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An Extended TOPSIS Method for the Multiple Attribute Decision Making Problems Based on double hierarchy linguistic soft sets (DHLSSs)

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Abstract In this paper, classical soft set theory is extended to develop Double Hierarchy Linguistic Soft Sets (DHLSSs) for addressing multi-attribute decision-making problems. DHLSSs provide an effective framework for handling qualitative information expressed through double hierarchy linguistic terms. Since the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) is a widely accepted method for multi-criteria decision analysis, this study integrates TOPSIS with DHLSSs to propose an extended decision-making approach. The proposed method is designed to handle situations in which attribute weights and attribute values are unknown and represented in the form of double hierarchy linguistic term elements. Background concepts related to DHLTSs, Hamacher t-norms and t-conorms, as well as fuzzy sets, rough sets, soft sets, and fuzzy soft sets are briefly reviewed. Attribute weights are determined using an entropy-based method, and an improved TOPSIS algorithm is employed to rank alternatives. An illustrative example is presented to demonstrate the feasibility and effectiveness of the proposed approach.

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1 Introduction

We seldom utilize mathematical methods to resolve societal problems because mathematics relies on precise, objective concepts. Numerous number theories, such as fuzzy set theory (FST) [12, 30], rough set theory (RST) [16, 29], and soft set theory (SST) [28], were created to address this lack of ambiguity.

FST provides a handy framework for expressing and managing fuzzy conceptions, and Zadeh initially developed it in 1965 [12, 31]. The fuzzy membership function (FMF) is the foundation of FST. The FMF, among other things, decides whether an element is a part of a set. Since its inception, mathematicians and computer scientists have both studied this subject.

For the purpose of researching intelligent systems with limited as well as erroneous data the RST is a set theory (ST) derivation. 1982 saw Pawlak release his first book [2, 12]. The traditional RST, which has been extended to include covering-based RSs, is built on equivalence relations [44, 45].

Molodtsov first presented an innovative mathematical method for addressing ambiguity in 1999 [28] and came up with the concept of the SSs. Because it does not impose any restrictions on approximate detail, SST is very useful and easily applied in practice. Maji et al. [26] implemented the Olodtsov approach by incorporating different types of SST methods. Ali et al. [1] many operations were performed on the SSs. Subsequently, Ge and Yang [11] examined these operational principles in more detail in [1, 26] and made several notable discoveries, including some additional approaches[17].

Ge et al. [10] assert that there are specific connections between topology and SST. Aktas and Cagman [21] investigated and contrasted the differences between SSs and the related concepts of FSs and RSs in order to better comprehend them.

Atanassov [3, 4, 16] developed Intuitionistic Fuzzy Sets (IFSs) by incorporating the degree of non-membership into FSs. People actually prefer to characterize their evaluation data with linguistic term sets (LTSs), such as "very terrible," "a little bad," "excellent," and so on, even if the FS and IFS tools have gained popularity. As a result, LTSs are capable of handling difficult circumstances. Since the computing with words (CW) paradigm was introduced by Zadeh [43] and its significance was made clear, different extension forms of LTSs [37, 38] have been suggested and examined.

DHL term sets (DHLTSs) were created by Ge et al. [14, 15] in order to faithfully reflect the expert speakers' utterances. When making investment decisions, almost every one of decision makers (DMs) find it difficult to appropriately analyses project features. However, for the vast majority of DMs, projecting traits through qualitative analysis is more feasible and suitable. Due to their diverse language expressions, DHLTSs are able to represent qualitative information more flexibly than the single linguistic term set [33]. It takes less time to get feedback since DMs are better able to intuitively provide an evaluation value when they evaluate the project's attribute data.

The DHLTS incorporates the 1st and 2nd hierarchy linguistic word sets, which conveys ambiguity and vagueness more flexibly. The essential operational principles are used to establish the Hamacher operations of DHLEs. The Hamacher t-norm and t-conorm (HTN, HTCN) are based on algebraic and Einstein t-norms and t-conorms, respectively [34, 35]. They form a broad and flexible family of continuous trigonometric rules. In a study on IFS, Xia et al.created Hamacher activities and examined the various traits. Language was assessed using a single-value neutrosophic 2 tuple by Wu et al. [36]. A few inventive Hamacher aggregation operators were provided.

Soft set models can be used in conjunction with other mathematical models. The notion behind FSSs, which was first proposed by Maji et al. [24], was constructed by combining the concepts of FSs and SSs. The concept of IFSs, which was predicated on a synthesis of the IFS [5, 6] and SSs models, was also developed by Maji et al. [19, 25]. Yang et al. [39] merged the interval-valued FS [11, 13] with the SS to introduce the Interval Value Concept of FSS. Jiang et al. [20] were the ones who initially proposed the concept of interval valued IFSS. As an IF extension of interval-valued FSST or IFSST as an interval-valued fuzzy extension of Feng et al. [8, 9] are the authors who initially developed the concepts of RSSs, SRSs, and

SRFSs.

One of the most well-liked techniques for making decisions is TOPSIS, which was put out by Hwang and Yoon [18]. Thanks to improvements and modifications made by many researchers, this strategy has been effectively applied in practice over the past 20 years to deal with a range of problems for decision makers in many industries. Chen [7] extended TOPSIS using linguistic variables for group decision-making situations to represent different criteria and solution evaluation weights with respect to these weights. Liu [23] suggested an updated TOPSIS approach to address multi-attribute decision-making scenarios where attribute weights and attribute values are all interval fuzzy values. Liu [41] proposed an improved TOPSIS technique for multi-attribute group decision making based on an interval-valued trapezoidal fuzzy number. The fuzzy group TOPSIS approach was used by Mohammadi et al. [27] to choose the best security measures for e-business operations. For the purpose of solving the facility detection problem, Verma et al. [32] proposed an interval-wise IF TOPSIS approach.

The remainder of this paper is organized as follows: In Section 2, we first consider some background information on DHLTS HTN and HTCN [22]. In Section 3, we discussed the Hamacher Operational Rules of DHLES. We briefly discuss the background and concepts of the FS, RS, SS, FSS, and soft covering based sets [42] in a section 4. In section 5, DHLSSs were defined and give some related example, discussing DHLSSs in detail. In section 6 In order to rank the alternatives, we extend the TOPSIS method and provide An entropy-based technique for calculating attribute weights. We also offer thorough decision-making processes. In section 7 to illustrate how to implement the recommended procedure and contrast it with the existing approach, we offer a case study. Ultimately, Section 8 brings us to a close.

2 Preliminaries

This section briefly discusses the HTN and HTCN as well as several of important DHLTS concepts.

2.1 Double hierarchy linguistic term sets (DHLTSs)

The DHLTS was created by Gou et al. [14] to improve the precision of linguistic term expression.

Definition 2.10 Gou et al (see [14]). Due to the 1st and 2nd hierarchical LTSs, respectively $S = \{S_m \mid m = -\Upsilon_1 \ldots - 1, -2, 0, 1, 2 \ldots \Upsilon_1\}$ and $O = \{O_n \mid n == -\Upsilon_2 \ldots - 1, -2, 0, 1, 2 \ldots \Upsilon_2\}$ therefore, the following is illustrated as the mathematical description of DHLTS:

$$\bar{S}_{(o)} = \{S_m \prec O_n \succ \mid m, n = -\Upsilon \ldots - 1, -2, 0, 1, 2 \ldots \Upsilon\} \tag{1}$$

When S_m the 1st HLT is, O_n are the 2nd SHLT, and the two together represent the DHLT, they indicate the different linguistic term levels S_m.

It is interesting that the values of m must be used to decide the order of the SHLT.

Remarks 2.10 Gou et al. (See 1[11]) four distinct situation categories are displayed according to various values of m:

1 when $m > 0$, indicating that the 1st HLT is positive, the 2nd HLT should be displayed in ascending order.

2 The SHLT must be presented in descending inverse order when is less than zero, as this results in a negative FHLT.

3 When $m = \Upsilon$ we just examine the front part of the SHLT. item[4] The second half of the SHLT is taken into account, where $m = -\Upsilon$.

Via Gou et al. [14]. Two transformed functions (TF) between numerical scale (NS) and subscript (S") were devised to deal with DHLT more efficiently.

Definition 2.20 Gou et al (See [14]) Two TF, f and f^{-1}, between the NS and the S' (m, n) of the DHLT $S_m \prec O_n \succ$ are shown as follows, assuming that $S = \{S_m \prec O_n \succ | \; m, n = -\Upsilon \ldots -1, -2, 0, 1, 2 \ldots \Upsilon\}$ is continuous DHLTS:

$$f : [-\Upsilon_1, \Upsilon_1] \times [-\Upsilon_2, \Upsilon_2] \rightarrow, \quad f(m, n) = \frac{n + (\Upsilon_1 - m)\Upsilon_2}{2\Upsilon_2\Upsilon_2} = \delta \tag{2}$$

$$\begin{aligned} f^{-1} &: [0, 1] \rightarrow [-\Upsilon_1, \Upsilon_1] \times [-\Upsilon_2, \Upsilon_2], \\ f^{-1}(\delta) &= [2\Upsilon_1\delta - \Upsilon_1] \prec O_{\Upsilon_2} \prec O_{(2\Upsilon_1\delta - \Upsilon_1 - [2\Upsilon_1\delta - \Upsilon_1])} \succ \\ &= [2\Upsilon_1\delta - \Upsilon_1] + 1 \prec O_{(\Upsilon_2(2\Upsilon_1\delta - \Upsilon_1 - [2\Upsilon_1\delta - \Upsilon_1]))} \end{aligned} \tag{3}$$

In this case, the integer part of the number is represented as $[2\Upsilon_1\delta - \Upsilon_1]$. The transformation functions F and F^{-1} between the DHLT $S_m \prec O_n \succ$ and the numerical scale δ may be built using Definition 2.20 as a starting point.

$$F : \bar{S}_{(o)} \rightarrow [0, 1], \quad F(S_m \prec O_n \succ) = f(m, n) = \delta \tag{4}$$

$$F^{-1} : [0, 1] \rightarrow \bar{S}_{(o)}, \quad F^{-1}(\delta) = S_m \prec O_n \succ \tag{5}$$

3 the Hamacher Operational Rules of DHLES

Let $u = S_{m_1} \prec O_{n_1} \succ$ and $v = S_{m_2} \prec O_{n_2} \succ$ be two DHLEs and $\varsigma \geq 0, 0 \leq \Theta \leq 1$ the follow is a definition of the Hamacher operations of DHLEs:

$$\begin{aligned} u \oplus_h v &= S_{m_1} \prec O_{n_1} \succ \oplus_h S_{m_2} \prec O_{n_2} \succ \\ &= F^{-1}\left(\frac{F(S_{m_1} \prec O_{n_1} \succ) + F(S_{m_2} \prec O_{n_2} \succ) - F(S_{m_1} \prec O_{n_1} \succ)F(S_{m_2} \prec O_{n_2} \succ) - (1-\varsigma)F(S_{m_1} \prec O_{m_1} \succ)F(S_{m_2} \prec O_{n_2} \succ)}{1 - (1-\varsigma)F(S_{m_1} \prec O_{n_1} \succ)F(S_{m_2} \prec O_{n_2} \succ)}\right) \end{aligned} \tag{6}$$

$$\begin{aligned} u \otimes_h v &= (S_{m_1} \prec O_{n_1} \succ) \otimes_h (S_{m_2} \prec O_{n_2} \succ) \\ &= F^{-1}\left(\frac{F(S_{m_1} \prec O_{n_1} \succ)F(S_{m_2} \prec O_{n_2} \succ)}{\varsigma + (1-\varsigma)F(S_{m_1} \prec O_{n_1} \succ) + F(S_{m_2} \prec O_{n_2} \succ) - F(S_{m_1} \prec O_{n_1} \succ)F(S_{m_2} \prec O_{n_2} \succ)}\right) \end{aligned} \tag{7}$$

$$\begin{aligned} \Theta \otimes_h u &= \Theta \odot_h (S_{m_1} \prec O_{n_1} \succ) \\ &= F^{-1}\left(\frac{(1 + (\varsigma - 1)F(S_{m_1} \prec O_{n_1} \succ))^{\Theta} - (1 - F(S_{m_1} \prec O_{n_1} \succ))^{\Theta}}{(1 + (\varsigma - 1)F(S_{m_1} \prec O_{n_1} \succ))^{\Theta} + (\varsigma - 1)(1 - F(S_{m_1} \prec O_{n_1} \succ))^{\Theta}}\right) \end{aligned} \tag{8}$$

$$(u)^{\Theta} = F^{-1}\left(\frac{\varsigma\left(F(S_{m_1} \prec O_{n_1} \succ)\right)^{\Theta}}{\left(1+(1-\varsigma)F(S_{m_1} \prec O_{n_1} \succ)\right)^{\Theta} + \left((\varsigma-1)(1-F(S_{m_1} \prec O_{n_1} \succ))\right)^{\Theta}}\right) \tag{9}$$

Example 3.20 let $S_O(\Upsilon = 4)$ be a DHLTS $u = (S_{m_1} \prec O_{n_1} \succ \mid m_1 = 2\ n_1 = 3)$ and $v = (S_{m_2} \prec O_{n_2} \succ \mid m_2 = 1\ n_2 = 2)$ be two DHLEs $\varsigma = 2, \Theta = 0.6$ considering the aforementioned operational principle,

(1) $u \oplus_h v = (S_m \prec O_n \succ \mid m = 3\ n = 3.01)$
(2) $u \otimes_h v = (S_m \prec O_n \succ \mid m = 0\ n = -1.3123)$
(3) $\Theta \otimes_h u_1 = (S_m \prec O_n \succ \mid m = 1\ n = 0.1408)$

$$(u_1)^{\Theta} = (S_m \prec O_n \succ \mid m = 2\ n = 0.72)$$

Theorem 3.30 Given that $\hat{u} = (S_m \prec O_n \succ \mid m, n \in [-\Upsilon, \Upsilon]) \in \bar{S}_{(o)}$ is the set of all DHLEs generated on the basis of $\bar{S}_{(o)}$, $u_1, u_2 \in \hat{u}$ and $\Theta \in [0,1]$ then operational rule on DHLEs specified by the HTN and HTCN are closed, that is,

(1) $u \oplus_h v \in \hat{u}$
(2) $u \otimes_h v \in \hat{u}$
(3) $\Theta \otimes_h u \in \hat{u}$
(4) $[\![(u)]\!]^{\flat^{\flat}} \in \hat{u}$

Definition 3.40 Given that $f_i = S_{m_i} \prec O_{n_i} \succ (i = 1, 2 \ldots n)$ are n DHLEs, $W = (w_1, w_2, \ldots, w_3)$ is the corresponding weighting vector, satisfying that $\sum_{i=1}^{n} w_i = 1$. The DH Hamacher weighted averaging operator is mapping DHLHWA : $\mathcal{H}^n \to \mathcal{H}$

$$\text{DHLHWA}(f_1, f_2, \ldots, f_i) = \bigoplus_{i=1}^{n} (W_i \odot f_i) \tag{10}$$

Where $\mathcal{H}$ stands for the collection of all DHLEs.

The following conclusion may be drawn from the Hamacher operating guidelines for DHLEs:

Definition 3.50 Given that $f_i = S_{m_i} \prec O_{n_i} \succ (i = 1, 2 \ldots n)$ are n DHLEs and then the HWA of n DHLEs According to the relative positive and negative ideal solutions of the DHL term Information System.

$$\text{DHLHWA}(f_1, f_2, \ldots, f_i) = F^{-1}\left(\frac{\prod_{i=1}^{n}\left(1+(\varsigma-1)F(S_{m_i} \prec O_{n_i} \succ)\right)^{w_i} - \prod_{i=1}^{n}\left(1-F(S_{m_i} \prec O_{n_i} \succ)\right)^{w_i}}{\prod_{i=1}^{n}\left(1+(\varsigma-1)F(S_{m_i} \prec O_{n_i} \succ)\right)^{w_i} + (\varsigma-1)\prod_{i=1}^{n}\left(1-F(S_{m_i} \prec O_{n_i} \succ)\right)^{w_i}}\right) \tag{11}$$

4 SOFT SET (SS)

Definition 2.1 (See [12]) Allow U being a universe. FS P in U is an ordered pair set.

$$P = \{(x, \mu_P(X)) : x \in U\} \tag{8}$$

Where $\mu_A(X)$ lists the grades of X belongings in A and $\mu_p(X) :\to [0,1] = \mid$ is a mapping. I^U Stands for the family of all FSs in U.

Definition 2.2. (See [28]). Let $P \subseteq E$ and U be a universal set. A set value mapping is $F : P \to F(U)$, and a pair $G = (F, A)$ is referred to as a SS over U. A SS over U is, in another sense, a family that has been parameterized of subsets of U for $\forall\, e \in A$, where $F(e)$ is the set of e-approximation elements of intersections with the set $G = (F, P)$. It's crucial to keep in mind that $F(e)$ could be unpredictable. While some of these containers could be empty, others might not.

Definition 2.3 [[24] let U be a universal set and $P \subseteq E$. $[\![(f_{(P)}, E)]\!]$ is considered a FSSs if a mapping such that $f_{(P)} : E \to I^U$

$$f_{(P)}(e) = \mu^e_{(f_P)}$$

where $\mu^e_{(f_P)} = \bar{o}$ if $e \in E - P$ and $\mu^e_{(f_P)} \neq \bar{o}$ if $e \in P$ where $\bar{o}(u) = 0$ for each $u \in U$ In 2010, Feng et al. [41] looked into the idea of SRSs, which combines SSs and RSs. A departure from PAS [10] resulted in SAS, which was chosen to use SSs against equivalence relation to granulize the universe of speech. We suggest readers who are interested in learning more about this topic to [27, 41]. Every proof is there for you to see.

5 Double Hierarchy Linguistic Soft and Soft Covering Based Sets (DHLSSs)

In order to demonstrate its fundamental qualities, this section develops a DHLSS using a SS with a DHLTS.

Definition 5.1 Let $P \subseteq E$. An a DHLSS $\breve{L}_p$ over U is a set defined by function $\breve{L}_p$ represent a mapping

$$\breve{L}_p : E \to \bar{S}_o \quad \text{such that} \quad \breve{L}_p(x) = s_m\langle o_n\rangle \text{ if } x \in p^c, \quad \breve{L}_p(x) = \emptyset.$$

$\breve{L}_A$ over U shown by the set of ordered pairs.

$$\breve{L}_p = \left\{ \left(x, \breve{L}_p(x)\right) : \breve{L}_p(x) \in F(U) \right\}.$$

EXAMPLE 5.2 Let $U = \{v_1, v_2, v_3, v_4, v_5\}$ be a universal set and $E = \{w_1, w_2, w_3, w_4\}$ be a set of parameters and $\{s_m\langle o_n\rangle \mid m, n \in [-4, 4]\}$. If $P = \{w_1, w_2, w_3\} \subseteq E$,

$$\breve{L}_P(w_1) = \left\{ \frac{v_1}{(S_2\langle O_2\rangle)}, \frac{v_4}{(S_3\langle O_1\rangle)} \right\}, \quad \breve{L}_P(w_2) = \left\{ \frac{v_1}{(S_o\langle O_2\rangle)}, \frac{v_2}{(S_1\langle O_3\rangle)}, \frac{v_3}{S_{3\langle O_0\rangle}} \right\}, \quad \breve{L}_P(w_3) = \left\{ \frac{v_3}{S_{2\langle O_3\rangle}}, \frac{v_4}{S_{1\langle O_3\rangle}} \right\}.$$

Then the DHLSS set can be expressed as

$$\breve{L}_P(w) = \left\{ \left(w_1, \left\{ \frac{v_1}{(S_2\langle O_2\rangle)}, \frac{v_4}{(S_3\langle O_1\rangle)} \right\} \right), \left(w_2, \left\{ \frac{v_1}{(S_o\langle O_2\rangle)}, \frac{v_2}{(S_1\langle O_3\rangle)}, \frac{v_3}{S_{3\langle O_0\rangle}} \right\} \right), \left(w_3, \left\{ \frac{v_3}{S_{2\langle O_3\rangle}}, \frac{v_4}{S_{1\langle O_3\rangle}} \right\} \right) \right\}.$$

6 An extended TOPSIS Method for multiple attribute decision making based on DHLSSs

Assume $K = (K_1, K_2, \ldots K_n)$ is a collection of options in the context of a multiple attributing choice problem. $L = (L_1, L_2, \ldots L_n)$ is a set of attributes, $W = (w_1, w_2, \ldots w_n)$ is a vector of weights for the attributes, and $\sum_{j=1}^{n} W_j = 1, W_j \geq 0$. With W_j being an unknown. Assume that $X = [x_{ij}]_{m\times n}$ represents the DHLSSs for option K_i with regard to attribute L_j in the decision matrix $x_{ij} = \Big([(x_i, \breve{L}_{p_i}(x)) : \breve{L}_{p_i}(x) \in F(U)], [(x_i, \breve{L}_{p_i}(x)) : \breve{L}_{p_i}(x) \in F(U)], [(x_i, \breve{L}_{p_i}(x)) : \breve{L}_{p_i}$

Following are the processes for rating the options based on these criteria.

Step 1: Standardized decision matrix

Decision-making challenges sometimes involve several criteria or attributes, some of which are critical while others are not. As a result, it is vital to select the suitable criteria or qualities for a specific decision-making circumstance. Professional judgment or another technically sound technique can be used to identify the most significant characteristics. In general, characteristics are divided into two categories: benefit type and cost type. The benefit type is named from the notion that the better the alternative, the larger the value of the property. The greater the value of the characteristic, the more costly the option.

Step 2: Determine the weights of experts.

We need to find out the matrix weights because they are absolutely unknown. Following that, we'll calculate W using the entropy technique. The equation to do so is shown below.

$$M_1 = \begin{pmatrix} a_{11} & \dots & a_{1j} \\ \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} \end{pmatrix}, \quad M_2 = \begin{pmatrix} a_{11} & \dots & a_{1j} \\ \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} \end{pmatrix}, \quad \dots, \quad M_n = \begin{pmatrix} a_{11} & \dots & a_{1j} \\ \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} \end{pmatrix}$$

$$S_i = \frac{1}{n}\sum_{i=1}^{n}\left[\left(\sqrt{2}\cos\left(\frac{(F(S_m\langle O_n\rangle) - .F(S_m\langle O_n\rangle)^c)}{4}\right) - 1\right) \times \frac{1}{\sqrt{2}-1}\right] \tag{12}$$

$$S_m^{+} = \sum_{i=1}^{n} S_i \tag{13}$$

$$W_{M_i} = \frac{S_m^{+}}{\sum_1^n S_m^{+}} \tag{14}$$

Step 3: using the double hierarchy Hamacher weighted averaging operator

$$= F^{-1}\left(\frac{\prod_{i=1}^{n}(1+(\varsigma-1)\cdot F(S_{m_i}\langle O_{n_i}\rangle))^{W_{M_i}} - \prod_{i=1}^{n}(1-F(S_{m_i}\langle O_{n_i}\rangle))^{W_{M_i}}}{\prod_{i=1}^{n}(1+(\varsigma-1)\cdot F(S_{m_i}\langle O_{n_i}\rangle))^{W_{M_i}} + (\varsigma-1)\prod_{i=1}^{n}(1-F(S_{m_i}\langle O_{n_i}\rangle))^{W_{M_i}}}\right) \tag{15}$$

Step 4: To rank alternatives, apply the extended TOPSIS method

TOPSIS primary principle is that the optimal option should be the one that is closest to the positive ideal solution while being the furthest away from the negative ideal solution. The processes for using extended TOPSIS to rank choices are as follows.

1: Calculate the weight of criteria using the entropy technique.

$$W_i = \left(\sqrt{2}\cos\left(\frac{(L_{ij} - L_{ij}^{C})}{4}\right) - 1\right) \times \frac{1}{\sqrt{2}-1} \tag{16}$$

$$A = (a_{ij})_{m\times n} = \begin{pmatrix} w_1a_{11} & w_2a_{12} & \dots & w_na_{1n} \\ w_1a_{21} & w_2a_{22} & \dots & w_na_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_1a_{n1} & w_2a_{n2} & \dots & w_na_{nn} \end{pmatrix} \tag{17}$$

2: Find the ideal solution in both the positive (P) and negative (N) ideal:

The following is the derivation of the P and the N:

$$P = K^{+} = \{r_1^{+}, r_2^{+}, \ldots r_n^{+}\}$$

$$P = K^{+} = \{\max(r_{ij}) \mid j = 1, 2, \ldots n\} \tag{18}$$

$$N = K^{-} = \{r_1^{-}, r_2^{-}, \ldots r_n^{-}\}$$

$$N = K^{-} = \{\min(r_{ij}) \mid j = 1, 2, \ldots n\} \tag{19}$$

3: Find the distance between option A_i and the P or N ideal solution.

The separation values between the P and N may be calculated using the n-dimensional Euclidean distance.

$$d_i^{+} = d(r_{ij}, r^{+}) = \sqrt{\sum_{j=1}^{n} (r_{ij} - r^{+})^2} \tag{20}$$

$$d_i^{-} = d(r_{ij}, r^{-}) = \sqrt{\sum_{j=1}^{n} (r_{ij} - r^{-})^2} \tag{21}$$

Determine the coefficient of relative proximity.

$$O_i = \frac{d_i^{-}}{d_i^{+} - d_i^{-}} \tag{22}$$

4: Ranking the alternatives

A higher value of O_i based on the proximity coefficient to the ideal alternative A_i indicates the superior alternative.

7 Extended TOPSIS method for multi-attribute decision making of DHLSSs an application example

We'll use an example regarding choosing a company to invest in to demonstrate how the suggested approach may be used (adapted from [40]). A business wishes to contribute a certain amount of money to a particular sector. There are five alternative options, all of which are listed in Table 1. The options are evaluated using four criteria, which are all included in table 2.

Table 1 A list of alternative

K_1	Car - company
K_2	Food - company
K_3	Computer - company
K_4	Arms - company
K_5	Cloths - company

Table 2 A lists of criteria

L_1	Risk
L_2	Growth
L_3	Environmental impact
L_4	Economic strength

STEP 2: Determine the weights of experts Following that, we'll calculate W_{M_i} using the (14) entropy technique

$M_1 =$

	L_1	L_2	L_3	L_4
K_1	$S2\langle O3\rangle$	$S1\langle O1\rangle$	$S2\langle O1\rangle$	$S3\langle O-2\rangle$
K_2	$S1\langle O2\rangle$	$S2\langle O2\rangle$	$S1\langle O-1\rangle$	$S0\langle O3\rangle$
K_3	$S3\langle O1\rangle$	$S2\langle O3\rangle$	$S3\langle O-2\rangle$	$S1\langle O2\rangle$
K_4	$S-2\langle O-1\rangle$	$S0\langle O2\rangle$	$S1\langle O1\rangle$	$S2\langle O-2\rangle$
K_5	$S2\langle O-1\rangle$	$S1\langle O1\rangle$	$S2\langle O1\rangle$	$S1\langle O3\rangle$

$M_2 =$

	L_1	L_2	L_3	L_4
K_1	$S2\langle O1\rangle$	$S2\langle O1\rangle$	$S1\langle O1\rangle$	$S0\langle O-1\rangle$
K_2	$S2\langle O3\rangle$	$S2\langle O3\rangle$	$S0\langle O3\rangle$	$S1\langle O2\rangle$
K_3	$S1\langle O2\rangle$	$S3\langle O-2\rangle$	$S-1\langle O2\rangle$	$S2\langle O-2\rangle$
K_4	$S0\langle O1\rangle$	$S0\langle O2\rangle$	$S2\langle O2\rangle$	$S2\langle O1\rangle$
K_5	$S3\langle O1\rangle$	$S1\langle O1\rangle$	$S1\langle O3\rangle$	$S1\langle O1\rangle$

$M_3 =$

	L_1	L_2	L_3	L_4
K_1	$S2\langle O1\rangle$	$S2\langle O3\rangle$	$S3\langle O-2\rangle$	$S-1\langle O2\rangle$
K_2	$S2\langle O1\rangle$	$S-2\langle O-1\rangle$	$S1\langle O2\rangle$	$S0\langle O2\rangle$
K_3	$S3\langle O1\rangle$	$S3\langle O-2\rangle$	$S0\langle O3\rangle$	$S1\langle O1\rangle$
K_4	$S0\langle O2\rangle$	$S2\langle O1\rangle$	$S2\langle O1\rangle$	$S1\langle O1\rangle$
K_5	$S3\langle O-2\rangle$	$S2\langle O2\rangle$	$S2\langle O-2\rangle$	$S0\langle O1\rangle$

$M_1 =$

	L_1	L_2	L_3	L_4
K_1	0.84375	0.65625	0.78125	0.8125
K_2	0.6875	0.8125	0.59375	0.59375
K_3	0.90625	0.84375	0.8125	0.6875
K_4	0.21875	0.5625	0.65625	0.6875
K_5	0.71875	0.65625	0.78125	0.71875

$M_2 =$

	L_1	L_2	L_3	L_4
K_1	0.78125	0.78125	0.65625	0.46875
K_2	0.84375	0.84375	0.59375	0.6875
K_3	0.6875	0.8125	0.4375	0.6875
K_4	0.53125	0.5625	0.8125	0.78125
K_5	0.90625	0.65625	0.71875	0.65625

$M_3 =$

	L_1	L_2	L_3	L_4
K_1	0.78125	0.84375	0.8125	0.4375
K_2	0.78125	0.21875	0.6875	0.5625
K_3	0.90625	0.8125	0.59375	0.65625
K_4	0.5625	0.78125	0.78125	0.65625
K_5	0.8125	0.8125	0.6875	0.53125

We found W_{M_i} using (14) $W_{M_1} = 0.328$, $W_{M_2} = 0.342$ and $W_{M_3} = 0.330$

Step: 3 using (15) the DHLHWA

Table 3: Using DHLHWA

	L_1	L_2	L_3	L_4
K_1	0.803756	0.770739	0.755648	0.600204
K_2	0.779271	0.690371	0.626546	0.618101
K_3	0.855477	0.8233	0.636462	0.677425
K_4	0.446012	0.646834	0.757667	0.713313
K_5	0.82881	0.716121	0.731212	0.640909

Step: 4 to rank alternatives, apply the extended TOPSIS method We use an entropy technique (16) and find the weight of Criteria. $(w_1, w_2, w_3, w_4) = (0.211, 0.243, 0.260, 0.286)$

Table 4 show that $(w_1c_1, w_2c_2, w_3c_3, w_4c_4)$

	L_1	L_2	L_3	L_4
K_1	0.169593	0.18729	0.196468	0.171658
K_2	0.164426	0.16776	0.162902	0.176777
K_3	0.180506	0.200062	0.16548	0.193743
K_4	0.094109	0.157181	0.196993	0.204007
K_5	0.174879	0.174018	0.190115	0.1833
K^+	0.855477	0.8233	0.757667	0.713313
K^-	0.446012	0.646834	0.626546	0.600204

In table 5 we find distance between the alternative K_i and the relative closeness coefficient

Table 5: Each alternative's distance and relative proximity coefficient

d_i^+	0.05656	0.235922	0.18011	0.27494	0.190133
d_i^-	0.13916	0.086016	0.153942	0.066441	0.136462
O_i	0.711017	0.267182	0.460831	0.194624	0.417832

The relative proximity coefficient values for each of the four options, as indicated in Table 5, determine the ranking order of the four alternatives.

$K_1 > K_3 > K_5 > K_2 > K_4$ The best choice is K_1, while the second best option is K_3.

$POS = \{K_1, K_3\}$, $BND = \{K_5\}$ and $NEG = \{K_2, K_4\}$

Compare with the existing method In this research, a technique is provided, and to further demonstrate its efficacy, we compare it with a method proposed by Deng Julong [41]. Use the GRA method to compute the conditional probability. The P and the N are the same as those of our technique, and the weights of the characteristics are set to

Table 6: the result of conditional probability

	K_1	K_2	K_3	K_4	K_5
G^+	0.803488	0.65481	0.860658	0.746842	0.787712
G^+	0.673634	0.803691	0.656502	0.796715	0.665695
H_i	0.543955	0.448961	0.567282	0.483845	0.541976
$p(A/X_i)$	0.543955	0.448961	0.567282	0.483845	0.541976

$W = (0.211, 0.243, 0.260, 0.286)^T$ to make comparisons easier. Next, Table 6 may be used to determine the conditional probability. The decision result may then be obtained:

$POS = \{K_1, K_3\}$, $BND = \{K_5\}$ and $NEG = \{K_2, K_4\}$.

Table 6 demonstrates that the GRA technique gets almost the same decision results as our suggested strategy. As a result, the effectiveness of our proposed approach may be demonstrated.

8 Conclusion

In this paper, current soft sets are expanded to create DHLSSs. The DHLSS is used to solve a decision-making problem, and TOPSIS is a common and successful strategy for making judgments based on several criteria. Then, the TOPSIS approach is extended to the double hierarchy linguistic soft set, and the attribute weights are determined using the entropy method. In the end, each component of the proposed approach has been demonstrated using an actual example. It demonstrates how simple this system is to use, how it constantly grows and improves multi-attribute decision-making concepts, and it proposes a new strategy for dealing with MADM challenges. We will continue to work on the design and execution of the recommended technique in the future.

Conflict of interest

The authors declares no conflict of interest.

Data availability

All the data are available within the manuscript.

Author Contributions

Muhammad Javed Khan: Writing main manuscript, Supervision,methodology. **Hamza Zafar**: Writting and Review the main manuscript, data curation, analysis.

Compliance with Ethical Standards

It is declared that all authors don't have any conflict of interest. Furthermore, informed consent was obtained from all individual participants included in this study.

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