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COUNTING OF BINARY MATRICES AVOIDING SOME 2 × 2 MATRICES

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ABSTRACT. *The number of matrices avoiding certain types of matrices is NP-hard in general. In this paper the binary matrices are considered. In particular, the problem of finding the total number of special binary matrices avoiding some types of 2 × 2 matrices is the main objective of this paper. The solution of the problem is given under some constraints as well as under general situation. The formula for the special binary matrices is obtained for total count of matrices of order n × k and also obtained the formula for special binary matrices avoiding some matrices of order 2 × 2. The formula is obtained in terms of the Catalan numbers.*

***Keywords**:* Catalan number; special binary matrices; lonesome matrices.

1. **Introduction.** Let M be the set of all binary matrices of order $n \times k$ then obviously the cardinality of M is 2^{n} . In the set of binary matrices M, matrices can be calculated which avoid some sub-matrix "γ" of order 2×2 and it can be written mathematically as $\phi\ (n, k; \gamma)$. In this paper we are going to introduce and calculate the total number of special Binary matrices and those special binary matrices which avoid the blocks $O_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Rayser [11] made the classes of binary matrices with the same column sum vector and row sum vector. Brewbaker [1, 2] calculated the number of binary matrices avoiding square identity matrix of order *2*. He used the classes defined by Rayser. Rayser worked on lonesome binary matrices and defined lonesome matrices as "the binary matrix which avoids $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is called a lonesome binary matrix". Brewbaker [1, 2] and Kim [5] also named the binary matrices avoiding *"I"* as lonesome matrices and also wrote that the number of lonesome matrices are equal to the Poly-Bernoulli number $B_n^{(k)}$ with $k=1$. Brewbaker [2] generated the second proof of Rayser's theorem and studied the properties of binary lonesome matrices. Kaneko [4] defined the Poly-Bernoulli numbers which were used by Kim and Brewbaker for their proof. Kitaev [8, 9, 10] worked on multi avoidance of right angled numbered polyomino patterns and also found number of matrices avoiding some patterns such as stair matrix or forbidden matrices. Klinz and Rudolf [7] spent time to permute the matrices avoiding some forbidden matrices latter on Kaneko [4] took this work forward. Heong-Kwan Ju and Seughyun Seo [3] worked on lonesome matrices and made some generating functions of the lonesome matrices. They also found the method to find the number of matrices avoiding some equivalence classes of matrices of order 2 × 2 with their generating functions as well. They used the work of Sanchez Pregreno [14] and the work of Brewbaker to find the important results. The study of "*A*-free matrix" was introduced by Spinrad [12, 13] where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, he dealt with a totally balanced matrix which has a permutation of the rows and columns that are *A*-free but in [3] Hyeong-Kwan Ju and Seunghyun Seo remarked that the set of totally balanced matrix is different from *M (A)*. They changed the original binary matrices into block matrices by making some change in rows and columns, each block containing 0 or 1

completely. They found the number of matrices avoiding some square matrices of order 2 with their generating functions. They found the number of matrices avoiding $A=\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$, $B=\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$, $E\cup O_2=\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\cup\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ and $B\cup D=\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\cup\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}$ under some conditions, also found the generating functions for the formulas.

2. **Materials and Methods:** Hyeong-Kwan Ju and Seunghyun Seo [3] used different methods to calculate the binary matrices avoiding some 2 × 2 matrices. We also worked to calculate the binary matrices which avoid the matrices of order 2 but the avoiding matrix is different from the matrices avoided in [3], we took the work forward by a different method. We used positions of the entries of matrix to make binary matrix then shuffled the rows and columns to make more binary matrices and calculate the total number of matrices made this way and named them as special binary matrices. We calculated special binary matrices avoiding null binary matrix of order 2 by using permutation of things not all different and Catalan number.

3. Definitions and Notations: Binary matrices are the matrices with entries 0 or 1 or both but no any other entry. *Block matrices* are those matrices in which some matrices are collected or which can be partitioned as small matrices, it is also called partition matrix. Given a matrix $M=\begin{bmatrix}1 & 0 & 1 & 1\\0 & 0 & 1 & 1\\0 & 0 & 1 & 0\end{bmatrix}$ is called block matrix in which $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ and $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ are partitions or blocks and $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ is not a partition of the matrix *M*. We also say that M avoids $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ and contains $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ a $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$. A matrix $M=[m_{ij}]$ is called a special binary matrix if it satisfies the following:

$$m_{ij} = \begin{cases} 0 & i.\ i+j = e \\ 1 & i.\ i+j = o \end{cases}$$

The number of special binary matrices of order n × k is denoted by χ (n, k). Given a square matrix $M = \begin{bmatrix}m_1 & m_1 & m_1\\m_2 & m_2 & m_2\\m_3 & m_3 & m_3\end{bmatrix}$ of order 3. By using conditions of special binary matrices we can make a special binary matrix "*N*" from "*M*" which is $N = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix}$. Now by exchanging the first column by second column in M, we get a new matrix $M_1 = \begin{bmatrix}m_1 & m_1 & m_1\\m_2 & m_2 & m_2\\m_3 & m_3 & m_3\end{bmatrix}$ and a special binary matrix $N_1 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 0 & 0\end{bmatrix}$. Similarly we can get special binary matrices from ordinary or general matrices. From now on we will use $\chi(n, p)$ for the total number of special binary matrices of order n × p and we will use the notation $\chi(n, k;\ \alpha)$ for the special binary matrices avoiding block α.

The nth *Catalan number* is defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1}\sum_{i=0}^{n}\binom{n}{i}^2 = \prod_{i=2}^{n}\frac{n+i}{i} = \frac{2(2n-1)}{n+1}C_{n-1}$$

4. Main Results By using the condition of special binary matrices and permutation of things not all different we can count the total number of special binary matrices of different order as well as special binary square matrices avoiding $O_{2\times 2}$, where $O_{2\times 2}$ is the square null matrix of order 2.

Theorem 3.1

Let $M = [m_{ij}]$ be a matrix of order $n \times p$, where $n = 2k$ and $p = 2m$, then the total special binary matrices are $\chi(n,p) = (k+1)(m+1)C_m C_k$, where C_m a C_k are Catalan numbers.

Proof:

As given that $M = [m_{ij}]$ be a matrix of order n × p, where $n = 2k$ and $p = 2m$, we take a set A containing first n=2k positive integers for column numbering and a set B containing first p=2m positive integers for row numbering then we must have *k* even and *k* odd numbers in set $A = \{1, 2, 3, \ldots, n = 2k\}$ similarly *m* even numbers and *m* odd numbers in set $B = \{1, 2, 3, \dots, p = 2m\}$. To make the special binary matrices, by associating all odd numbers with "*o*" and all even numbers with "*e*" we get A= B ={o,e}. Now let *C* be a word of length *2k* containing *k* number of *e*'s and *k* number of *o*'s and let *D* be a word of length *2m* containing m number of *o*'s and *m* number of *e*'s. Now by using the "permutation of things not all different" we can make $\frac{(2k)!}{k!k!}$ words of length *2k* from the word *C*, that is

$$\frac{(2k)!}{k!\,k!} = (k+1)\ \frac{(2k)!}{(k+1)k!\,k!} = (k+1)\ \frac{(2k)!}{(k+1)!\,k!} = (k+1)\ C_k$$

Which is the number of words to label the rows. Similarly we can make $\frac{(2m)!}{m!m!}$ = (m+1) $C_m$words of length *2m+1* from the word *D*. Which is the number of words to label the columns. Now we use *(m+1)* C_m number of words as column label and *(k+1)* C_k words as row label to make special binary matrices. When we use one word from (m+1) C_m words to label the column and use all words one by one from *(k+1)* C_k words as row label we get *(k+1)* C_k number of matrices now we use second word from *(m+1)* C_m as column label and again using all *(k+1)* C_k words as row label we again get *(k+1)*C_k number of matrices and by carrying on this process we get total number of special binary matrices

$$[(k+1)\ C_k]\ .\ [(m+1)\ C_m] = (k+1)(m+1)C_m C_k$$

Corollary 1:
Let $M = [m_{ij}]$ be a square matrix of order $n = 2k$, then total number of special binary matrices is
χ (n, n) = $[(k+1)C_k]^2$, where C_k is catalan number.

Theorem 3.2
Let $M = [m_{ij}]$ be a square matrix of order $n \times p$ where $n = 2k+1$ and $p = 2m+1$, then the total special binary matrices are χ (n, p) = $n\ \ C_m C_k$

Proof:
Given that $M = [m_{ij}]$ be a matrix of order $n \times p$ where $n = 2k+1$ and $p=2m+1$, then we must have k even and *k+1* odd numbers in set $A = \{1, 2, 3, \dots, n = 2k+1\}$ and also there exist *m* even numbers and *m+1* odd numbers in set $B = \{1,2,3,\dots, p = 2m+1\}$. Let us associate all odd numbers with "*o*" and all even numbers with "*e*" in both of sets so that we may avoid the repetition of matrices during their use as labels for matrices. Then set *A* and set *B* becomes the set *{o,e}*. Now let *C* be a word of length *2k+1* containing *k* number of e's and *(k+1)* number of *o*'s and let *D* be a word of length 2m containing *m+1* number of *o's* and m number of *e's*. Now by using the "permutation of things not all different" we can make $\frac{(2k+1)!}{k!(k+1)!}$ words of length *2k+1*from the word *C*, that is

$$\frac{(2k+1)!}{(k+1)!\,k!} = (2k+1)\ \frac{(2k)!}{(k+1)!\,k!} = (2k+1)\ C_k = n\ \ C_k$$

Which is the number of words to label the rows, similarly we can make $\frac{(2m+1)!}{m!(m+1)!}$ words of length *2m+1* from the word *D*, that is

$$\frac{(2m+1)!}{(m+1)!\,m!} = (m+1)\ \frac{(2m+1)!}{(m+1)m!\,m!} = (2m+1)\ \frac{(2m)!}{(m+1)!\,m!} = (2m+1)\ C_m = p\ \ C_m$$

Which is the number of words to label the columns. Now we use "*p* C_m" number of words as column label and "*n* C_k" words as row label to make special binary matrices. When we use one word from "*p* C_m" words to label the column and use all words one by one from "*n* C_k" words as row label we get "*n* C_k" number of matrices now use 2nd word from *p* C_m as column label and again using all "*n* C_k" as row label we again get "*n* C_k" number of matrices and so on this way we will get total number of special binary matrices

$$\chi(n, p) = [(2k+1)\ C_k]\ .\ [(2m+1)\ C_m] = n\ \ C_m C_k$$

Corollary 1:
Let $M = [m_{ij}]$ be a square special binary matrix of order $n = 2k+1$, then total number of special binary matrices is $\chi(n, n) = \frac{(k+2)}{2} C_{k+1}$

Theorem 3.3

Let $M = [m_{ij}]$ be a matrix of order $n \times p$ where $n = 2k+1$ and $p = 2m$, then the total special binary matrices are $\chi(n, p) = n(m+1)C_m C_k$

Proof:

By given $M = [m_{ij}]$ be a matrix of order $n \times p$, where $n = 2k+1$ and $p = 2m$, then we must have k even and $k+1$ odd numbers in set $A = \{1,2,3,\ldots, n = 2k+1\}$ and also we have m even numbers and m odd numbers in set $B = \{1,2,3,\ldots,p = 2m\}$. Let us associate all odd numbers with "*o*" and all even numbers with "e" in both of sets so that we may avoid the repetition of matrices. Then set *A* and set *B* becomes the set $\{o,e\}$. Now let *C* be a word of length $2k+1$ containing *k* number of *e*'s and $(k+1)$ number of *o's* and let *D* be a word of length $2m$ containing m number of *o's* and m number of *e's*. Now by using the "permutation of things not all different" we can make $\frac{(2k+1)!}{k!(k+1)!}$ words of length $2k+1$ that is

$$\frac{(2k+1)!}{(k+1)!\,k!} = (2k+1)\ \frac{(2k)!}{(k+1)!\,k!} \qquad = (2k+1)\ C_k = n\ C_k$$

Similarly we can make $\frac{(2m)!}{m!\,m!}$ words of length $2m$ that is

$$\frac{(2m)!}{m!\,m!} = (m+1)\ \frac{(2m)!}{(m+1)m!\,m!} \qquad = (m+1)\ \frac{(2m)!}{(m+1)!\,m!} \qquad = (m+1)\ C_m$$

Now we use $(m+1)\ C_m$ number of words as column label and $n\ C_k$ words as row label to make special binary matrices. When we use one word from $(m+1)\ C_m$ words to label the column and use all words one by one from $n\ C_k$ words as row label we get $n\ C_k$ number of matrices now use second word from $(m+1)\ C_m$ as column label and again using all "$n\ C_k$" words as row label we again get $n\ C_k$ number of matrices and so on this way we will get total number of special binary matrices

$$\chi(n, p) = [n\ C_k]\ .\ [(m+1)\ C_m \quad = n(m+1)C_m C_k$$

Proposition

Let $M = [m_{ij}]$ be a matrix of order $n \times p$ where $n = 2m$ and $p = 2k+1$, then the total special binary matrices are $\chi(n, p) = p\ (m+1)C_m C_k$

Theorem 3.4

Let $M = [m_{ij}]$ be a square matrix of order $n \times n$ where n = 2k, then the special binary matrices avoiding partition $O_{2\times 2}$ are $\chi(n, n; O_{2\times 2}) = 4(k+1)\ C_k - 2k+6$

Proof:

Given that $M = [m_{ij}]$ be a square matrix of order $n = 2k$. Then according to corollary 1 of theorem 3.1 we have $[(k+1)C_k]^2$ total number of special binary matrices and according to theorem 3.1 we have $(k+1)C_k$ words to label the columns as well as rows. Now when we use one word which do not contain sub-word "*oo*" or "*ee*" from $(k+1)C_k$ words to label the column and use all words one by one as row label we will get $(k+1)C_k$ matrices avoiding block "$O_{2\times 2}$". Now when we use same word as row label and use all $(k+1)\ C_k$ words as column label one by one we will get $(k+1)\ C_k$ matrix again but one matrix avoiding $O_{2\times 2}$ will be repeated so number of matrices avoiding block $O_{2\times 2}$ are $2(k+1)\ C_k - 1$. Now similarly when we use 2nd word as column label which do not contain two *e's* or two *o's* together we will get $(k+1)\ C_k - 1$ matrices avoiding $O_{2\times 2}$ and while using same word as row label we will get $(k+1)C_k - 2$ matrices because two matrices are repeated. So matrices avoiding $O_{2\times 2}$ are

$$\chi(n, n; O_{2\times 2}) = 2(k+1)\ C_k - 1 + (k+1)\ C_k - 1 + (k+1)\ C_k - 2 = 4(k+1)\ C_k - 4$$

Now to calculate the number of words from (k+1) C_k words which contain only one sub-word "*oo*". Here we will consider "*oo*" as a single letter so we have "*n-1*" letters and to calculate words containing only one sub-word "*oo*" we must know the positions between letters for this we know that one "e" must be at start and one at last. Now "*k-2*" e's and "*k-1*" *o's* are between these two *e*'s and there are "*k-1*" positions to put "*o*" *or* "*oo*" between each two *e*'s in a word because we are calculating the words which contain exactly one pair "oo" and not containing "ee". So we can say that there are "*k-1*" words containing "*oo*" Similarly we can count the words containing "ee" but not containing "oo" which will be same in number that is *k-1*. So we have $(k-1) + (k-1) = 2k-2$ Matrices avoiding partition "$O_{2\times 2}$"

Hence total number of matrices avoiding "$O_{2\times 2}$" is

$$\chi(n, n; O_{2\times 2}) = 4(k+1)\ C_k - 4 + 2k - 2 \qquad = 4(k+1)\ C_k + 2k - 6$$

Theorem 3.5
Let $M = \left[m_{ij}\right]$ be a square matrix of order $n \times n$ where n = 2k+1, then the special binary matrices avoiding partition $O_{2\times 2}$ are
$\chi(n, n; O_{2\times 2}) = 2n\, C_k - 1$

Proof:
Given that $M = \left[m_{ij}\right]$ be a special binary square matrices of order n = 2k+1, Then according to the corollary 1 of theorem 3.2 we have $\frac{(k+2)}{2} C_{k+1}$ number of total special binary matrices and according to theorem 3.2 we must have $n\ C_k$ words to label rows and columns. Now when we use one word which do not contain sub-word "*oo*" or "*ee*" together from $n\ C_k$ words to label the column and use all words one by one as row label we will get $n\ C_k$ matrices avoiding partition "$O_{2\times 2}$" Now when we use same word as row label and use all $n\ C_k$ words as column label one by one we will get $n\ C_k$ matrices again but one matrix avoiding $O_{2\times 2}$ will be repeated so number of matrices avoiding partition $O_{2\times 2}$ is
$(2k+1)C_k+(2k+1)C_k-1= 2(2k+1)\ C_k\ -1 = 2\ n\ C_k - 1$. In all possible words "$n\ C_k$" there exist only one word which do not contain sub-word "*oo*" or "*ee*".
Hence total number of matrices avoiding "$O_{2\times 2}$" is $\chi(n, n; O_{2\times 2}) = 2n\ C_k\ -1$

3. **Conclusion**. Different mathematicians worked on binary matrices and counted the number of matrices avoiding some matrices, we also worked in the same idea but we introduced the special binary matrices with their total count and also calculated the special binary matrices avoiding the null matrix of order 2. Special binary matrices avoiding some different matrices can also be calculated. This work can be took forward with different avoiding matrix or matrices.

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