseteq Orth(A) \subseteq M(A)$ holds for every Archimedean f-algebra A and, in the case that A is a Banach ℓ-space, $Z(A) = Orth(A) \subseteq M(A)$.

Theorem 3.7. *If $(A, *)$ is a faithful Banach algebra, then the following are equivalent.*

(1) *$(A, *)$ is an AM-f-algebra.*

(2) *The representation of* $(A,*)$ *into* $M(A)$ *is a contractive algebraic* ℓ*-isomorphism of* $(A,*)$ *into* $(Z(A),\circ)$.

(3) $(A,*)$ *can be identified with a closed* ℓ*-subalgebra of* $(C(K),\cdot)$ *for some compact Hausdorff topological space* K.

Proof. **(1)⇒(2)** Suppose that $(A,*)$ is an AM-f-algebra and Define the representation $R:(A,*)\to(Z(A),\circ)\subseteq M(E)$ by $R(a)=\pi_a$, where $\pi_a b=a*b$, $b\in A$. For each $a\in A$, the equation $\pi_a b=a*b$ defines a bounded linear operator on A; for,

$$\|\pi_a b\|=\|a*b\|\leq\|a\|\,\|b\|.$$

Moreover, since A is an f-algebra, if $a\perp b$ in A, then $a*c\perp b$ for all $c\in A$; that is, $\pi_a c\perp b$. This shows that $\pi_a\in Z(A)$ for all $a\in A$, as A is a Banach ℓ-space. In other words, we have $R(A)\subseteq Z(A)$.

Suppose that $a,b\in A$. Then

$$\pi_{a\vee b}c=(a\vee b)*c=a*c\vee b*c=\pi_a c\vee\pi_b c$$

for all $c\in A^+$, which shows that $R(a\vee b)=Ra\vee Rb$; that is, R is an ℓ-homomorphism. Moreover, R is an algebra homomorphism; for, by the associativity of A,

$$\pi_{a*b}c=(a*b)*c=a*(b*c)=a*(\pi_b c)=\pi_a(\pi_b c)=(\pi_a\circ\pi_b)c$$

for all $c\in A$. Thus $R(a*b)=Ra\circ Rb$.

Since A is faithful, if $Ra=0$ for all $a\in A$, then $\pi_a b=a*b=0$ for all $b\in A$, which implies that $a=0$. Hence R is injective. This proves that R is an algebraic ℓ-isomorphism.

(2)⇒(3) By Proposition 3.5, $(Z(A),\circ)\cong(C(K),\cdot)$ for some compact Hausdorff topological space K. Hence, since A is an arbitrary Banach ℓ-algebra, $(A,*)$ is isometrically and algebraically ℓ-isomorphic to a closed ℓ-subalgebra of a $(C(K),\cdot)$ algebra by Theorem 3.4.

(3)⇒(1) Suppose that D is a closed ℓ-subalgebra of $(C(K),\cdot)$. On the other hand, $(A,*)=(D,\cdot)$ by the hypothesis, from which the result follows.

4. **The identification of** $C_b(L,Z(A))$**.** We consider the center $Z(C_0(L,A))$ of the space $C_0(L,A)$ of all continuous mappings from L into A vanishing at infinity, where (L,ξ) is a locally compact Hausdorff topological space and A is a Banach ℓ-space. The pair $(C_0(L,A),\cdot)$ is a Banach ℓ-space with respect to the natural partial ordering and the supremum norm. Moreover, it is a Banach f-algebra whenever A is so. Suppose that $\mathcal{L}_b(A)$ is endowed with the strong operator topology τ, and that $C_b(L,\mathcal{L}_b(A))$ denotes the space of all norm bounded continuous mappings from (L,ξ) into $(\mathcal{L}_b(A),\tau)$. It has been proved in [7, Theorem 6.2] that $C_b(L,\mathcal{L}_b(A))$ can be identified with a subalgebra of $\mathcal{L}_b(C_0(L,A))$. Moreover, this identification is an isometric algebraic ℓ-isomorphism. The following result shows that $C_b(L,Z(A))$ can be identified with the center $Z(C_0(L,A))$ of $C_0(L,A)$.

We consider the center $Z(C_0(L,A))$ of the space $C_0(L,A)$ of all continuous mappings from L into A vanishing at infinity, where (L,ξ) is a locally compact Hausdorff topological space and A is a Banach ℓ-space. The pair $(C_0(L,A),\cdot)$ is a Banach ℓ-space with respect to the natural partial ordering and the supremum norm. Moreover, it is a Banach f-algebra whenever A is so. Suppose that $\mathcal{L}_b(A)$ is endowed with the strong operator topology τ, and that $C_b(L, \mathcal{L}_b(A))$ denotes the space of all norm bounded continuous mappings from (L, ξ) into $(\mathcal{L}_b(A), \tau)$. Ercan and Wickstead have proved in [7, Theorem 6.2] that $C_b(L, \mathcal{L}_b(A))$ can be identified with a subalgebra of $\mathcal{L}_b(C_0(L, A))$. Moreover, this identification is an isometric algebraic ℓ-isomorphism. The following result shows that $C_b(L, Z(A))$ can be identified with the center $Z(C_0(L, A))$ of $C_0(L, A)$.

Theorem 4.1. *Let L and A be as above, and suppose that $\mathcal{L}_b(A)$ is endowed with the strong operator topology. Then $C_b(L, Z(A))$ and $Z(C_0(L, A))$ are isometrically and algebraically ℓ-isomorphic as AM-f-algebras.*

Proof. We consider the mapping $T : C_b(L, Z(A)) \to Z(C_0(L, A))$ defined by $T(\phi) = \pi_\phi$ for each $\phi \in C_b(L, Z(A))$, where $\pi_\phi f(x) = \phi(x)(f(x))$ for all $f \in C_0(L, A)$ and $x \in L$.

If $\phi \in C_b(L, Z(A))$, then, for each $0 \leq f \in C_0(L, A)$,

$$\|\pi_\phi f\| = \sup_{x \in L} \|\phi(x)(f(x))\| \leq \|\phi\| \, \|f\|,$$

where $\|\phi\| = \sup_{x \in L} \|\phi(x)\|$, which implies that $\|\pi_\phi\| \leq \|\phi\|$. Hence T is bounded. Moreover, if $f \wedge g = 0$ in $C_0(L, A)$, then it follows immediately that $\pi_\phi f \wedge g = 0$ in $C_0(L, A)$. This shows that $\pi_\phi \in Z(C_0(L, A))$ for each $\phi \in C_b(L, Z(A))$.

It is routine to show that T is an algebraic ℓ-isomorphism. We show only that T is surjective. Suppose that $\Phi \in Z(C_0(L, A))$. For $x \in L$ and $f \in C_0(L, A)$, the value $(\Phi f)(x)$ is unambiguously defined by the value of $f(x)$ (for, if $f_1(x) = f(x)$ for some $f_1 \in C_0(L, A)$, then, by linearity,

$$|(\Phi(f_1 - f))(x)| \leq \|\Phi\| |(f_1 - f))(x)| = 0,$$

and so $\Phi f_1(x) = \Phi f(x)$).

Let $f \in C_0(L, A)$, $x \in L$ and define a mapping $\varphi : L \to Z(C_0(L, A))$ by $\varphi(x)f(x) = \Phi f(x)$, where $\Phi = \pi_\varphi$; i.e., $\varphi(x)f(x) = \pi_\varphi f(x)$. In the same way above it is easily seen that $\varphi(x) \in Z(C_0(L, A))$ and that φ is unambiguously defined. It remains to show that $\varphi \in C_b(L, Z(A))$. As $\Phi \in Z(C_0(L, A))$, we have $\Phi(C_0(L, A)) \subseteq C_0(L, A)$; in other words, $\pi_\varphi(C_0(L, A)) \subseteq C_0(L, A)$. It follows from Theorem 6.1 of [7] that φ is norm bounded and continuous with respect to the strong operator topology. So $\varphi \in C_b(L, Z(A))$, as required.

The isometric property follows from $\|\pi_\phi\| = \|\phi\|$, as follows. As already established, $\|\pi_\phi\| \leq \|\phi\|$. Since

$$\|\phi(x)\| = \sup_{a \in A} \|\phi(x)(a)\|,$$

we have that, for all $x \in L$,

$$\|\phi(x)\|_{f(L)} = \sup_{f(x) \in A} \|\phi(x)(f(x))\| = \sup_{f(x) \in A} \|\pi_\phi f(x)\| \leq \|\pi_\phi\|,$$

and so $\|\phi\|_{f(L)} \leq \|\pi_\phi\|$. It follows that $\|\pi_\phi\| = \|\phi\|$; that is, $\|T(\phi)\| = \|\phi\|$.

Since $C_0(L, A)$ is a Banach ℓ-space, $Z(C_0(L, A))$ is an AM-f-algebra with unit by Theorem 2.4. We therefore have $Z(C_0(L, A)) \cong C_b(L, Z(A))$ as AM-f-algebras, as required.

Remark 4.2. **(1)** The mapping $\mathtt{1} : L \to Z(A)$, defined by $\mathtt{1}(x) = I$ for all $x \in L$ is a multiplicative identity of $C_b(L, Z(A))$, and so is a norm order unit in $C_b(L, Z(A))$ as observed in Remark 2.5, where I is the identity operator in $Z(A)$.

(2) If we denote the identity operator in $Z(C_0(L, A))$ by II (that is, $II(f) = f$ for all $f \in C_0(L, A)$), then $\pi_{\mathtt{1}} = II$ since

$$\pi_{\mathtt{1}} f(x) = \mathtt{1}(x)(f(x)) = I(f(x)) = f(x)$$

for all $x \in L$, and so $\pi_{\mathtt{1}} f = f$ for all $f \in C_0(L, A)$. It follows that $T(\mathtt{1}) = II$; in other words, T maps the multiplicative identity of $C_b(L, Z(A))$ into that of $Z(C_0(L, A))$.

Corollary 4.3. *For every locally compact Hausdorff topological space L,* $Z(C_0(L)) \cong C_b(L)$ *as AM-f-algebras.*

REFERENCES

[1] Aliprantis, C. D. and Burkinshaw, O. (1981). Principles of Real Analysis. Elsevier North-Holland, New York and Oxford.

[2] Aliprantis, C. D. and Burkinshaw, O. (1985). Positive Oparators. Academic Press.

[3] Bernau, S. J. and Huijsmans, C. B. (1990). Almost f-algebras and d-algebras. Math. Proc. Cambridge Philos. Soc. 107, 287-308.

[4] Birkhoff, G. (1967). Lattice Theory. Amer. Math. Soc. Colloq. Publ. No. 25.

[5] Birkhoff, G. and Pierce, R. S. (1956). Lattice-ordered rings. An. Acad. Brasil. Ciénc. 28, 41-49.

[6] Conwayi, J. B. (1985). A Course in Funtional Analysis. Spinger-Verlag New York nc.

[7] Ercan, Z. and Wickstead, A. W. (1996). Banach lattices of continuous Banach lattices-valued functions. J. Math. Anal. Appl. 198, 121-136.

[8] Huijsmans, C. B. and de Pagter, B. (1984). The order bidual of lattice algebras. J. Funct. Anal. 59 , 41-64.

[9] Kakutani, S. (1941) Concrete represantation of abstract M-spaces. Ann. of Mat. 42, 994-1024.

[10] Krein, M. and Krein, S. (1940). On an inner characteristic of the set of all continuos functions definrd on a bicompact Hausdorff spaces. C. R. (Dokl.) Acad. Sci. URSS (N.S.) 27, 427-430.

[11] Krein, M. and Krein, S. (1943). Sur l'espace des fonctions continues définies sur un bicompact de on a Hausdorff et ses sous-espaces semi-ordonnés. Mat. Sb. 13, 18-38.

[12] Krein, M. and Milman, D. (1940). On extreme points of regular convex sets. Studia Math. 9, 133-138.

[13] Kudláček, V. (1962). On some types of ℓ-rings. Sborni Vysokého Učeni Techn v Brně 1-2, 179-181.

[14] Luxemburg, W. A. J. and Zaanen, A. C. (1971). Riesz Spaces I. North-Holland.

[15] Martignon, L. (1980). Banach f-algebras and Banach lattice algebras with unit. Bol. Soc. Bras. Mat. 11, 11-18.

[16] Schefold, H. H. (1974). Banach Lattices and Positive Operators. Spinger-Verlag Berlin Heidelberg New York.

[17] Wickstead, A. W. (1977). Representation and duality of multiplication operators on Archimedean Riesz spaces. Compositio Math. 35, 225-238.

[18] Yilmaz, R. and Rowlands, K. (2006). On orthomorphisms, quasi-orthomorphisms and quasi-multipliers. J. Math. Anal. Appl. 313, 120-131.

[19] Zaanen, A. C. (1983). Riesz Spaces II. North-Holland.

[20] Zaanen, A. C. (1997). Introduction to Operator Theory in Riesz Spaces. Springer.