formulation of the method is presented in [6]. A remarkable number of applications are available in the literature. These areas include linear, non-linear, homogeneous and non homogeneous systems of differential equations. These systems results from modelling physical phenomena from science and engineering. non-linearity and implementation of physical boundary conditions imposes a challenging task for solving such models mathematically. The homotopy analysis method (HAM) is applied for this purpose. The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. In this paper, an alternative approach based on HAM is presented to approximate the solution on nonlinear coupled system of PDEs.

The article is organized as follows: Section II presents a brief introduction to the homotopy analysis method. In section III, the series analytic solution is presented for the system. The finite difference method is formulated in section IV. Section IV comprises of the comparison of analytical and numerical solutionsfor some test problems. A concluding remark is given in the final section.

2. **Basic Ideas of HAM**. Let us consider the differential equations given as,

$$N_i[z_i(x,t)] = 0, \qquad i = 1, 2, \ldots, n, \tag{2}$$

where N_i stands for nonlinear operators of the whole equations, x and t denote the independent variables and $z_i(x,t)$ are the unknown functions in variables x and t respectively Adapting the generalized homotopy method, Liao [13], the zero-order deformation equations are obtined as follows,

$$(1-q)L\left[\ddot{}_i\left(x,t:q\right) - z_{i,0}\left(x,t\right)\right] = qh_i N_i\left[\ddot{}_i\left(x,t:q\right)\right], \tag{3}$$

where $q \in [0,1]$ is an embedding parameter, h_i are nonzero auxiliary functions, L is an auxiliary linear operator, $z_{i,0}(x,t)$ are initial guesses of $z_i(x,t)$ and ¤$_i(x,t;q)$ are unknown functions. It is important to note that, one has great freedom to choose auxiliary objects in HAM. Obviously, when $q = 0$ and $q = 1$, both ¤$_i(x,t;0) = z_{i,0}(x,t)$ and ¤$_i(x,t;1) = z_i(x,t)$ hold. Hence the approximate solution approaces the exact solution as q increases from 0 to 1. Taylor series expansion of ¤$_i(x,t;q)$ with respect to q , is

$$¤_i(x,t;q) = z_{i,0}(x,t) + \sum_{m=1}^{+\infty} z_{i,m}(x,t), \tag{4}$$

Where

$$z_{i,m} = \frac{i}{m!}\frac{\partial^m ¤_i(x,t;q)}{\partial q^m}\Big|_{q=0} \tag{5}$$

The series equation [4] converges at $q = 1$ subject to suitble choices of the auxiliary linear operator, the initial guess, the auxiliary parameters and the auxiliary functions and the auxiliary functions. So proceed as follows under this assumption of suitable choice and we have

$$¤_i(x,t;1) = z_{i,0}(x,t) + \sum_{m=1}^{+\infty} z_{i,m}(x,t), \tag{6}$$

as one of solutions of the original nonlinear equations, as proved by Liao [2]. As $h_i = -1$, Eq [3] becomes

$$(1-q)L[\text{¤}_i(x,t;q)-z_{i,0}(x,t)]=qN_i[\text{¤}_i(x,t;q],\tag{7}$$

which are frequently used in the homotopy-perturbation method [29].

According to [5], the governing equations can be deduced from the zero-order deformation equations [3]. Define the vectors
,

$$Z_{i,n}=\{z_{i,0}(x,t),z_{i,1}(x,t),\ldots,z_{i,n}(x,t)\}.\tag{8}$$

Differentating [3] m times with respect to the embedding parameter q and then setting $q=0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equations,

$$L[z_{i,m}(x,t)-\text{¡}_m z_{i,m}(x,t)]=h_i R_{i,m}(z_{i,m-1}),\tag{9}$$

Where

$$R_{i,m}(z_{i,m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N_i[\text{¤}_i(x,t;q)]}{\partial q^{m-1}}|_{q=0}\tag{10}$$

and

$\text{¡}_m=0$ for $m\le 1$ while, $\text{¡}_m=1$ for $m>1$.

3. Finite Difference System. For the numerical solution of system [1] using finite difference method a uniform mesh is considered. The interval $[0,x]$ is discretized into uniform intervals each having length h. let $h=\Delta x=l/M$ and $k=\Delta k=T/N$ be th space and time discretizations and take $x_i=ih$ and, $t_n=nk$ as a grid point in the domain $[0,1]\times[0,\mathrm{t}]$. here M and N are the total number of intervals along space and time axis respectively. Define $u_{i,n}=u(x_i,t_n)$ as the discrete analogue of the continuous density function $u(x,t)$. The forward difference approximation is used for time derivative discretization and a backward difference approximation is employed for the space derivative discretization. For the treatment of nonlocal boundary condition, the trapezoidal rule has been incorporated. This yield a finite difference system which is presented in the matrix form as follows.

$$u_i^{k+1}=A^k u_i^k+b_i^k,\tag{11}$$

where,

$$A^k=\begin{bmatrix} -_k & 0 & 0 & 0 & 0 & \ldots & 0 & 0\\ r & \text{¢}_2 & 0 & 0 & 0 & \ldots & 0 & 0\\ 0 & r & \text{¢}_3 & 0 & 0 & \ldots & 0 & 0\\ 0 & 0 & r & \text{¢}_4 & 0 & \ldots & 0 & 0\\ 0 & 0 & 0 & r & \text{¢}_5 & \ldots & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & 0 & 0 & r & \text{¢}_{M-2} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & r & \text{¢}_{M-1} \end{bmatrix}\tag{12}$$

Is a matrix of order $(M-1)\times(N-1)$. The column vector b_i^k corresponds to the constant values from initial and boundary conditions. Here, $-^{k} = -r + r.\frac{(1)}{2}.^{-k-1} - \Delta t.«(x_1)+1$, $r=\frac{\Delta t}{\Delta x}$ and $-_0 = u_0^0 ._0 / (2-2\ \ (x_1))$ and $¢_1 = -r+1-\Delta t.«(x_1)$. A matlab code is designed for the numerical implementation of these methods. The results are shown in Figure 1 and Figure 2.

4. Numerical Tests

4.1. Problem 1:

Consider the following differential system:

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+«(x)u=0, \qquad t<T<0 \qquad 0<x\leq 1 \tag{13}$$

$$«(x) = 10\,e^{-100(1-x)},$$
$$u(0,t) = 2-e^{-t}, \qquad t<T<0$$
$$u(x,0) = w_0 e^{\sim x}, \qquad (x) = 20x(1-x), \qquad 0<x\leq 1.$$

In order to get solution of [13] by homotopy analysis method, the initial approximations are choosen as

$$u_0(x,t) = w_0 e^{\sim x}, \tag{14}$$

while the linear operator is taken as follows

$$L[¤_i(x,t;q)] = \frac{\partial ¤_i(x,t;q)}{\partial t}, \tag{15}$$

with $i=1$ and the property

$$L[c_i] = 0, \tag{16}$$

where c_i are the integral constants. According to these definitions, the *zeroth-order deformation*equations are

$$(1-q)L[¤_i(x,t;q)] = \frac{\partial ¤_i(x,t;q)}{\partial t}, \qquad i=1 \tag{17}$$

For $q=0$ and $q=1$, we have

$$u(x,t;0) = z_{i,0}(x,t) = u_0(x,t), \quad ¤_1(x,t;1) = u(x,t). \tag{18}$$

Thus as q from 0 to 1, the solution varies from the initial guess to the exact solutions. Expanding in Taylor series with respect to q, one has

$$¤_i(x,t;q) = z_{i,0}(x,t) + \sum_{m=1}^{+\infty} z_{i,m}(x,t)q^m, \tag{19}$$

where

$$z_{i,m} = \frac{i}{m!}\frac{\partial^m ¤_i(x,t;q)}{\partial q^m}\Big|_{q=0}. \tag{20}$$

For suitable choices of the auxiliary linear operator, the initial guess, the uxiliary parameters and the auxiliary functions, the series equation [19] converges at $q = 1$ and we have

$$\mathrm{u}(x,t) = z_{1,0}(x,t) + \sum_{m=2}^{+\infty} z_{i,m}(x,t)\,, \tag{21}$$

Which must be one of the solutions of the system. Consider a vector which is defined as

,

$$Z_{i,n} = \{\mathrm{z}_{i,0}(x,t), z_{i,1}(x,t), \ldots, z_{i,n}(x,t)\}. \tag{22}$$

So the *mth-order deformation* equations are

$$L[z_{i,m}(x,t) - \text{¡}_m z_{i,m}(x,t)] = h_i R_{i,m}(z_{i,m-1}), \tag{23}$$

With the initial conditions

$z_{i,m}(x,0) = 0$.

And

$$R_{1,m}(z_{i,m-1}) = (\mathrm{z}_{1,m-1})_t + (z_{1,m-1})_x + \text{«}(x,t)(z_{1,m-1}). \tag{24}$$

The solution of the *mth-order deformation* equation [23] is given as

$$z_{i,m}(x,t) = \text{¡}_m z_{i,m-1}(x,t) + h_i \int_0^t R_{i,m}(z_{i,m-1}) d\text{²} + c_i\,, \tag{25}$$

Where the integration constant c_i are obtained by the initial conditions given in [13]. The series solution by HAM can be written in the form as follows

$$u(x,t) = z_{1,o}(x,t) + z_{1,1}(x,t) + z_{1,2}(x,t) + z_{1,3}(x,t) + \cdots. \tag{26}$$

For $h = -1$, the approximated series solutions are computed as follows:

$$u(x,t) = w_0 \sim e^{\sim x} + 2h[w_0 \sim e^{\sim x}.t + w_0 \text{«}(x) e^{\sim x}.t + S\,, \tag{27}$$

$$S = h^2[w_0 \sim e^{\sim x} + w_0 \text{«}(x) e^{\sim x}.t + w_0 \sim^2 e^{\sim x} t^2 / 2 + w_0 \sim^2 e^{\sim x} t^2 / 2(1+\sim). \tag{28}$$

A plot is shown below for $u(x,t)$ at $t = 0.3$ and $h = -1$.

4. 2. Problem 2.

To demonstrate the application of HAM for a larger class of reaction functions, a more general system is taken as follows:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \text{«}(x)u = 0, \qquad \mathrm{t} < \mathrm{T} < 0 \qquad 0 < x \le 1 \tag{29}$$

$$\text{«}(\mathrm{x}) = 1/(1-x)$$

$$u(0,t) = (1-x)e^{-x}\,, \qquad 0 < t < T$$

$$u(x,0) = w_0 e^{\sim x}\,, \qquad (x) = 20x(1-x), \qquad 0 < x \le 1.$$

The series solution is computed by HAM presented as:

$$u(x,t) = (1-x)e^{-x} + 3htxe^{-x} + h^2[xte^{-x} - xt^2 e^{-x}/2 + t^2 xe^{-x}/(2 = 2x) + \cdots. \tag{30}$$

It is important to note that the series solutions are valid with the assumption that series [30] converges for $q = 1$. The auxiliary parameter plays a basic role for this assumption. For different values of h, the curves are plotted and checked the region of h for which solution series is convergent. For that proper values of h, the solution is best approximation of the exct solution of the problem.

The plots shown in Figure 1 and Figure 2 shows the solution obtained by HAM and finite difference method.

The time is taken to be $t = 0.3$.

5. Conclusion. This study is based upon the approximate series solutions and numerical solutions for some problems from literature. Series solutions are obtained by the well known homotopy analysis method, while the numericl results are computed by implementation of finite difference algorithm. Both of these methods have applied first time in the literature for the presented problems. The plotted results agree with those already available. The techniques encures an efficient and alternative way for mathematical analysis of the problems from different aspects. The numerical methods gives a stable solution that is checked by stability analysis of the scheme, while approximate solutions abtained by HAM are convergent series solution which is confirmed by the appropriate convergence region of the auxiliary parameter.
Future study is in progress for developing non-standard finite difference algorithms and their stability analysis.

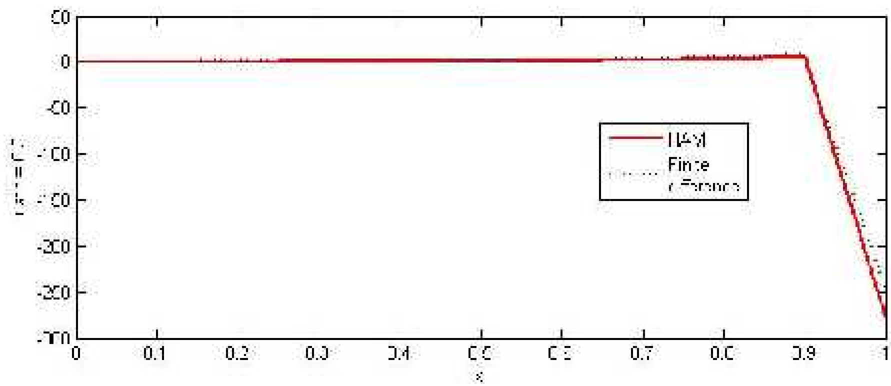

Figure 1. Result of Problem 1.

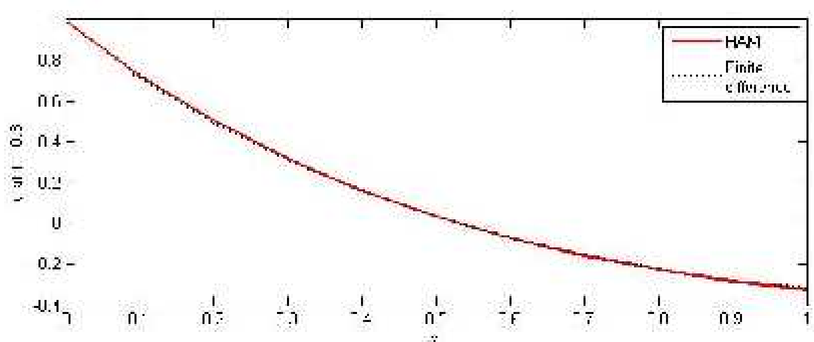

Figure 2. Result of Problem 2.

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