VFAST Transactions on Mathematics
http://vfast.org/journals/index.php/VTM@ 2013 ISSN(e): 2309-0022; ISSN(p): 2411-6343
Volume 1, Number 1, January-December, 2013

pp. 01-07

SOME CONVEXITY PROPERTIES FOR A GENERAL INTEGRAL OPERATOR

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Abstract. *In the present article, we consider some subclasses of analytic functions of complex order. By using three different methods we study the mapping properties of these classes under an integral operator.*
Keywords: Convex functions; starlike functions; convoluion; integral operator.

1. **Introduction.** Let $A(n)$ denote the class of functions $f(z)$ analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ and of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k \, z^k \quad (p \in \mathbb{N}). \tag{1.1}$$

In particular, $A(1) = A$. By $\mathcal{S}^*(n, b)$ and $\mathcal{C}(n, b)$, $(n \in \mathbb{N}$ and $b \in \mathbb{C} \setminus \{0\})$, we mean the suclasses of $A(n)$ which are defined, respectively, by

$$Re\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0, \quad (z \in \mathbb{U}), \tag{1.2}$$

$$Re\left\{1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right\} > 0, \quad (z \in \mathbb{U}). \tag{1.3}$$

We note that for $0 < b \leq 1$, these classes coincide with the well known classes of starlike and convex of order $1 - b$. Also for $b = 1$, $n = 1$, the above two classes defined in (1.2) and (1.3) reduce to the well known classes of starlike $\mathcal{S}^*$ and convex $\mathcal{C}$ respectively, for details of the above two classes see [1, 2].

For functions $f(z)$, $g(z) \in A(n)$ of the form (1.1), We define the convolution (Hadamard product) of $f(z)$ and $g(z)$ by

$$(f \star g)(z) = z + \sum_{k=n+1}^{\infty} a_k \, b_k \, z^k, \quad (z \in \mathbb{U}).$$

Using the concept of convolution many authors generalized Breaz operator in several directions, see [3, 4] for example. Here, we consider a generalized integral operator $I(f_j, g_j)(z)$ as follows:

$$I(f_j, g_j)(z) =_0^z {}_{j=1}^m \left(\frac{(f_j \star g_j)(t)}{t}\right)^{\alpha_j} dt, \tag{1.4}$$

where $f_j, g_j \in A(n)$ with $(f_j \star g_j)(z) \neq 0$ and $\alpha_j > 0$ for all $1 \leq j \leq m$. The operator $I(f_j, g_j)(z)$ reduces to many well-known integral operators by varying the parameters α_j and by choosing suitable functions instead of $g_j(z)$. For example if we take $\frac{z}{1-z}$ and $\frac{z}{(1-z)^2}$ instead of $g_j(z)$ with $m=1$, we obtain the integral operators introduced and studied by Breaz and Breaz [5] and Breaz et al. [6], for details see [7, 8, 9, 10, 11, 12, 13]. Also for $m=1$, $g_1(z)=\frac{z}{1-z}$, $\alpha_1=\alpha\in[0,1]$ in (1.4), we obtain the integral operator studied in [14] given as

$$ {}_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt, $$

and for $m=1$, $g_1(z)=\frac{z}{(1-z)^2}$, $\alpha_1=\delta\in\mathbb{C}$, $|\delta|\le\frac{1}{4}$ in (1.4), we obtain the integral operator

$$ {}_0^z \left(f'(t)\right)^{\delta} dt, $$

discussed in [15, 16].

We will assume throughout our discussion, unless otherwise stated, that $n\in\mathbb{N}$, $b\in\mathbb{C}\setminus\{0\}$, $\alpha_j>0$ such that

$$ {}_{j=1}^{m}\alpha_j<1, \tag{1.5} $$

for all $1\le j\le m$.

In this article, we investigate some mapping propeties of the integral operator $I(f_j,g_j)(z)$ for the class $\mathcal{C}(n,b)$.

2. Preliminary Results

To obtain our main results, we need the following Lemma's.

Lemma 2.1 [17]**.** If $q(z)\in A(n)$ with $n\ge1$ and satisfies the condition

$$ |q'(z)-1|<\frac{n+1}{\sqrt{(n+1)^2+1}} \quad (z\in\mathbb{U}), $$

then

$$ q(z)\in\mathcal{S}^*. $$

Lemma 2.2 [18]**.** If $q(z)\in A(n)$ satisfies the condition

$$ |\arg q'(z)|<\frac{\pi}{2}\,\delta_n \quad (z\in\mathbb{U}), $$

where δ_n is the unique root of the equation

$$ 2\tan^{-1}[n(1-\delta_n)]+\pi(1-2\delta_n)=0, \tag{2.1} $$

then

$$ q(z)\in\mathcal{S}^*. $$

Lemma 2.3 [19]**.** Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that Ψ is a mapping from $\mathbb{C}^2\times\mathbb{U}$ to $\mathbb{C}$ which satisfies $\Psi(ix,y,z)\notin\Omega$ for $z\in\mathbb{U}$, and for all real x,y such that $y\le\frac{-n}{2}\left(1+x^2\right)$. If $q(z)=1+c_nz^n+\ldots$ is analytic in $\mathbb{U}$ and $\Psi(q(z),zq'(z),z)\in\Omega$ for all $z\in\mathbb{U}$, then $Req(z)>0$.

3. Main Results

Theorem 3.1. If $f_j(z)\in A(n)$ satisfies

$$ \left|\left(\frac{f_j(z)*g_j(z)}{z}\right)^{\frac{1}{b}}\left\{\frac{z\left(f_j(z)*g_j(z)\right)'}{f_j(z)*g_j(z)}+b-1\right\}-b\right| < \frac{n+1}{\sqrt{(n+1)^2+1}}|b| \quad (z\in\mathbb{U}), \tag{3.1} $$

then the integral operator $I(f_j,g_j)(z)\in\mathcal{C}(n,b)$.

Proof. Let us set a function $p(z)$ by

$$ p(z)=z\left(\frac{f_j(z)*g_j(z)}{z}\right)^{\frac{1}{b}}=z+\frac{a_n\,b_n}{b}z^n+\ldots \tag{3.3} $$

for $f_j(z)\in A(n)$. Then clearly (3.3) shows that $p(z)\in A(n)$.

Differentiating (3.3) logarithmically, we have

$$\frac{p'(z)}{p(z)} = \frac{1}{b}\left[\frac{f_j(z) * g_j(z)'}{f_j(z) * g_j(z)} - \frac{1}{z}\right] + \frac{1}{z} \tag{3.4}$$

which gives

$$|p'(z) - 1|$$
$$= \left|\left(\frac{f_j(z) * g_j(z)}{z^p}\right)^{\frac{1}{b}} \frac{1}{b}\left\{\frac{z\left(f_j(z) * g_j(z)\right)'}{f_j(z) * g_j(z)} + b - 1\right\} - 1\right|.$$

Thus using (3.1), we have

$$|p'(z) - 1| \leq \frac{n+1}{\sqrt{(n+1)^2 + 1}}, \quad (z \in \mathbb{U}).$$

Hence, using Lemma 2.1, we have $p(z) \in \mathcal{S}^*$.

From (3.4), we can write

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\left[\frac{z\left(f_j(z) * g_j(z)\right)'}{f_j(z) * g_j(z)} - 1\right] + 1.$$

Since $p(z) \in \mathcal{S}^*$, it implies that $Re\frac{zp'(z)}{p(z)} > 0$. Therefore, we get

$$Re\left\{1 + \frac{1}{b}\left(\frac{z\left(f_j(z) * g_j(z)\right)'}{f_j(z) * g_j(z)} - 1\right)\right\} = Re\frac{zp'(z)}{p(z)} > 0,$$

and this implies that

$$Re\left\{1 + \frac{1}{b}\left(\frac{z\left(f_j(z) * g_j(z)\right)'}{f_j(z) * g_j(z)} - 1\right)\right\} > 0. \tag{3.5}$$

From (1.4) we can write

$$I'\left(f_j, g_j\right) =_{j=1}^{\infty} \left(\frac{f_j(z) * g_j(z)}{z}\right)^{\alpha_j},$$

Differentiating logarithmically and then simple computation gives

$$1 + \frac{zI''\left(f_j, g_j\right)}{I'\left(f_j, g_j\right)} =_{j=1}^{m} \alpha_j \left(\frac{z\left(f_j(z) * g_j(z)\right)'}{f_j(z) * g_j(z)} - 1\right) + 1.$$

Equivalently, we have

$$\left\{1 + \frac{1}{b}\left(\frac{zI''\left(f_j, g_j\right)}{I'\left(f_j, g_j\right)}\right)\right\} =_{j=1}^{m} \alpha_j \left\{\frac{1}{b}\left(\frac{z\left(f_j(z) * g_j(z)\right)'}{f_j(z) * g_j(z)} - 1\right)\right\} + 1.$$

Taking real part and then using (3.5) and (1.5), we obtain

$$Re\left\{1 + \frac{1}{b}\left(\frac{zI''\left(f_j, g_j\right)}{I'\left(f_j, g_j\right)}\right)\right\} > 0,$$

and hence $I\left(f_j, g_j\right)(z) \in \mathcal{C}(n, b)$.

Setting $n = 1$ and $g_j(z) = \frac{z}{1-z}$ in Theorem 3.1, we get

Corollary 3.2. If $f(z) \in A$ satisfies

$$\left|\left(\frac{f_j(z)}{z}\right)^{\frac{1}{b}}\left\{\frac{zf_j'(z)}{f_j(z)} + b - 1\right\} - b\right| < \frac{2|b|}{\sqrt{5}} \quad (z \in \mathbb{U}),$$

then $I\left(f_j\right) \in \mathcal{C}(b)$, the class of convex functions of complex order b.

Putting $n = 1$ and $g(z) = \frac{z}{(1-z)^2}$ in Theorem 3.1, we have

Corollary 3.3. If $f(z) \in A$ satisfies

$$\left|\left(f_j'(z)\right)^{\frac{1-b}{b}}\left\{zf_j''(z) + bf_j'(z)\right\} - b\right| < \frac{2|b|}{\sqrt{5}} \quad (z \in \mathbb{U}),$$

then $I(f_j) \in \mathcal{C}(b)$, the class of convex functions of complex order b.

Theorem 3.4. If $f_j(z) \in A(n)$ satisfies

$$\left|\arg\left(\frac{f_j(z) * g_j(z)}{z}\right)^{\frac{1}{b}} + \arg\left\{\frac{1}{b}\left(\frac{z\left(f_j(z) * g_j(z)\right)'}{f_j(z) * g_j(z)} + b - 1\right)\right\}\right| < \frac{\pi}{2}\,\delta_n \quad (z \in \mathbb{U}), \tag{3.6}$$

where δ_n is the unique root of (2.1), then $I(f_j, g_j)(z) \in \mathcal{C}(n, b)$.

Proof. Let $p(z)$ be given by (3.3), which clearly belongs to the class $A(n)$.

Now differentiating (3.3), we have

$$p'(z) = \left(\frac{f_j(z) * g_j(z)}{z}\right)^{\frac{1}{b}} \frac{1}{b}\left\{\frac{z\left(f_j(z) * g_j(z)\right)'}{f_j(z) * g_j(z)} + b - 1\right\} \tag{3.7}$$

which gives

$$|\arg p'(z)| = \left|\arg\left(\frac{f_j(z) * g_j(z)}{z}\right)^{\frac{1}{b}} + \arg\left\{\frac{1}{b}\left(\frac{z\left(f_j(z) * g_j(z)\right)'}{f_j(z) * g_j(z)} + b - 1\right)\right\}\right|.$$

Thus using (3.6), we have

$$|\arg p'(z)| \le \frac{\pi}{2}\,\delta_n \ \ (z \in \mathbb{U}),$$

where δ_n is the root of (2.1). Hence, using Lemma 2.2, we have $p(z) \in \mathcal{S}^*$.

From (3.7), we can write

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\left[\frac{z\left(f_j(z) * g_j(z)\right)'}{f_j(z) * g_j(z)} - 1\right] + 1.$$

Since $p(z) \in \mathcal{S}^*$, it implies that $Re\frac{zp'(z)}{p(z)} > 0$. Therefore, we get (3.5).

From (1.4) we can write

$$I'(f_j, g_j) =_{j=1}^{\infty} \left(\frac{f_j(z) * g_j(z)}{z}\right)^{\alpha_j},$$

Differentiating logarithmically and then simple computation gives

$$1 + \frac{zI''(f_j, g_j)}{I'(f_j, g_j)} =_{j=1}^{m} \alpha_j\left(\frac{z\left(f_j(z) * g_j(z)\right)'}{f_j(z) * g_j(z)} - 1\right) + 1.$$

Equivalently, we have

$$\left\{1 + \frac{1}{b}\left(\frac{zI''(f_j, g_j)}{I'(f_j, g_j)}\right)\right\} =_{j=1}^{m} \alpha_j\left\{\frac{1}{b}\left(\frac{z\left(f_j(z) * g_j(z)\right)'}{f_j(z) * g_j(z)} - 1\right)\right\} + 1.$$

Taking real part and then using (3.5) and (1.5), we obtain

$$Re\left\{1 + \frac{1}{b}\left(\frac{zI''(f_j, g_j)}{I'(f_j, g_j)}\right)\right\} > 0,$$

and hence $I(f_j, g_j)(z) \in \mathcal{C}(n, b)$.

Making $n = 1$, $b = 1 - \alpha$ with $0 \le \alpha < 1$ and $g(z) = \frac{z}{1-z}$, we have

Corollary 3.5. If $f(z) \in A$ satisfies

$$\left|\arg\left(\frac{f_j(z)}{z}\right) + (1-\alpha)\arg\left\{\frac{zf_j'(z)}{f_j(z)} - \alpha\right\}\right| < \frac{\pi}{2}\,\delta_1(1-\alpha) \quad (z \in \mathbb{U}),$$

where δ_1 is the unique root of (2.1) with $n = 1$, then $I(f_j(z)) \in \mathcal{C}(\alpha)$, the class of convex functions of order α.

Also if we take $n = 1$, $b = 1 - \alpha$ with $0 \le \alpha < 1$ and $g(z) = \frac{z}{(1-z)^2}$ in Theorem 3.4, we obtain the following result.

Corollary 3.6. If $f(z) \in A$ satisfies

$$\left| \arg f_j'(z) + (1-\alpha) \arg \left\{ \frac{z f_j''(z)}{f_j'(z)} + 1 - \alpha \right\} \right| < \frac{\pi}{2}\ \delta_1 (1-\alpha) \quad (z \in \mathbb{U}),$$

where δ_1 is the unique root of (2.1) with $n = 1$, then $I(f_j)(z) \in \mathcal{C}(\alpha)$, the class of convex functions of order α.

Theorem 3.7. If $f(z) \in A(n)$ satisfies

$$Re \left[\frac{1}{b} \left\{ \frac{z (f_j(z) * g_j(z))'}{f_j(z) * g_j(z)} \left(\rho \frac{z (f_j(z) * g_j(z))''}{(f_j(z) * g_j(z))'} + 1 \right) \right\} + b - 1 \right] > \frac{M^2}{4L} + N,$$

where $0 \leq \alpha \leq 1$ and

$$\begin{aligned} L &= \rho (Reb + \frac{n}{2}) \\ M &= 2\rho Imb3.7 \\ N &= \rho \left(\frac{\left((Reb)^2 - (Imb)^2 - Reb\right)(Reb) + (Imb)^2 (2Reb - 1)}{(Reb)^2 + (Imb)^2} - \frac{n}{2} \right), \end{aligned} \tag{1}$$

then $I(f_j, g_j)(z) \in \mathcal{C}(n, b)$.

Proof. Let us set

$$\frac{z (f_j(z) * g_j(z))'}{f_j(z) * g_j(z)} = bp(z) - b + 1. \tag{3.8}$$

Then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$.

Taking logarithmic differentiation of (3.8) and then by simple computation, we obtain

$$\frac{1}{b} \left\{ \frac{z (f_j(z) * g_j(z))'}{f_j(z) * g_j(z)} \left(\rho \frac{z (f_j(z) * g_j(z))''}{(f_j(z) * g_j(z))'} + 1 \right) + b - 1 \right\}$$

$$= Azp'(z) + Bp^2(z) + Cp(z) + D = \Psi(p(z), zp'(z), z)$$

with

$$A = \rho,\ B = \rho b,\ C = -2\rho b + \rho + 1,\ D = \rho (b-1).$$

Now for all real x and y satisfying $y \leq -\frac{n}{2}(1 + x^2)$, we have

$$\Psi(ix, y, z) = Ay - Bx^2 + C(ix) + D.$$

Reputing the values of A, B, C, D and then taking real part, we obtain

$$\begin{aligned} Re\Psi(ix, y, z) &\leq -Lx^2 + Mx + N \\ &= -\left(\sqrt{L}x - \frac{M}{2\sqrt{L}} \right)^2 + \frac{M^2}{4L} + N \\ &< \frac{M^2}{4L} + N, \end{aligned}$$

where L, M, N are given in (1).

Let $\Omega = \left\{ w : Rew > \frac{M^2}{4L} + N \right\}$. Then $\Psi(h(z), zh'(z), z) \in \Omega$ and $\Psi(ix, y, z) \notin \Omega$, for all real x and y satisfying $y \leq -\frac{n}{2}(1 + x^2)$, $z \in \mathbb{U}$. Using Lemma 2.3, we have $Rep(z) > 0$. This implies that

$$Re \left\{ 1 + \frac{1}{b} \left(\frac{z (f_j(z) * g_j(z))'}{f_j(z) * g_j(z)} - 1 \right) \right\} > 0,$$

and hence by using the same procedure as in the above Theorems we obtain that $I(f_j, g_j)(z) \in \mathcal{C}(n, b)$.

Acknowledgment: The work here is fully supported by LRGS/TD/2011/UKM/ICT/03/02.

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