

Upper Bound of the Third Hankel Determinant for a Subclass of Analytic Functions Connected with Sine Functions

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Abstract

Let \mathcal{RL}_{\sin} and \mathcal{SL}_{\sin} represents the families of multivalent bounded turning and multivalent starlike functions that are subordinate with sine function in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. For these families our aim is to find the bounds of Hankel determinant of order three. Further, the estimate of third order Hankel determinant for the family \mathcal{SL}_{\sin} in this work improve the bounds which was investigated recently. Moreover, the same bounds have been investigated for 2-fold symmetric and 3-fold symmetric functions. Also we discuss $H_2(3)$ determinant for the above mentioned families.

2020 Mathematics Subject Classification. Primary 30C45; 30C50; 30C80;
Secondary 11B65, 47B38

Key Words and Phrases. Subordinations, Multivalent functions, Trigonometric function, Hankel determinant problem.

1 Introduction and Definitions

The representation of the region is denoted by the notation $\mathbb{E} = \{z : |z| < 1\} \subset \mathbb{C}$ while \mathcal{A}_p indicated the class of multivalent functions (say $f \in \mathcal{A}_p$) which are holo-

morphic in \mathbb{E} and obeys the following Taylor series form:

$$f(z) = z^p + \sum_{l=p+1}^{\infty} a_l z^l, \text{ with } p \in \mathbb{N}. \quad (1.1)$$

We seen that $\mathcal{A}_1 = \mathcal{A}$, the class of all normalized holomorphic functions. Let the two functions f and g be holomorphic in \mathbb{E} . Then f is subordinate to g , symbolically represented as $f \prec g$, if a holomorphic function w can be find with the properties $w(0) = 0$ and $|w(z)| < |z|$ such that $f(z) = g(w(z))$ ($z \in \mathbb{E}$). In addition, if the function g in \mathbb{E} is univalent, then we got:

$$f \prec g \Leftrightarrow f(0) = g(0) \ \& \ f(\mathbb{E}) \subseteq g(\mathbb{E}).$$

In 1989, Shanmugam [47] introduced a unified subfamily of holomorphic functions with the use of subordinations and it will be given as:

$$\mathcal{S}_\phi^*(h) = \left\{ f \in \mathcal{A} : \frac{z(f * \phi)'}{(f * \phi)} \prec h(z), \quad (z \in \mathbb{E}) \right\}, \quad (1.2)$$

where "*" stands for convolution, h is a convex and ϕ is a fixed function in \mathcal{A} . By putting $z/(1-z)$ and $z/(1-z)^2$ instead of ϕ in $\mathcal{S}_\phi^*(h)$, we get the classes $\mathcal{S}^*(h)$ and $\mathcal{C}(h)$ respectively, introduced by Ma and Minda [33] with the restriction that h is a function, with $\text{Re} h > 0$ in \mathbb{E} , whose image domain is symmetric about the real axis and starlike with respect to $h(0) = 1$ with $h'(0) > 0$ and then investigated certain useful problems of these classes.

Now taking some particular functions instead of h in $\mathcal{S}^*(h)$, we obtain many subfamilies of the collection \mathcal{A} which have different geometric interpretation, see [19, 14, 34, 48, 45, 49, 20, 44].

The class $\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z})$ was contributed by Sokól et.al [52], consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the Bernoulli's lemniscate given by $|w^2 - 1| < 1$. That is;

$$\mathcal{S}_L^* := \left\{ f : f \in \mathcal{A} \text{ and } \left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1, \quad z \in \mathbb{E} \right\},$$

and as for subordination, we have

$$\mathcal{S}_L^* := \left\{ f : f \in \mathcal{A} \text{ and } \frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, \quad z \in \mathbb{E} \right\}.$$

The above class \mathcal{S}_L^* was studied further by different authors as Ali et.al [1], Kumar et. al [27], Raza and Malik [42] and Sokól [53]. We now extend the above concepts to the subfamilies of multivalent functions which are defined is the following;

$$\mathcal{RL}_p : = \left\{ f : f \in \mathcal{A}_p \text{ and } \frac{f'(z)}{pz^{p-1}} \prec 1 + \sin(z), \quad (z \in \mathbb{E}) \right\}, \quad (1.3)$$

$$\mathcal{SL}_p : = \left\{ f : f \in \mathcal{A}_p \text{ and } \frac{zf'(z)}{pf(z)} \prec 1 + \sin(z), \quad (z \in \mathbb{E}) \right\}, \quad (1.4)$$

$$\mathcal{CL}_p : = \left\{ f : f \in \mathcal{A}_p \text{ and } \frac{(zf'(z))'}{pf'(z)} \prec 1 + \sin(z), \quad (z \in \mathbb{E}) \right\}. \quad (1.5)$$

For more contribution on p -valent analytic functions with different image domains, see [22, 51, 56, 4, 5, 6, 8, 36, 41].

In 1966-67, Pommerenke [38, 39] described the Hankel determinant, $H_{q,n}(f)$ with $q, n \in \mathbb{N} = \{1, 2, \dots\}$, of q th order for a function $f \in \mathcal{A}$ as follows;

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \dots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}. \quad (1.6)$$

For some of the known subfamily of analytic functions this determinant has been extensively studied. For instance; The sharp bounds of Hankel determinant of order second $|H_{2,2}(f)| = |a_2a_4 - a_3^2|$ have been investigated in [17, 18] for families of starlike \mathcal{S}^* , convex \mathcal{C} and bounded turning \mathcal{R} functions. We also found the exact bounds as:

$$|H_{2,2}(f)| \leq \begin{cases} 1, & \text{for } f \in \mathcal{S}^*, \\ \frac{1}{8}, & \text{for } f \in \mathcal{C}, \\ \frac{4}{9}, & \text{for } f \in \mathcal{R}. \end{cases}$$

For the Bazilevič set [26], the exact value was obtained for this determinant. After that, for several other subclasses of holomorphic functions various researchers tried to study this particular problems. Recently, Srivastava [54] and his coauthors used q -derivative operator to evaluate the bounds of second Hankel determinant for symmetric bi-valent function and in [55], the researchers looked at Hankel and Toeplitz determinants for the starlike functions subclasses in q -analogue associated with conic domain. For more literature see [43, 12, 10, 16, 30, 3, 31, 35, 37]. The order 3 determinant $H_{3,1}(f)$ with entries from (1.1) is specified by:

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = -a_5a_2^2 + 2a_2a_3a_4 - a_3^3 + a_5a_3 - a_4^2. \quad (1.7)$$

The estimation of the determinant $|H_{3,1}(f)|$ is very difficult to obtain instead of calculating the value of $|H_{2,2}(f)|$ and that is why no one tried to study the problem of third order Hankel determinant until the year 2010. In [9], Babalola made the first attempt and established bounds of $H_{3,1}(f)$ for families \mathcal{S}^* , \mathcal{C} and \mathcal{R} . Even though this estimates are not sharp, but it motivated all the researchers to study this problem for holomorphic functions subfamilies. For example see [2, 7, 11, 14, 24, 42, 50]. In 2017, Zaprawa [57] enhance the work of Babalola [9] in which he obtained the following:

$$|H_{3,1}(f)| \leq \begin{cases} 1, & \text{for } f \in \mathcal{S}^*, \\ \frac{49}{540}, & \text{for } f \in \mathcal{C}, \\ \frac{41}{60}, & \text{for } f \in \mathcal{R}. \end{cases}$$

and noted that these inequalities are still not sharp yet. In addition to the sharpness, he thinking about subfamilies of \mathcal{S}^* , \mathcal{C} and \mathcal{R} containing of mappings with the m -fold

symmetry and reached the sharp bounds. Later Kwon et.al [28] obtain a superior approximation of $|H_{3,1}(f)|$ than Zaprawa [57] and for starlike functions he achieved $|H_{3,1}(f)| \leq 8/9$. Eventually in 2018, two well-known researcher Kowalczyk et.al [23] and Lecko et.al [29] achieved the best possible values of $|H_{3,1}(f)|$ and are given by

$$|H_{3,1}(f)| \leq \begin{cases} 4/135, & \text{for } f \in \mathcal{C} \\ 1/9, & \text{for } f \in \mathcal{S}^*(1/2) \end{cases}$$

where $\mathcal{S}^*(1/2)$ represents starlike functions of order half. In this article, our amis is to investigate the estimate of $|H_{3,1}(f)|$ for the subfamilies \mathcal{RL}_{\sin} and \mathcal{SL}_{\sin} of analytic functions connected with sine function. We also study this problem for femalies of m -fold symmetric functions connected with sine functions. Moreover, also we discuss $H_2(3)$ determinnat for the above mentioned femalies.

2 A SET OF LEMMAS

Definition 1. A function $p \in \mathcal{P}$ iff it has the series expansion

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, (z \in \mathbb{E}) \quad (2.1)$$

along with the $\Re p(z) \geq 0, z \in \mathbb{E}$.

Lemma 1. If $p \in \mathcal{P}$ and has the form, then

$$|c_{m+n} - \lambda c_m c_n| \leq \begin{cases} 2 & \text{for } 0 \leq \lambda \leq 1, \\ 2|2\lambda - 1| & \text{otherwise.} \end{cases} \quad (2.2)$$

$$|c_n| \leq 2 \quad (2.3)$$

and for complex number μ , we have

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\}. \quad (2.4)$$

The result (2.2), (2.3) and (2.4) are taken from [13], [40] and [21] respectively.

Lemma 2. [46] Let α, β, γ and a satisfies that $a, \alpha \in (0, 1)$ and

$$8a(1-\alpha) \left[(\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2 \right] + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \leq 4\alpha^2(1-\alpha)^2 \alpha(1-a).$$

If $p \in \mathcal{P}$, then

$$\left| \gamma c_1^2 + a c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2} \beta c_1^2 c_2 - c_4 \right| \leq 2.$$

Lemma 3. [15] Let $p \in \mathcal{P}$ and has the form (2.1), then

$$\begin{aligned} 2c_2 &= c_1^2 + (4 - c_1^2)x, \\ 4c_3 &= c_1^3 + (4 - c_1^2)c_1x(2-x) + 2(4 - c_1^2)(1 - |x|^2)z. \end{aligned}$$

for the certain x and z such that $|x| \leq 1$ and $|z| \leq 1$.

Lemma 4. [40] If $p \in \mathcal{P}$ and has the form (2.1), then

$$|Jc_1^3 - Kc_1c_2 + Lc_3| \leq 2|J| + 2|K - 2J| + 2|K - J + L|.$$

Lemma 5. If $p \in \mathcal{P}$, then the sharp estimate

$$|c_n - vc_kc_{n-k}| \leq 2$$

hold for $n, k = 1, 2, \dots, n > k, v \in [0, 1]$.

From this Lemma we obtain the very well known bound for coefficients of $p \in \mathcal{P}$, i.e $|c_n| \leq 2$, as well as the inequality $|c_n - c_kc_{n-k}| \leq 2$ proved by Livingston [32].

Lemma 6. If $p \in \mathcal{P}$, then $|c_2c_4 - c_3^2| \leq 4$. The equality holds only for functions

$$p(z) = \frac{1+z^3}{1-z^3}, \quad (2.5)$$

$$p(z) = \frac{1+z^2}{1-z^2} \quad (2.6)$$

and their rotations.

3 BOUNDS OF $|H_{3,1}(f)|$ FOR THE SET \mathcal{RL}_{\sin}

Theorem 1. Let $f \in \mathcal{RL}_{\sin}$ and has the form (1.1), then

$$|a_{p+1}| \leq \frac{p}{(p+1)},$$

$$|a_{p+2}| \leq \frac{p}{(p+2)},$$

$$|a_{p+3}| \leq \frac{p}{(p+3)},$$

$$|a_{p+4}| \leq \frac{p}{(p+4)}.$$

The first bound is best possible.

Proof. If $f \in \mathcal{RL}_{\sin}$, then (1.3) yields

$$\frac{f'(z)}{pz^{p-1}} = 1 + \sin w(z), \quad (z \in \mathbb{E}). \quad (3.1)$$

If $p \in \mathcal{P}$, then

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + \dots \quad (3.2)$$

Equivalently, we have

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1z + c_2z^2 + c_3z^3 + \dots}{2 + c_1z + c_2z^2 + c_3z^3 + \dots}.$$

From (1.1), we easily have

$$\frac{f'(z)}{pz^{p-1}} = 1 + \frac{p+1}{p}a_{p+1}z + \frac{p+2}{p}a_{p+2}z^2 + \frac{p+3}{p}a_{p+3}z^3 + \dots \quad (3.3)$$

and with support of the series expansion of $w(z)$ along with simple calculations, we obtain

$$1 + \sin(w(z)) = 1 + w(z) - \frac{(w(z))^3}{3!} + \frac{(w(z))^5}{5!} - \frac{(w(z))^7}{7!} + \dots$$

further simple computations and rearranging the terms, we have

$$\begin{aligned} 1 + \sin(w(z)) &= 1 + \frac{1}{2}c_1z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + \left(\frac{5}{48}c_1^3 + \frac{c_3}{2} - \frac{c_1c_2}{2}\right)z^3 + \\ &\quad \left(\frac{c_4}{2} + \frac{5}{16}c_1^2c_2 - \frac{c_1^4}{32} - \frac{c_1c_3}{2} - \frac{c_2^2}{4}\right)z^4 + \dots \end{aligned} \quad (3.4)$$

From (3.3) and (3.4), we easily achieve

$$a_{p+1} = \frac{p}{2(p+1)}c_1, \quad (3.5)$$

$$a_{p+2} = \frac{p}{2(p+2)}\left(c_2 - \frac{1}{2}c_1^2\right), \quad (3.6)$$

$$a_{p+3} = \frac{p}{2(p+3)}\left(c_3 - c_1c_2 + \frac{5}{24}c_1^3\right), \quad (3.7)$$

$$a_{p+4} = \frac{p}{2(p+4)}\left(c_4 - \frac{c_2^2}{2} - c_1c_3 + \frac{5}{8}c_1^2c_2 - \frac{c_1^4}{16}\right). \quad (3.8)$$

Using (2.3) and applying triangle inequality in (3.5), we have

$$|a_{p+1}| \leq \frac{p}{(p+1)}.$$

Similarly using (2.2) and triangle inequality in (3.6), we achieve

$$|a_{p+2}| \leq \frac{p}{(p+2)}.$$

Using Lemma 4 and triangle inequality in (3.7), we get

$$|a_{p+3}| \leq \frac{p}{(p+3)}.$$

By re-arranging (3.8), we get

$$a_{p+4} = \frac{p}{2(p+4)} \left| \frac{c_1^4}{16} + \frac{c_2^2}{2} + c_1c_3 - \frac{5}{8}c_1^2c_2 - c_4 \right|$$

Utilizing Lemma 2 and triangle inequality, and taking

$$a = \frac{1}{2}, \alpha = \frac{1}{2}, \beta = \frac{5}{12}, \gamma = \frac{1}{16}$$

The above values fulfilled all the condition of Lemma 2, we obtain

$$|a_{p+4}| \leq \frac{p}{(p+4)}.$$

□

Theorem 2. Let $f \in \mathcal{RL}_{\sin}$ and of the form (1.1). Then

$$|a_{p+2} - \lambda a_{p+1}^2| \leq \frac{p}{(p+2)} \max \left\{ 1, \left| \frac{p^2\lambda + 2p\lambda}{(p+1)^2} \right| \right\}.$$

Proof. From (3.5) and (3.6), we have

$$\begin{aligned} |a_{p+2} - \lambda a_{p+1}^2| &= \left| \frac{(2p^3 + 4p^2 + 2p)}{4(p+1)^2(p+2)} c_2 - \frac{(p^3 + 2p^2 + p)}{4(p+1)^2(p+2)} c_1^2 - \frac{\lambda(p^3 + 2p^2)}{4(p+1)^2(p+2)} c_1^2 \right|, \\ &= \frac{p(2p^2 + 4p + 2)}{4(p+1)^2(p+2)} \left| c_2 - \left(\frac{1}{2} \frac{(p^2 + 2p + 1 + p^2\lambda + 2p\lambda)}{(p+1)^2} \right) c_1^2 \right|. \end{aligned}$$

where

$$\mu = \left(\frac{p^2 + 2p + 1 + p^2\lambda + 2p\lambda}{(p+1)^2} \right)$$

Application of (2.4) and triangle inequality, lead us to

$$|a_{p+2} - \lambda a_{p+1}^2| \leq \frac{p}{(p+2)} \max \left\{ 1, \left| \frac{p^2\lambda + 2p\lambda}{(p+1)^2} \right| \right\}.$$

□

For $\lambda = 1$, we achieve the corollary below.

Corollary 1. Let $f \in \mathcal{RL}_{\sin}$ and given the form (1.1). Then

$$|a_{p+2} - a_{p+1}^2| \leq \frac{p}{(p+2)} \tag{3.9}$$

Theorem 3. Let $f \in \mathcal{RL}_{\sin}$ be the series form (1.1). Then

$$|a_{p+1}a_{p+2} - a_{p+3}| \leq \frac{6p}{(p+1)(p+2)(p+3)}.$$

Proof. From (3.5), (3.6) and (3.7), we have

$$|a_{p+1}a_{p+2} - a_{p+3}| = \left| \frac{p(11p^2c_1^3 - 36p^2c_1c_2 + 33pc_1^3 + 24p^2c_3 - 108pc_1c_2 + 10c_1^3 + 72pc_3 - 48c_1c_2 + 48c_3)}{48(p+1)(p+2)(p+3)} \right|$$

By re-arranging, we get

$$\begin{aligned} |a_{p+1}a_{p+2} - a_{p+3}| &= \frac{p}{48(p+1)(p+2)(p+3)} \left| (11p^2 + 33p + 10) c_1^3 \right. \\ &\quad \left. - (36p^2 + 108p + 48) c_1c_2 + (24p^2 + 72p + 48) c_3 \right|. \end{aligned}$$

The implementation of triangle inequality together with Lemma 4, give us

$$|a_{p+1}a_{p+2} - a_{p+3}| \leq \frac{6p}{(p+1)(p+2)(p+3)}.$$

The proof is ended. \square

Theorem 4. *Let $f \in \mathcal{RL}_{\sin}$ be the series form (1.1). Then*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{p^2}{(p+2)^2}.$$

Proof. By the virtue of (3.5), (3.6) and (3.7), we get

$$|a_{p+1}a_{p+3} - a_{p+2}^2| = \left| -\frac{p^2}{96(p+1)(p+2)^2(p+3)} (p^2c_1^4 + 4pc_1^4 - 24p^2c_1c_3 + 24p^2c_2^2 - 2c_1^4 - 96pc_1c_3 + 96pc_2^2 + 24c_1^2c_2 - 96c_1c_3 + 72c_2^2) \right|.$$

Use of the Lemma 3, we obtain

$$|a_{p+1}a_{p+3} - a_{p+2}^2| = \frac{p^2}{96(p+1)(p+2)^2(p+3)} \left| (-12p^2y - 48py - 48y)c_1(4 - c_1^2)(1 - |x|) + (6p^2x^2 + 24px^2 + 24x^2)c_1^2(4 - c_1^2) + (6p^2x^2 + 24px^2 + 18x^2)(4 - c_1^2)^2 + (p^2 + 4p + 4)c_1^4 \right|.$$

Applying triangle inequality, we get

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{p^2}{96(p+1)(p+2)^2(p+3)} \left((p^2 + 4p + 4)|c_1^4| + (6p^2x^2 + 24px^2 + 24x^2)|c_1^2|(4 - c_1^2) + (6p^2x^2 + 24px^2 + 18x^2)(4 - c_1^2)^2 + (12p^2y + 48py + 48y)|c_1|(4 - c_1^2)(1 - |x|) \right). \quad (3.10)$$

Let $|c_1| = c$, & $|x| = x$, Therefore, the above equation (3.10) will become

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{p^2}{96(p+1)(p+2)^2(p+3)} \left((p^2 + 4p + 4)c^4 + (6p^2x^2 + 24px^2 + 24x^2)c^2(4 - c^2) + (6p^2x^2 + 24px^2 + 18x^2)(4 - c^2)^2 + (12p^2y + 48py + 48y)c(4 - c^2)(1 - x) \right). \quad (3.11)$$

The above equation (3.11) attains its maximum value at $c = 0$, and $x = 1$, we have

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{p^2}{(p+2)^2}.$$

This is our required result. \square

Theorem 5. *Let $f \in \mathcal{RL}_{\sin}$ be given the form (1.1). Then*

$$|H_{3,1}(f)| \leq \frac{2p^2(p^5 + 11p^4 + 48p^3 + 107p^2 + 126p + 66)}{(p+1)(p+2)^3(p+3)^2(p+4)}.$$

Proof. We can easily achieve

$$|H_{3,1}(f)| \leq |a_{p+2}| |a_{p+1}a_{p+3} - a_{p+2}^2| + |a_{p+3}| |a_{p+3} - a_{p+1}a_{p+2}| + |a_{p+4}| |a_{p+2} - a_{p+1}^2|.$$

Now, from Theorem 1, Theorem 3, Theorem 4 and (3.9) we obtain

$$|H_{3,1}(f)| \leq \frac{2p^2(p^5 + 11p^4 + 48p^3 + 107p^2 + 126p + 66)}{(p+1)(p+2)^3(p+3)^2(p+4)}.$$

□

Taking $p = 1$, we obtain the following Corollary.

Corollary 2. *Let $f \in \mathcal{RL}_{\sin}$ be given by (1.1). Then*

$$|H_{3,1}(f)| \leq \frac{359}{2160} = 0.16620.$$

4 BOUNDS OF $|H_{3,1}(f)|$ FOR THE SET \mathcal{SL}_{\sin}

Theorem 6. *Let $f \in \mathcal{SL}_{\sin}$ and has the form (1.1). Then*

$$\begin{aligned} |a_{p+1}| &\leq p, \\ |a_{p+2}| &\leq \begin{cases} \frac{p}{2} & \text{for } p = 1, \\ \frac{p^2}{2} & \text{otherwise.} \end{cases}, \\ |a_{p+3}| &\leq \begin{cases} \frac{2p}{144} \Psi(p) & \text{where } \Psi(p) = (|3p^2 - 9p + 5| + 2|3p^2 - 7| + |3p^2 + 9p + 5|) , \\ \frac{p}{288} (12p^3 + 72p^2 + 68p - 109) & \text{for } p \leq 2, \\ \frac{p}{24} (p^3 + 6p^2 + 22p + 33) & \text{otherwise.} \end{cases} \end{aligned}$$

The first two bounds are best possible.

Proof. If $f \in \mathcal{SL}_{\sin}$, then (1.3) gives

$$\frac{zf'(z)}{pf(z)} = \sqrt{1 + w(z)}, (z \in \mathbb{E}). \quad (4.1)$$

Also if $p \in \mathcal{P}$, then from (3.2) and (4.1) we have

$$\frac{zf'(z)}{pf(z)} = \sqrt{\frac{2p(z)}{1+p(z)}}. \quad (4.2)$$

By using (1.1), we obtain

$$\begin{aligned} \frac{zf'(z)}{pf(z)} &= 1 + \frac{a_{p+1}z}{p} + \left(\frac{2a_{p+2}}{p} - \frac{a_{p+1}^2}{p} \right) z^2 + \left(\frac{3a_{p+3}}{p} - \frac{3a_{p+1}a_{p+2}}{p} + \frac{a_{p+1}^3}{p} \right) z^3 \\ &+ \left(\frac{4a_{p+1}^2a_{p+2}}{p} - \frac{4a_{p+1}a_{p+3}}{p} + \frac{4a_{p+4}}{p} - \frac{a_{p+1}^4}{p} - \frac{2a_{p+2}^2}{p} \right) z^4 + \dots \end{aligned} \quad (4.3)$$

Comparing (3.4) and (4.3), we easily get

$$a_{p+1} = \frac{p}{2}c_1, \quad (4.4)$$

$$a_{p+2} = \frac{p}{8}(pc_1^2 - c_1^2 + 2c_2), \quad (4.5)$$

$$\begin{aligned} a_{p+3} &= \frac{p}{144}(3p^2c_1^3 - 9pc_1^3 + 5c_1^3 + 18pc_1c_2 - 24pc_1c_2 \\ &+ 24c_3), \end{aligned} \quad (4.6)$$

$$\begin{aligned} a_{p+4} &= \frac{p}{1152}((3p^3 - 18p^2 + 29p - 9)c_1^4 \\ &+ (36p^2 - 132p + 90)c_1^2c_2 + (96p - 144)c_1c_3 \\ &+ (36p - 72)c_2^2 + 144c_4). \end{aligned} \quad (4.7)$$

Applying (2.3) and triangle inequality in (4.4), we get

$$|a_{p+1}| \leq p.$$

By re-arranging (4.5), we have

$$|a_{p+2}| = \frac{p}{4} \left| c_2 - \frac{(1-p)}{2}c_1^2 \right|.$$

Use of the (2.2) and triangle inequality, we achieve

$$|a_{p+2}| \leq \begin{cases} \frac{p}{2} & \text{for } p = 1, \\ \frac{p^2}{2} & \text{otherwise.} \end{cases} .$$

Re-arranging (4.6), we have

$$|a_{p+3}| = \frac{p}{144} |(3p^2 - 9p + 5)c_1^3 + (18p - 24)c_1c_2 + 24c_3|.$$

Use of the Lemma 4 and triangle inequality, we get

$$|a_{p+3}| \leq \frac{2p}{144}\Psi(p),$$

where

$$\Psi(p) = (|3p^2 - 9p + 5| + 2|3p^2 - 7| + |3p^2 + 9p + 5|). \quad (4.8)$$

Rearranging (4.7), we get

$$\begin{aligned}
|a_{p+4}| &= \frac{p}{1152} \left((168p - 145) c_1 \left(c_3 - \frac{(52p - 40)}{(168p - 145)} c_1 c_2 \right) + (36p - 72) c_2 \right. \\
&\quad \left(c_2 - \frac{(12p - 32)}{(36p - 72)} c_1^2 \right) + (36p^2 - 68p + 18) c_1^2 \left(c_2 - \frac{(18p^2 - 29p + 9)}{(36p^2 - 68p + 18)} c_1^2 \right) \\
&\quad \left. + 144 \left(c_4 - \frac{(72p - 1)}{144} c_1 c_3 \right) + 3p^3 c_1^4 \right).
\end{aligned}$$

Utilization of triangle inequality along with (2.2) and (2.3), we obtain

$$|a_{p+4}| \leq \frac{p}{288} (12p^3 + 72p^2 + 68p - 109). \text{ for } p \leq 2$$

Now, for the other value of p , we have

$$\begin{aligned}
|a_{p+4}| &= \frac{p}{1152} \left(3p^3 c_1^4 + \left(-36c_1^2 \left(\frac{1}{2} c_1^2 - c_2 \right) p^2 \right) \right. \\
&\quad \left. + \left(66c_1^2 \left(\frac{29}{66} c_1^2 - c_2 \right) - 96c_1 \left(\frac{11}{16} c_1 c_2 - c_3 \right) + 36c_2^2 \right) p \right. \\
&\quad \left. - 90c_1^2 \left(\frac{1}{10} c_1^2 - c_2 \right) - 144 \left(\frac{72}{144} c_2^2 - c_4 \right) - 144c_1 c_3 \right).
\end{aligned}$$

The implementation of triangle inequality and (2.2) and (2.3), we obtain

$$|a_{p+4}| \leq \frac{p}{24} (p^3 + 6p^2 + 22p + 33).$$

□

Theorem 7. Let $f \in \mathcal{SL}_{\sin}$ and has given by (1.1). Then for $\lambda \in \mathbb{C}$

$$|a_{p+2} - \lambda a_{p+1}^2| \leq \frac{p}{2} \max \{1, |2\lambda p - p|\}.$$

Proof. From (4.4) and (4.5), we have

$$\begin{aligned}
|a_{p+2} - \lambda a_{p+1}^2| &= \left| \frac{2p}{8} c_2 - \frac{2\lambda p^2}{8} c_1^2 + \frac{(p^2 - p)}{8} c_1^2 \right|, \\
&= \frac{p}{4} \left| c_2 - \frac{c_1^2}{2} (2\lambda p - p + 1) \right|.
\end{aligned}$$

with

$$\mu = (2\lambda p - p + 1).$$

Applying (2.4) and triangle inequality

$$|a_{p+2} - \lambda a_{p+1}^2| \leq \frac{p}{2} \max \{1, |2\lambda p - p|\}.$$

□

For $\lambda = 1$, we obtain the following Corollary.

Corollary 3. *Let $f \in \mathcal{SL}_{\sin}$ and be given by (1.1). Then*

$$|a_{p+2} - a_{p+1}^2| \leq \frac{p}{2}. \quad (4.9)$$

Theorem 8. *Let $f \in \mathcal{SL}_{\sin}$ and has the form (1.1). Then*

$$|a_{p+1}a_{p+2} - a_{p+3}| \leq \frac{p(6p^2 + 1)}{18}.$$

Proof. From (4.4), (4.5) and (4.6), we have

$$|a_{p+1}a_{p+2} - a_{p+3}| = \left| \frac{p^3}{24}c_1^3 - \frac{5}{144}pc_1^3 + \frac{p}{6}c_1c_2 - \frac{p}{6}c_3 \right|. \quad (4.10)$$

Re-arranging (4.10), we obtain

$$|a_{p+1}a_{p+2} - a_{p+3}| = \frac{p}{144} |(6p^2 - 5)c_1^3 + 24c_1c_2 - 24c_3|.$$

Use of the Lemma 4 and triangle inequality, we get

$$|a_{p+1}a_{p+2} - a_{p+3}| \leq \frac{p(6p^2 + 1)}{18}.$$

□

Theorem 9. *Let $f \in \mathcal{SL}_{\sin}$ and be given by (1.1). Then*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{p^2}{4}.$$

Proof. From (4.4), (4.5), and (4.6), we have

$$\begin{aligned} |a_{p+1}a_{p+3} - a_{p+2}^2| &= \frac{1}{576} |p^2c_1^4 - 3p^4c_1^4 - 12p^2c_1^2c_2 + 48p^2c_1c_3 - 36p^2c_2^2|, \\ &= \frac{p^2}{576} |3p^2c_1^4 - c_1^4 + 12c_1^2c_2 - 48c_1c_3 + 36c_2^2|. \end{aligned}$$

Using Lemma 3, we get

$$|a_{p+1}a_{p+3} - a_{p+2}^2| = \frac{p^2}{576} |(3p^2 + 2)c_1^4 + 12c_1^2Qx^2 - 24c_1QWy + 9Q^2x^2|,$$

Utilizing triangle inequality, we have

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{p^2}{576} ((3p^2 + 2)c_1^4 + 12c_1^2Qx^2 + 24c_1QWy + 9Q^2x^2).$$

with

$$Q = (4 - c_1^2) \quad \& \quad W = (1 - |x|^2). \quad (4.11)$$

The function above has the maximum value at $c_1 = 0$, and $|x| = 1$ for all values of p . Therefore

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{p^2}{4}.$$

which is the required proof. □

Theorem 10. *Let $f \in \mathcal{SL}_{\sin}$ and has the form (1.1). Then*

$$|H_{3,1}(f)| \leq \begin{cases} \frac{p^2}{5184} (24p^2\Psi(p) + 108p^3 + 648p^2 + 4\Psi(p) + 1260p - 981) & \text{for } p \leq 2, \\ \frac{p^2}{1296} (6p^2\Psi(p) + 27p^3 + 324p^2 + \Psi(p) + 594p + 891) & \text{otherwise.} \end{cases}$$

Proof. We easily obtain

$$|H_{3,1}(f)| \leq |a_{p+2}| |a_{p+1}a_{p+3} - a_{p+2}^2| + |a_{p+3}| |a_{p+3} - a_{p+1}a_{p+2}| + |a_{p+4}| |a_{p+2} - a_{p+1}^2|.$$

Now, from Theorem 6, Theorem 8, Theorem (9), and (4.9), we get

$$|H_{3,1}(f)| \leq \begin{cases} \frac{p^2}{5184} (24p^2\Psi(p) + 108p^3 + 648p^2 + 4\Psi(p) + 1260p - 981) & \text{for } p \leq 2, \\ \frac{p^2}{1296} (6p^2\Psi(p) + 27p^3 + 324p^2 + \Psi(p) + 594p + 891) & \text{otherwise.} \end{cases}$$

where $\Psi(p)$ is given by (4.8). \square

Taking $p = 1$, the following corollary obtain.

Corollary 4. *Let $f \in \mathcal{SL}_{\sin}$ be given by (1.1). Then*

$$|H_{3,1}(f)| \leq \frac{1763}{5184} = 0.34008.$$

5 Bounds of $|H_{3,1}(f)|$ for 2-fold and 3-fold functions

Let $m \in \mathbb{N} = \{1, 2, \dots\}$. If a rotation Δ about the origin through an angle $2\pi/m$ carries \mathbb{E} on itself, then such a domain \mathbb{E} is called m -fold symmetric. An analytic function f is m -fold symmetric in \mathbb{E} , if

$$f(e^{2\pi i/m}z) = e^{2\pi i/m}f(z), \quad (z \in \mathbb{E}).$$

By $\mathcal{S}^{(m)}$, we define the set of m -fold univalent functions having the following Taylor series form

$$f(z) = z + \sum_{k=p}^{\infty} a_{mk+1} z^{mk+1}, \text{ where } (p \in \mathbb{N}). \quad (5.1)$$

The sub-families $\mathcal{RL}_{\sin}^{(m)}$ and $\mathcal{SL}_{\sin}^{(m)}$ of $\mathcal{S}^{(m)}$ are the sets of m -fold symmetry bounded turning and starlike functions respectively associated with sine functions. More intuitively, an analytic function f of the form (5.1), belongs to the families $\mathcal{RL}_{\sin}^{(m)}$ and $\mathcal{SL}_{\sin}^{(m)}$, if and only if

$$\frac{f'(z)}{pz^{p-1}} = 1 + \sin\left(\frac{p(z)-1}{p(z)+1}\right), \quad p \in \mathcal{P}^{(m)}, \quad (5.2)$$

$$\frac{zf'(z)}{pf(z)} = 1 + \sin\left(\frac{p(z)-1}{p(z)+1}\right), \quad p \in \mathcal{P}^{(m)}. \quad (5.3)$$

where the set $\mathcal{P}^{(m)}$ is defined by

$$\mathcal{P}^{(m)} = \left\{ p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} c_{mk} z^{mk}, \quad (z \in \mathbb{E}) \right\}. \quad (5.4)$$

Here we prove some theorems related to 2-fold and 3-fold symmetric functions.

Theorem 11. If $f \in \mathcal{RL}_{\sin}^{(2)}$ and has the form given in (5.1), then

$$|H_{3,1}(f)| \leq \frac{p^2}{(p+2)(p+4)}.$$

Proof. Let $f \in \mathcal{RL}_{\sin}^{(2)}$. Then, there exists a function $p \in \mathcal{P}^{(2)}$, such that

$$\frac{f'(z)}{pz^{p-1}} = 1 + \sin\left(\frac{p(z)-1}{p(z)+1}\right).$$

Using the series form (5.1) and (5.4), when $m = 2$ in the above relation, we can get

$$a_{p+2} = \frac{p}{2(p+2)}(c_2), \quad (5.5)$$

$$a_{p+4} = \frac{p}{2(p+4)}\left(c_4 - \frac{c_2^2}{2}\right). \quad (5.6)$$

Now,

$$H_{3,1}(f) = a_{p+2}a_{p+4} - a_{p+2}^3.$$

Utilizing (5.5) and (5.6), we get

$$H_{3,1}(f) = -\frac{p^2}{4(p+2)^3(p+4)}c_2(p^2c_2^2 - p^2c_4 + 4pc_2^2 - 4pc_4 + 2c_2^2 - 4c_4).$$

By realterting, it yields

$$H_{3,1}(f) = \frac{p^2(p^2+4p+4)}{4(p+2)^3(p+4)}(c_2)\left(c_4 - \frac{(p^2+4p+2)}{(p^2+4p+4)}c_2^2\right).$$

Using triangle inequality long with (2.2) and (2.3), gives us

$$|H_{3,1}(f)| \leq \frac{p^2}{(p+2)(p+4)}.$$

Hence, the proof is done. \square

Theorem 12. If $f \in \mathcal{RL}_{\sin}^{(3)}$ and has the form given in (5.1), then

$$|H_{3,1}(f)| \leq \frac{p^2}{(p+3)^2}.$$

Proof. As, $f \in \mathcal{RL}_{\sin}^{(3)}$, therefore there exists a function $p \in \mathcal{P}^{(3)}$, such that

$$\frac{f'(z)}{pz^{p-1}} = 1 + \sin\left(\frac{p(z)-1}{p(z)+1}\right).$$

Utilizing the series form (5.1) and (5.4), when $m = 3$ in the above relation, we can obtain

$$a_{p+3} = \frac{p}{2(p+3)}(c_3).$$

Then,

$$H_{3,1}(f) = -a_{p+3}^2.$$

$$H_{3,1}(f) = -\left(\frac{p^2}{4(p+3)^2} (c_3^2)\right).$$

Utilizing (2.3) and triangle inequality, we have

$$|H_{3,1}(f)| \leq \frac{p^2}{(p+3)^2}.$$

Thus, the Theorem is proved. \square

Theorem 13. *If $f \in \mathcal{SL}_{\sin}^{(2)}$ and has the form given in (5.1), then*

$$|H_{3,1}(f)| \leq \frac{p^2}{(p+1)(p+3)}.$$

Proof. Let $f \in \mathcal{SL}_{\sin}^{(2)}$. Then, there exists a function $p \in \mathcal{P}^{(2)}$, such that

$$\frac{zf'(z)}{pf(z)} = 1 + \sin\left(\frac{p(z)-1}{p(z)+1}\right).$$

Using the series form (5.1) and (5.4), when $m = 2$ in the above relation, we can get

$$a_{p+2} = \frac{p}{2(p+1)} (c_2), \quad (5.7)$$

$$a_{p+4} = \frac{p}{4(p+1)(p+3)} ((2p+2)c_4 - c_2^2). \quad (5.8)$$

Now,

$$H_3(f) = a_{p+2}a_{p+4} - a_{p+2}^3.$$

Utilizing (5.7) and (5.8), we get

$$H_{3,1}(f) = -\frac{p^2}{8(p+1)^3(p+3)} c_2 (p^2 c_2^2 - 2p^2 c_4 + 4pc_2^2 - 4pc_4 + c_2^2 - 2c_4).$$

By realterting, it yields

$$H_{3,1}(f) = \frac{p^2(2p^2+4p+2)}{8(p+1)^3(p+3)} (c_2) \left(c_4 - \frac{(p^2+4p+1)}{(2p^2+4p+2)} c_2^2 \right).$$

Using triangle inequality long with (2.2) and (2.3), gives us

$$|H_{3,1}(f)| \leq \frac{p^2}{(p+1)(p+3)}.$$

Hence, the proof is done. \square

Theorem 14. *If $f \in \mathcal{SL}_{\sin}^{(3)}$ and has the form given in (5.1), then*

$$|H_{3,1}(f)| \leq \frac{p^2}{(p+2)^2}.$$

Proof. As, $f \in \mathcal{SL}_{\sin}^{(3)}$, therefore there exists a function $p \in \mathcal{P}^{(3)}$, such that

$$\frac{zf'(z)}{pf(z)} = 1 + \sin\left(\frac{p(z)-1}{p(z)+1}\right).$$

Utilizing the series form (5.1) and (5.4), when $m = 3$ in the above relation, we can obtain

$$a_{p+3} = \frac{p}{2(p+2)}(c_3).$$

Then,

$$\begin{aligned} H_{3,1}(f) &= -a_{p+3}^2. \\ H_{3,1}(f) &= -\left(\frac{p^2}{4(p+2)^2}(c_3^2)\right). \end{aligned}$$

Utilizing (2.3) and triangle inequality, we have

$$|H_{3,1}(f)| \leq \frac{p^2}{(p+2)^2}.$$

Thus, the Theorem is proved. \square

6 Determinant $H_2(3)$ for \mathcal{RL}_{\sin} and \mathcal{SL}_{\sin}

Let f, g , be multivalent. Then

$$f \in \mathcal{RL}_{\sin} \Leftrightarrow \frac{f'(z)}{pz^{p-1}}. \quad (6.1)$$

$$g \in \mathcal{SL}_{\sin} \Leftrightarrow \frac{zf'(z)}{pf(z)}. \quad (6.2)$$

Therefore, if $f \in \mathcal{RL}_{\sin}$, then $H_2(3) = F(c_1, c_2, c_3, c_4)$, from (3.6), (3.7) and (3.8), we get

$$\begin{aligned} F(c_1, c_2, c_3, c_4) &= -\frac{p^2}{2304(p+2)(p+4)(p+3)^2} (7p^2c_1^6 - 24p^2c_1^4c_2 + 42pc_1^6 - 48p^2c_1^3c_3 \\ &+ 72p^2c_1^2c_2^2 - 144pc_1^4c_2 + 38c_1^6 + 288p^2c_1^2c_4 - 576p^2c_1c_2c_3 + 288p^2c_2^3 - 288pc_1^3c_3 \\ &+ 432pc_1^2c_2^2 + 24c_1^4c_2 - 576p^2c_2c_4 + 576p^2c_3^2 + 1728pc_1^2c_4 - 4032c_1c_2c_3 + 1728pc_2^3 \\ &- 3456pc_1c_2c_3 - 672c_1^3c_3 + 72c_1^2c_2^2 - 3456pc_2c_4 + 3456pc_3^2 + 2592c_1^2c_4 - 5184c_2c_4 \\ &+ 2592c_2^3 + 4608c_3^2). \end{aligned} \quad (6.3)$$

Therefore, if $g \in \mathcal{SL}_{\sin}$, then $H_2(3) = G(c_1, c_2, c_3, c_4)$, from (4.5), (4.6) and (4.7), we obtain

$$\begin{aligned} G(c_1, c_2, c_3, c_4) = & \frac{1}{82944} \left(-9p^6 c_1^6 + 27p^5 c_1^6 - 54p^5 c_1^4 c_2 - 21p^4 c_1^6 + 36p^4 c_1^4 c_2 + 18p^3 c_1^6 \right. \\ & + 288p^4 c_1^3 c_3 - 324p^4 c_1^2 c_2^2 + 72p^3 c_1^4 c_2 - 19p^2 c_1^6 - 432p^3 c_1^3 c_3 + 108p^3 c_1^2 c_2^2 \\ & - 12p^2 c_1^4 c_2 + 1296p^3 c_1^2 c_4 - 1728p^3 c_1 c_2 c_3 + 648p^3 c_2^3 + 336p^2 c_1^3 c_3 - 36p^2 c_1^2 c_2^2 \\ & \left. - 1292p^2 c_1^2 c_4 + 2016p^2 c_1 c_2 c_3 - 1296p^2 c_2^3 + 2592p^2 c_2 c_4 - 2304p^2 c_3^2 \right). \quad (6.4) \end{aligned}$$

Having regard to the explicit formulae of $H_2(3)$, finding the exact bounds of those expressions is a difficult issue. For this purpose, we begin with a specific event.

Theorem 15. *Let f be given by (1.1) with additional assumption that $a_{p+1} = 0$. If $f \in \mathcal{RL}_{\sin}$, then $|H_2(3)| \leq \frac{2p^2}{(p+4)(p+2)}$.*

Proof. Assume that for f given by (1.1) there is $a_{p+1} = 0$. From (6.3) it follows that $c_1 = 0$.

If $f \in \mathcal{RL}_{\sin}$, then

$$\begin{aligned} F(c_1, c_2, c_3, c_4) = & \frac{p^2}{2304(p+2)(p+4)(p+3)^2} \left(-(288p^2 + 1728p + 2592) c_2^3 \right. \\ & \left. + 576c_2 c_4 + (576p^2 + 3456p + 4608) (c_2 c_4 - c_3^2) \right). \quad (6.5) \end{aligned}$$

The desired estimate follows directly from the the triangle inequality and Lemmas 5 and 6. So

$$|H_2(3)| \leq \frac{2p^2}{(p+4)(p+2)}. \quad (6.6)$$

holds only if $|c_2 c_4 - c_3^2| = 4$ and $|c_2| = |c_4| = 2$. By Lemma 6, it is possible only for $p(z) = \frac{1+z^2}{1-z^2}$. As a consequence, we obtain the extremal functions $f \in \mathcal{RL}_{\sin}$ for which (6.6) holds. \square

Theorem 16. *Let g be given by (1.1) with additional assumption that $a_{p+1} = 0$. If $g \in \mathcal{SL}_{\sin}$, then $|H_2(3)| \leq \frac{p^3}{16}$.*

Proof. Assume that for g given by (1.1) there is $a_{p+1} = 0$. From (6.4) it follows that $c_1 = 0$.

If $g \in \mathcal{SL}_{\sin}$, then

$$|H_2(3)| = \frac{1}{82944} \left| (648p^3 - 1296p^2) c_2^3 + 288p^2 c_2 c_4 + 2304p^2 (c_2 c_4 - c_3^2) \right|. \quad (6.7)$$

It is enough to apply Lemmas 5 and 6 to obtain the bounds $\frac{p^3}{16}$ for \mathcal{SL}_{\sin} . The expression (6.7) is equal to $\frac{p^3}{16}$ if and only if $|c_2| = 2$, $|c_4| = 2$ and $|c_2 c_4 - c_3^2| = 4$. It is possible only for rotations of (2.6) This means that the extremal starlike functions are $g(z) = \frac{z}{1-z^2}$ and its rotations. \square

7 Conclusions

In this article, we studied Hankel determinant $H_{3,1}(f)$ for the families \mathcal{RL}_{\sin} and \mathcal{SL}_{\sin} subordinated with sine functions. Furthermore, we improve the bound of third Hankel determinant for the family \mathcal{SL}_{\sin} . These bounds are also discussed for 2-fold symmetric and 3-fold symmetric functions. Also we discussed $H_2(3)$ determinant for the families \mathcal{RL}_{\sin} and \mathcal{SL}_{\sin} .

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