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Improved exponential ratio-cum-regression type estimators under stratified random sampling for population mean

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Abstract This research introduces a distinctive class of exponential ratio-cum-regression estimators under the technique of stratified random sampling (STRS), designed for the efficient assessment of the population mean by incorporating dual concomitant variables (CV). Six refined estimators are introduced, by deriving their mean square error (MSE) through the approximation of first-order. These derivations are carried out through Taylor expansion technique. Their performance is systematically assessed against existing alternatives based on the MSE criterion. Theoretical developments supported by empirical evidence, and reveal that the proposed estimators consistently offer reduced MSE and enhanced percentage relative efficiency (PRE) over conventional methods.

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Introduction

In survey sampling, concomitant variables (CV) indicate supplementary data that can substantially improve the precision and efficacy of estimators when inferring unknown population parameters. This concept was first introduced by several researchers, Cochran [2], who advocated for incorporating auxiliary data through ratio estimators. Estimating population metrics using CV is a crucial element of sampling techniques in statistical surveys. The application of a single auxiliary variable can substantially enhance the efficiency of estimators by increasing precision and reducing bias. For this purpose, estimation techniques including product, regression and ratio estimators are employed to exploit the relationship between the auxiliary and study variables to generate more accurate and dependable results. However, employing two auxiliary variables instead of one can further strengthen the reliability and accuracy of population estimates. Although a single auxiliary variable offers useful insights into population characteristics, the integration of multiple auxiliary variables provides a more comprehensive and effective approach. In this context, Muneer et al. proposed an efficient technique for calculating the population mean using auxiliary variables [16].

Stratification is a vital technique in modern survey methodologies, employed to reduce population heterogeneity and boost the reliability of estimates. In this method, diverse population is segmented into separate strata, with sampling executed independently within each stratum, such that each stratum is internally homogeneous and externally heterogeneous. Samples are then systematically and independently drawn from these strata using appropriate allocation procedures i.e equal, proportional or optimum. Stratified sampling offers notable advantages, including administrative convenience and reduced survey costs [4].

Substantial research has highlighted the pivotal role of stratification and CV in enhancing the efficiency of estimators in survey sampling. For this purpose, numerous authors have proposed a range of estimators within the context of stratified random sampling (STRS). A significant contribution by Zaman and Kadilar [31] revealed that stratification markedly improves the estimators efficiency when the variability between strata surpasses that within strata. Expanding on this concept, Zaman and Kadilar [30], as well as Zaman [27], laid the groundwork for more refined methods of estimating the population mean by leveraging auxiliary variables. Building upon these advancements, Zaman [28] proposed an exponential-type estimator that contributed to a notable improvement in the accuracy of population mean estimation. Rather and Kadilar [17] extended this line of research by developing a dual form of the ratio-cum-product exponential estimator, contributing to greater adaptability and efficiency. Mradula et al. [15] designed an efficient estimator that incorporated a linear cost function. In a simulation-based investigation, Javed et al. [7] examined progressive strategies for population mean estimation, applying both classical and innovative approaches. Continuing in this direction, Javed and Irfan [6] proposed optimal estimators utilizing dual auxiliary variables and simulation methodologies to enhance estimation performance. Yadav and Tailor [24] explored the assessment of finite population means through the application of two CV, revealing substantial improvements in estimation reliability. Zaman and Bulut [29] introduced a revised regression estimator that incorporates robust regression strategies and employs covariance matrices to strengthen estimation reliability. Similarly, Kumar and Vishwakarma [14] suggested a comprehensive class of regression-cum-ratio estimators aimed at enhancing the accuracy of population mean estimation. Addressing sensitive variables, Zahid et al. [26] developed a generalized class of estimators that addresses non-response and measurement errors. Additional key contributions to population mean estimation within STRS have been provided by Aladag and Cingi [1], Grover and Kaur [3], Shabbir and Gupta [19], Khalid [20, 21], Kadilar and Cingi [8, 9], Koyuncu and Kadilar [11, 12], and Singh and Vishwakarma [22], each offering methodological enhancements and analytical insights. Hussain et al. [5] introduced methods for estimating the distribution function of a finite population using dual auxiliary information within both simple random sampling and STRS frameworks. Zaagan et al. [25] suggested a unique estimator that combines product, exponential, ratio and logarithmic forms to improve the efficiency of population mean estimation utilizing an auxiliary variable.

The primary aim of this research is to introduce novel and precise class of estimators for the population mean by employing an exponential ratio-cum-regression approach. These estimators incorporate known statistical indicators such as correlation coefficients and coefficients of variation associated with auxiliary variables to improve the precision of population mean. Consequently, the study explores the application of dual CV and assesses the performance of these estimators within the framework of STRS.

Subsampling And Notation Structures

The finite population $\mathcal{S} = \{1, 2, \ldots, L\}$ of N distinguishable units, categorized into L strata. The size of h^{th} stratum i.e. N_h (h=1,2,...,L) such that

$$\sum_{h=1}^{L} N_h = N.$$

A sample of size n_h is selected from the each stratum using simple random sampling without replacement (SRSWOR) to estimate the population mean, ensuring that the total sample size satisfies

$$\sum_{h=1}^{L} n_h = n.$$

The study variable is denoted by Y and dual concomitant information through X and Z, whereas y_{hi}, x_{hi} and z_{hi} denote the values of these variables in h^{th} stratum for h=1,2,...,L and i=1,2,3,...,N_h units.

For h^{th} stratum, The population and sample means, coefficient of variation, population standard deviations are represented as

$$\bar{Y}_h = \frac{\sum_{i=1}^{N_h} y_{hi}}{N_h}, \quad \bar{y}_h = \frac{\sum_{i=1}^{n_h} y_{hi}}{n_h},$$

$$\bar{X}_h = \frac{\sum_{i=1}^{N_h} x_{hi}}{N_h}, \quad \bar{x}_h = \frac{\sum_{i=1}^{n_h} x_{hi}}{n_h}, \quad \bar{Z}_h = \frac{\sum_{i=1}^{N_h} z_{hi}}{N_h}, \quad \bar{z}_h = \frac{\sum_{i=1}^{n_h} z_{hi}}{n_h}, \quad C_{yh} = \frac{S_{yh}}{\bar{Y}_h}, \quad C_{xh} = \frac{S_{xh}}{\bar{X}_h}, \quad C_{zh} = \frac{S_{zh}}{\bar{Z}_h},$$

$$S_{yh} = \sqrt{\frac{\sum_{i=1}^{N_h}(y_{hi} - \bar{Y}_h)^2}{N_h - 1}}, \quad S_{xh} = \sqrt{\frac{\sum_{i=1}^{N_h}(x_{hi} - \bar{X}_h)^2}{N_h - 1}} \quad \text{and} \quad S_{zh} = \sqrt{\frac{\sum_{i=1}^{N_h}(z_{hi} - \bar{Z}_h)^2}{N_h - 1}}$$

are respectively. The population correlation coefficients between the pairs (y, z), (y, x) and (x, z) of Y, X and Z can be represented as ρ_{yzh}, ρ_{yxh} and ρ_{xzh}. The overall population and sample means of Y, X and Z within STRS are represented as $\bar{Y} = \bar{Y}_{st} = \sum_{h=1}^{L} W_h \bar{Y}_h$, $\bar{X} = \bar{X}_{st} = \sum_{h=1}^{L} W_h \bar{X}_h$, $\bar{Z} = \bar{Z}_{st} = \sum_{h=1}^{L} W_h \bar{Z}_h$ and

$$\bar{y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_h, \quad \bar{x}_{st} = \sum_{h=1}^{L} W_h \bar{x}_h, \quad \bar{z}_{st} = \sum_{h=1}^{L} W_h \bar{z}_h,$$

where $W_h = \frac{N_h}{N}$ is the weight associated with each stratum. Additionally, the error terms are introduced as $\epsilon_{0h} = \frac{\bar{y}_{st} - \bar{Y}_{st}}{\bar{Y}_{st}}$, $\epsilon_{1h} = \frac{\bar{x}_{st} - \bar{X}_{st}}{\bar{X}_{st}}$ and $\epsilon_{2h} = \frac{\bar{z}_{st} - \bar{Z}_{st}}{\bar{Z}_{st}}$ such that $E(\epsilon_{ih}) = 0$ for $(i = 1, 2, 3)$. Also $E(\epsilon_{0h}^2) = \theta_h C_{yh}^2$, $E(\epsilon_{1h}^2) = \theta_h C_{xh}^2$, $E(\epsilon_{2h}^2) = \theta_h C_{zh}^2$, $E(\epsilon_{0h}\epsilon_{1h}) = \theta_h \rho_{yxh} C_{yh} C_{xh}$, $E(\epsilon_{0h}\epsilon_{2h}) = \theta_h \rho_{yzh} C_{yh} C_{zh}$, $E(\epsilon_{1h}\epsilon_{2h}) = \theta_h \rho_{xzh} C_{xh} C_{zh}$ where $\theta_h = \left(\frac{1}{n_h} - \frac{1}{N_h}\right)$.

Current estimators in STRS:

1) The unbiased sample estimators and its variances are given below:

$$\bar{y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_h \tag{1}$$

$$V(\bar{y}_{st}) = \sum_{h=1}^{L} W_h^2 \theta_h S_{yh}^2 \tag{2}$$

$$\bar{x}_{st} = \sum_{h=1}^{L} W_h \bar{x}_h \tag{3}$$

$$V(\bar{x}_{st}) = \sum_{h=1}^{L} W_h^2 \theta_h S_{xh}^2 \tag{4}$$

Koyuncu and Kadilar [13] proposed modified estimator of Gupta and Shabbir under STRS, as specified below along with its MSE.

$$\bar{y}_{GS} = \left[k_1 \bar{y}_{st} + k_2 (\bar{X}_{st} - \bar{x}_{st})\right] \frac{a\bar{X}_{st} + b}{a\bar{x}_{st} + b} \tag{5}$$

Where

$\ell_{120} = \frac{\sum_{h=1}^{L} W_h^2 \theta_h S_{yh}^2}{\bar{Y}_{st}^2}$, $\ell_{102} = \frac{\sum_{h=1}^{L} W_h^2 \theta_h S_{xh}^2}{\bar{X}_{st}^2}$, $\ell_{111} = \frac{\sum_{h=1}^{L} W_h^2 \theta_h S_{yxh}}{\bar{Y}_{st}\bar{X}_{st}}$, $\alpha = \frac{a\bar{X}_{st}}{a\bar{X}_{st}+b}$, $a = 0$ and $b = 1$.

$$k_1 = \frac{(-1 + \alpha^2 \ell_{102})\ell_{102}}{\alpha^2 \ell_{102}^2 + \ell_{111}^2 - \ell_{102}(1 + \ell_{120})}$$

$$k_2 = -\frac{\bar{Y}_{st}\{\ell_{111} + \alpha(\alpha^2 \ell_{102}^2 - \ell_{111}^2 + \ell_{102}(-1 - \alpha\ell_{111} + \ell_{120}))\}}{\bar{X}_{st}\{\alpha^2 \ell_{102}^2 + \ell_{111}^2 - \ell_{102}(1 + \ell_{120})\}}$$

$$MSE(\bar{y}_{GS}) = \frac{\bar{Y}_{st}^2(-1 + \alpha^2 \ell_{102})(-\ell_{111}^2 + \ell_{102}\ell_{120})}{\alpha^2 \ell_{102}^2 + \ell_{111}^2 - \ell_{102}(1 + \ell_{120})} \tag{6}$$

Koyuncu and Kadilar [10] proposed a range of combined ratio-product estimators for determining the population mean, with the objective of enhancing the performance of the estimators by incorporating coefficients of variation (CVs), estimators are represented by $\bar{y}_{RKP_1}, \bar{y}_{RKP_2}, \bar{y}_{RKP_3}$ and $\bar{y}_{RKP_4}$, as outlined below:

$$\bar{y}_{RKP_1} = \bar{y}_{st}\left(\frac{\bar{X}_{st}}{\bar{x}_{st}}\right)\left(\frac{\bar{Z}_{st}}{\bar{z}_{st}}\right) \tag{7}$$

$$MSE(\bar{y}_{RKP_1}) = \sum_{h=1}^{L} W_h^2 \theta_h \{S_{yh}^2 + D_1^2 S_{xh}^2 + D_2^2 S_{zh}^2 - 2D_1 S_{yxh} + 2D_1 D_2 S_{xzh} - 2D_2 S_{yzh}\} \tag{8}$$

Where $D_{11} = \frac{\bar{Y}_{st}}{\bar{X}_{st}}$ and $D_{12} = \frac{\bar{Y}_{st}}{\bar{Z}_{st}}$,

$$\bar{y}_{RKP_2} = \bar{y}_{st}\left(\frac{\bar{x}_{st}}{\bar{X}_{st}}\right)\left(\frac{\bar{z}_{st}}{\bar{Z}_{st}}\right) \tag{9}$$

$$MSE(\bar{y}_{RKP_2}) = \sum_{h=1}^{L} W_h^2\theta_h\{S_{yh}^2 + D_{11}^2S_{xh}^2 + D_{12}^2S_{zh}^2 + 2D_{11}S_{yxh} + 2D_{12}S_{yzh} + 2D_{11}D_{12}S_{xzh}\} \tag{10}$$

$$\bar{y}_{RKP_3} = \bar{y}_{st}\left(\frac{\bar{X}_{st}}{\bar{x}_{st}}\right)\left(\frac{\bar{z}_{st}}{\bar{Z}_{st}}\right) \tag{11}$$

$$MSE(\bar{y}_{RKP_3}) = \sum_{h=1}^{L} W_h^2\theta_h\{S_{yh}^2 + D_{11}^2S_{xh}^2 + D_{12}^2S_{zh}^2 - 2D_{11}S_{yxh} - 2D_{12}S_{yzh} + 2D_{11}D_{12}S_{xzh}\} \tag{12}$$

$$\bar{y}_{RKP_4} = \bar{y}_{st}\left(\frac{\bar{x}_{st}}{\bar{X}_{st}}\right)\left(\frac{\bar{Z}_{st}}{\bar{z}_{st}}\right) \tag{13}$$

$$MSE(\bar{y}_{RKP_4}) = \sum_{h=1}^{L} W_h^2\theta_h\{S_{yh}^2 + D_{11}^2S_{xh}^2 + D_{12}^2S_{zh}^2 + 2D_{11}S_{yxh} - 2D_{12}S_{yzh} + 2D_{11}D_{12}S_{xzh}\} \tag{14}$$

Koyuncu and Kadilar [13] introduced an improved estimator of Gupta and Shabbir.

$$\bar{y}_{GS2} = [p_1\bar{y}_{st} + p_2(\bar{X}_{st} - \bar{x}_{st})]\left(\frac{c\bar{X}_{st} + d}{c\bar{x}_{st} + d}\right) \tag{15}$$

Where

$\ell_{120} = \frac{\sum_{h=1}^{L} W_h^2\theta_h S_{yh}^2}{\bar{Y}_{st}^2}$, $\ell_{102} = \frac{\sum_{h=1}^{L} W_h^2\theta_h S_{xh}^2}{\bar{X}_{st}^2}$, $\ell_{111} = \frac{\sum_{h=1}^{L} W_h^2\theta_h S_{yxh}}{\bar{Y}_{st}\bar{X}_{st}}$, $\lambda = \frac{c\bar{X}_{st}}{c\bar{X}_{st}+d}$, $c = 1$ and $d = \sum_{h=1}^{L} W_h C_{xh}$.

$$p_1 = \frac{(-1 + \lambda^2\ell_{102})\ell_{102}}{\lambda^2\ell_{102}^2 + \ell_{111}^2 - \ell_{102}(1 + \ell_{120})}$$

$$p_2 = -\frac{\bar{Y}_{st}\{\ell_{111} + \lambda(\lambda^2\ell_{102}^2 - \ell_{111}^2 + \ell_{102}(-1 - \lambda\ell_{111} + \ell_{120}))\}}{\bar{X}_{st}\{\lambda^2\ell_{102}^2 + \ell_{111}^2 - \ell_{102}(1 + \ell_{120})\}}$$

$$MSE(\bar{y}_{GS2}) = \frac{\bar{Y}_{st}^2(-1 + \lambda^2\ell_{102})(-\ell_{111}^2 + \ell_{102}\ell_{120})}{\lambda^2\ell_{102}^2 + \ell_{111}^2 - \ell_{102}(1 + \ell_{120})} \tag{16}$$

Tailor et al. [23] applied the STRS technique in conjunction with the ratio-cum-product exponential estimator introduced by Singh et al.

$$\bar{y}_{TP} = \bar{y}_{st}\exp\left[\frac{\sum_{h=1}^{L} W_h(\bar{X}_h - \bar{x}_h)}{\sum_{h=1}^{L} W_h(\bar{X}_h + \bar{x}_h)}\right]\exp\left[\frac{\sum_{h=1}^{L} W_h(\bar{Z}_h - \bar{z}_h)}{\sum_{h=1}^{L} W_h(\bar{Z}_h + \bar{z}_h)}\right] \tag{17}$$

$$MSE(\bar{y}_{TP}) = \sum_{h=1}^{L} W_h^2 \theta_h \left[S_{yh}^2 + \frac{1}{4}(D_{11}^2 S_{xh}^2 + D_{12}^2 S_{zh}^2 - 2D_{11}D_{12}S_{xzh}) + D_{12}S_{yzh} - D_{11}S_{yxh} \right] \quad (18)$$

The research flow, illustrated in Figure 1, follows a structured approach to develop improved estimators for population mean estimation in stratified random sampling.

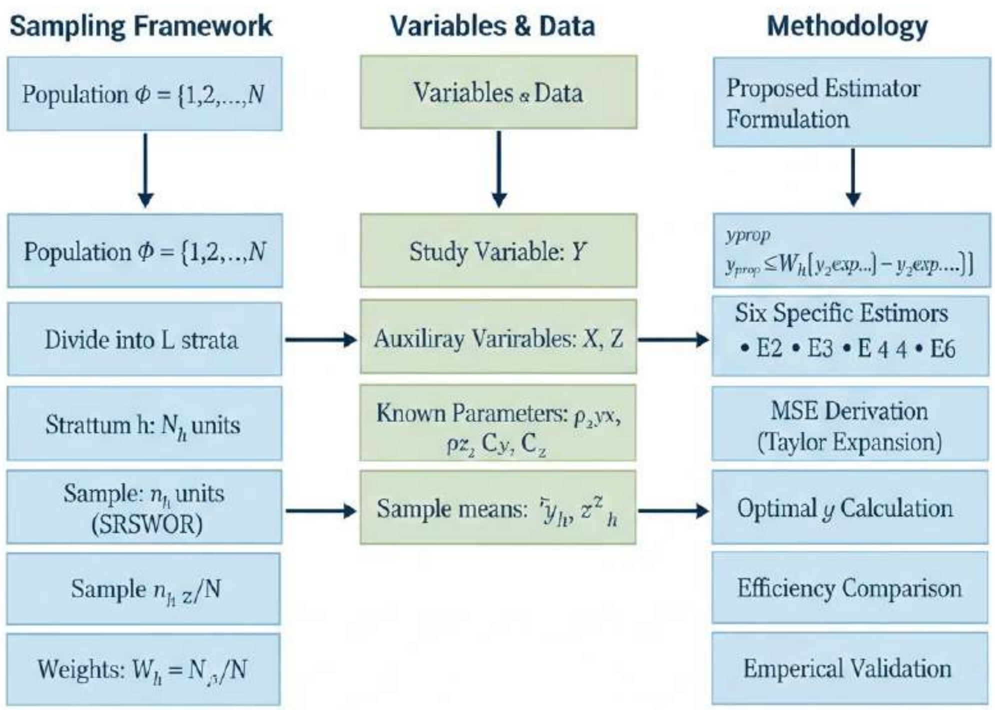

Figure 1. Complete methodology flowchart for developing exponential ratio-cum-regression estimators in stratified random sampling.

Proposed Estimator

A novel set of modified exponential ratio-cum-regression estimators are introduced for evaluating the population mean within the context of STRS, incorporating known parameter values of auxiliary variables. The proposed estimators demonstrate greater efficiency compared to current methods, as reflected by lower variance or MSE and a higher PRE as outlined below.

$$\bar{y}_{prop} = \sum_{h=1}^{L} W_h[\alpha_h \bar{y}_h + \beta_h \exp\left(\frac{\bar{X}_h - \bar{x}_h}{\bar{X}_h + \bar{x}_h}\right) - \gamma_h \exp\left(\frac{\bar{Z}_h - \bar{z}_h}{\bar{Z}_h + \bar{z}_h}\right)] \quad (19)$$

In this framework, α_h, β_h and γ_h are optimally chosen to reduce the MSE of the estimator. The constant "a" acts as a generalizing factor and assumes values derived from the known parameters of the CV. By leveraging various sets of CV, a diverse class of estimators can be constructed.

The estimator described above can be represented in terms of its error as follows:

$$\bar{y}_{prop} = \sum_{h=1}^{L} W_h \left[\gamma_{1h}\bar{Y}_h(1+e_{0h}) + \gamma_{2h}\exp\left(\frac{\bar{X}_h - \bar{x}_h}{\bar{X}_h + \bar{x}_h}\right) - \gamma_{3h}\exp\left(\frac{\bar{Z}_h - \bar{z}_h}{\bar{Z}_h + \bar{z}_h + 2a}\right)\right] \tag{20}$$

Eq(21) represented simplified form of error term

$$\bar{y}_{prop} - \bar{Y}_{st} = \sum_{h=1}^{L} W_h \left[\gamma_{1h}\bar{Y}_h(1+e_{0h}) + \gamma_{2h}\exp\left(-\frac{1}{2}e_{1h}\right) - \gamma_{3h}\exp\left(-\frac{\psi e_{2h}}{2}\right)\right] - \bar{Y}_{st} \tag{21}$$

Difference with $\bar{Y}_{st}$

$$\bar{y}_{prop} - \bar{Y}_{st} = \sum_{h=1}^{L} W_h \left[\gamma_{1h}\bar{Y}_h(1+e_{0h}) + \gamma_{2h}\left(1 - \frac{1}{2}e_{1h} + \frac{3}{8}e_{1h}^2\right) - \gamma_{3h}\left(1 - \frac{\psi e_{2h}}{2} + \frac{3}{8}\psi^2 e_{2h}^2\right) - \bar{Y}_h\right] \tag{22}$$

Where $\psi = \frac{\bar{Z}_h}{\bar{Z}_h + a}$

Eq(23) represented the bias of proposed estimator

$$Bias(\bar{y}_{prop}) = \sum_{h=1}^{L} W_h \left[\gamma_{1h}\bar{Y}_h + \gamma_{2h}\left(1 + \frac{3}{8}\theta_h C_{xh}^2\right) - \gamma_{3h}\left(1 + \frac{3}{8}\psi^2\theta_h C_{zh}^2\right) - \bar{Y}_h\right] \tag{23}$$

Proposed estimator's MSE expressed in eq (24)

$$\begin{aligned} MSE(\bar{y}_{prop}) = \sum_{h=1}^{L} W_h \Big[&\gamma_{1h}^2\bar{Y}_h^2(1+\theta_h C_{yh}^2) + \gamma_{2h}^2\left(1 + \frac{25}{4}\theta_h C_{xh}^2\right) \\ &+ \gamma_{3h}^2\left(1 + \psi^2\theta_h C_{zh}^2\right) + \bar{Y}_h^2 \\ &+ \gamma_{1h}\gamma_{2h}D_h - \gamma_{1h}\gamma_{3h}E_h - \gamma_{2h}\gamma_{3h}F_h \\ &- \gamma_{1h}G_h - \gamma_{2h}H_h + \gamma_{3h}I_h\Big] \end{aligned} \tag{24}$$

Simplified version of MSE

$$MSE(\bar{y}_{prop}) = \gamma_{1h}^2 A_h + \gamma_{2h}^2 B_h + \gamma_{3h}^2 C_h + \bar{Y}_h^2 + \gamma_{1h}\gamma_{2h}D_h - \gamma_{1h}\gamma_{3h}E_h - \gamma_{2h}\gamma_{3h}F_h - \gamma_{1h}G_h - \gamma_{2h}H_h + \gamma_{3h}I_h \tag{25}$$

Where

$$A_h = \bar{Y}_h^2[1 + \theta_h C_{yh}^2],$$

$$B_h = \left[1 + \frac{25}{4}\theta_h C_{xh}^2\right],$$

$$C_h = \left[1 + \psi^2\theta_h C_{zh}^2\right],$$

$$D_h = \bar{Y}_h \left[5 + 5\theta_h C_{yh}^2 - 4\theta_h \rho_{yxh} C_{yh} C_{xh}\right],$$

$$E_h = \bar{Y}_h \left[5 + \frac{15}{8}\psi^2\theta_h C_{zh}^2 + 2\psi\theta_h \rho_{yzh} C_{yh} C_{zh} + 5\theta_h C_{yh}^2\right],$$

$$F_h = \bar{Y}_h \left[2 + \frac{3}{4}\psi^2\theta_h C_{zh}^2 - \psi\theta_h \rho_{yzh} C_{yh} C_{zh}\right],$$

$$G_h = 2\bar{Y}_h^2,$$

$$H_h = \frac{5}{2}\bar{Y}_h \left[1 + \theta_h C_{xh}^2\right],$$

$$I_h = \bar{Y}_h \left[1 + \frac{3}{8}\psi^2\theta_h C_{zh}^2\right]$$

To obtain the most efficient values of γ_{1h}, γ_{2h} and γ_{3h} eq (25) has been differentiated and equate it result to zero.

$$\frac{\partial MSE(\bar{y}_{prop})}{\partial \gamma_{1h}} = 0 \tag{26}$$

$$2A_h\gamma_{1h} + D_h\gamma_{2h} - E_h\gamma_{3h} - G_h = 0 \tag{27}$$

Similarly

$$\frac{\partial MSE(\bar{y}_{prop})}{\partial \gamma_{2h}} = 0 \tag{28}$$

$$2B_h\gamma_{2h} + D_h\gamma_{1h} - F_h\gamma_{3h} - H_h = 0 \tag{29}$$

Additionally

$$\frac{\partial MSE(\bar{y}_{prop})}{\partial \gamma_{3h}} = 0 \tag{30}$$

$$2C_h\gamma_{3h} - E_h\gamma_{1h} - F_h\gamma_{2h} + I_h = 0 \tag{31}$$

By simultaneously solving Equations (27), (29), and (31), the corresponding values are obtained and presented in Equations (32), (33), and (34).

$$\gamma_{1h} = \frac{1}{2} \cdot \frac{(4B_hC_hG_h - 2I_hB_hE_h - 2C_hD_hH_h + I_hD_hF_h - F_h^2G_h + E_hF_hH_h)}{(4A_hB_hC_h - A_hF_h^2 - B_hE_h^2 - C_hD_h^2 + D_hE_hF_h)} \tag{32}$$

$$\gamma_{2h} = \frac{1}{2}\left(\frac{4A_hC_hH_h - 2I_hA_hE_h - 2C_hD_hG_h + I_hD_hF_h - E_hF_hG_h - F_h^2H_h}{4A_hB_hC_h - A_hE_h^2 - B_hF_h^2 - C_hD_h^2 + D_hE_hF_h}\right) \tag{33}$$

$$\gamma_{3h} = \frac{1}{2}\left(\frac{4I_hA_hB_h - 2A_hE_hH_h - 2B_hF_hG_h - I_hD_h^2 + D_hE_hG_h + D_hF_hH_h}{4A_hB_hC_h - A_hE_h^2 - B_hF_h^2 - C_hD_h^2 + D_hE_hF_h}\right) \tag{34}$$

By putting γ_{1h}, γ_{2h} and γ_{3h} values in eq (25), MSE ($\bar{y}_{prop}$) is obtained shown in eq (35)

Simplified form of MSE

$$MSE(\bar{y}_{prop}) = \sum_{h=1}^{L} W_h \left\{ \frac{1}{4} \left[\frac{1}{4A_hB_hC_h - A_hE_h^2 - B_hF_h^2 - C_hD_h^2 + D_hE_hF_h} \right] \right\} \tag{35}$$

$$MSE(\bar{y}_{prop}) = \sum_{h=1}^{L} W_h \left[\frac{1}{4} [\Omega_h][\Omega_{2h}] \right] \tag{36}$$

$$\Omega_h = \left[\frac{1}{4A_hB_hC_h - A_hE_h^2 - B_hF_h^2 - C_hD_h^2 + D_hE_hF_h} \right]$$

$$\begin{aligned}\Omega_{2h} = \Big[& 16A_hB_hC_h\bar{Y}_h^2 - 4A_hE_h^2\bar{Y}_h^2 - 4B_hF_h^2\bar{Y}_h^2 - 4C_hD_h^2\bar{Y}_h^2 + 4D_hE_hF_h\bar{Y}_h^2 \\ & + 4A_hB_h - 4A_hC_hH_h^2 - 2I_hD_hE_hG_h - 4B_hC_hG_h + 4I_hB_hF_hG_h \\ & + 4C_hD_hG_hH_h - D_h^2 - 2I_hD_hF_hH_h + 4I_hA_hE_hH_h \\ & + E_h^2G_h^2 - 2E_hF_hG_hH_h + F_h^2H_h^2 \Big]\end{aligned}$$

Proposed Estimator's Family

Table 1, presents newly developed and refined class of estimators for the population mean that integrate regression, exponential and ratio estimation techniques. This approach incorporates complementary population parameters including coefficients of variation and correlation to refine the performance of the proposed estimators.

Table 1. Class of the proposed estimator.

Estimators	α
$\bar{y}_{p1} = [\gamma_{1h}\bar{y}_h + \gamma_{2h} \exp\left(\frac{X_h - \bar{x}_h}{X_h + \bar{x}_h}\right) - \gamma_{3h} \exp\left(\frac{Z_h - \bar{z}_h}{Z_h + \bar{z}_h}\right) + 2\rho_{yzh}]$	ρ_{yzh}
$\bar{y}_{p2} = [\gamma_{1h}\bar{y}_h + \gamma_{2h} \exp\left(\frac{X_h - \bar{x}_h}{X_h + \bar{x}_h}\right) - \gamma_{3h} \exp\left(\frac{Z_h - \bar{z}_h}{Z_h + \bar{z}_h}\right) + 2C_{xh}]$	C_{xh}
$\bar{y}_{p3} = [\gamma_{1h}\bar{y}_h + \gamma_{2h} \exp\left(\frac{X_h - \bar{x}_h}{X_h + \bar{x}_h}\right) - \gamma_{3h} \exp\left(\frac{Z_h - \bar{z}_h}{Z_h + \bar{z}_h}\right) + 2\rho_{yzh}C_{xh}]$	$\rho_{yzh}C_{xh}$
$\bar{y}_{p4} = [\gamma_{1h}\bar{y}_h + \gamma_{2h} \exp\left(\frac{X_h - \bar{x}_h}{X_h + \bar{x}_h}\right) - \gamma_{3h} \exp\left(\frac{Z_h - \bar{z}_h}{Z_h + \bar{z}_h}\right) + 2C_{xh}^2]$	C_{xh}^2
$\bar{y}_{p5} = [\gamma_{1h}\bar{y}_h + \gamma_{2h} \exp\left(\frac{X_h - \bar{x}_h}{X_h + \bar{x}_h}\right) - \gamma_{3h} \exp\left(\frac{Z_h - \bar{z}_h}{Z_h + \bar{z}_h}\right) + 2\rho_{yzh}C_{xh}^2]$	$\rho_{yzh}C_{xh}^2$
$\bar{y}_{p6} = [\gamma_{1h}\bar{y}_h + \gamma_{2h} \exp\left(\frac{X_h - \bar{x}_h}{X_h + \bar{x}_h}\right) - \gamma_{3h} \exp\left(\frac{Z_h - \bar{z}_h}{Z_h + \bar{z}_h}\right) + 2C_{zh}^2]$	C_{zh}^2

Theoretical Comparisons:

A theoretical analysis was performed to evaluate the efficiency of the proposed estimator relative to conventional estimators under the STRS scheme. The results indicate that the newly developed estimator demonstrates improved performance over previously established methods in terms of **effectiveness**. The list of evaluated estimators is provided below.

(i) Comparing eq (36) with eq (2):

$$MSE(\bar{y}_{p1}) \leq MSE(\bar{y}_{st})$$

iff

$$\frac{1}{4}[\Omega_h][\Omega_{2h}] - \sum_{h=1}^{L} W_h^2 \theta_h S_{yh}^2 \leq 0$$

(ii) Comparing eq (36) and eq (4):

$$MSE(\bar{y}_{p1}) \leq MSE(\bar{x}_{st})$$

iff

$$\frac{1}{4}[\Omega_h][\Omega_{2h}] - \sum_{h=1}^{L} W_h^2 \theta_h S_{xh}^2 \leq 0$$

(iii) Comparing eq (36) and eq (6):

$$MSE(\bar{y}_{p1}) \leq MSE(\bar{y}_{GS})$$

$$\frac{1}{4}[\Omega_h][\Omega_{2h}] - \frac{\bar{Y}_{st}^2(-1+\alpha^2\ell_{102})(-\ell_{111}^2+\ell_{102}\ell_{120})}{\alpha^2\ell_{102}^2+\ell_{111}^2-\ell_{102}(1+\ell_{120})} \leq 0$$

(iv) Comparing eq (36) and eq (8):

$$MSE(\bar{y}_{p1}) \leq MSE(\bar{y}_{RKP_1})$$

$$\frac{1}{4}[\Omega_h][\Omega_{2h}] - \sum_{h=1}^{L} W_h^2 \theta_h \{S_{yh}^2 + D_1^2 S_{xh}^2 + D_2^2 S_{zh}^2 - 2D_1 S_{yxh} + 2D_1 D_2 S_{xzh} - 2D_2 S_{yzh}\} \leq 0$$

(v) Comparing eq (36) and eq (10):

$$MSE(\bar{y}_{p1}) \leq MSE(\bar{y}_{RKP_2})$$

$$\frac{1}{4}[\Omega_h][\Omega_{2h}] - \sum_{h=1}^{L} W_h^2 \theta_h \{S_{yh}^2 + D_1^2 S_{xh}^2 + D_2^2 S_{zh}^2 + 2D_1 S_{yxh} + 2D_1 D_2 S_{xzh} + 2D_2 S_{yzh}\} \leq 0$$

(vi) Comparing eq (36) and eq (12):

$$MSE(\bar{y}_{p1}) \leq MSE(\bar{y}_{RKP_3})$$

$$\frac{1}{4}[\Omega_h][\Omega_{2h}] - \sum_{h=1}^{L} W_h^2 \theta_h \{S_{yh}^2 + D_1^2 S_{xh}^2 + D_2^2 S_{zh}^2 - 2D_1 S_{yxh} - 2D_1 D_2 S_{xzh} + 2D_2 S_{yzh}\} \leq 0$$

(vii) Comparing eq (36) and eq (14):

$$MSE(\bar{y}_{p1}) \leq MSE(\bar{y}_{RKP_4})$$

$$\frac{1}{4}[\Omega_h][\Omega_{2h}] - \sum_{h=1}^{L} W_h^2 \theta_h \{S_{yh}^2 + D_1^2 S_{xh}^2 + D_2^2 S_{zh}^2 + 2D_1 S_{yxh} - 2D_1 D_2 S_{xzh} - 2D_2 S_{yzh}\} \leq 0$$

(viii) Comparing eq (36) and eq (16):

$$MSE(\bar{y}_{p1}) \leq MSE(\bar{y}_{GS2})$$

$$\frac{1}{4}[\Omega_h][\Omega_{2h}] - \frac{\bar{Y}_{st}^2(-1+\lambda^2\ell_{102})(-\ell_{111}^2+\ell_{102}\ell_{120})}{\lambda^2\ell_{102}^2+\ell_{111}^2-\ell_{102}(1+\ell_{120})} \leq 0$$

(ix) Comparing eq (36) and eq (18):

$$MSE(\bar{y}_{p1}) \leq MSE(\bar{y}_{TP})$$

$$\frac{1}{4}[\Omega_1][\Omega_2] - \sum_{h=1}^{L} W_h^2 F_h \left[S_{yh}^2 + \frac{1}{4}\left(D_{11}^2 S_{xh}^2 + D_{12}^2 S_{zh}^2 - 2D_{11}D_{12}S_{xzh} \right) + D_{12}S_{yzh} - D_{11}S_{yxh} \right] \leq 0$$

Empirical Real Data Sets

Data Set 1: Source: Koyuncu and Kadilar [10]. The dataset was obtained from the Turkish Ministry of Education in 2007 and covers 923 districts across six regions. It contains information on the number of teachers (Y), the number of students enrolled in primary and secondary schools (X), and the number of classrooms (Z).

Data Set 2: Source: Sarndal et al. (1992), p. 529 [18]. This dataset provides population figures (in thousands) for the years 1985 (y) and 1975 (x), along with the total number of seats in municipal councils (z).

Numerical Evaluation

Two distinct datasets are employed to assess the performance of the proposed estimators. Each dataset consists of a study variable (Y) and two auxiliary variables (X, Z). Data Set 1 is taken from Koyuncu and Kadilar [10], while Data Set 2 is obtained from Sarndal et al. [18]. The Mean Squared Error (MSE) and Percentage Relative Efficiency (PRE) of both the existing and proposed estimators are presented in Table 2. The numerical results indicate that the newly proposed estimators are more efficient than the conventional estimators.

Table 2. MSE and PRE comparison across estimators

2*S.No.	2*Estimators	Data Set 1		Data Set 2	
		MSE	PRE	MSE	PRE
1	ϖ_1	2309395	100	16248040	100
2	ϖ_2	1865652069	0.0012378	1.7103×10^{10}	0.0009500109
3	ϖ_3	187993	12.28448	173584.7	93.60294
4	ϖ_4	32617479429	0.0000708	86138586393	0.0001886267
5	ϖ_5	32646818500	0.0000707	86255740331	0.0001883705
6	ϖ_6	32609720219	0.0000708	86121069004	0.0001886651
7	ϖ_7	86121069004	0.0000268	86121069004	0.0001886651
8	ϖ_8	32639055049	0.0000707	86238211090	0.0001884088
9	ϖ_9	8147950616	0.000283432	21515632105	0.0007551737
10	ϖ_{p1}	15484.41	149.1432	88478.17	183.639
11	ϖ_{p2}	15640.83	147.6517	89032.98	182.4946
12	ϖ_{p3}	15624.61	147.805	88998.84	182.5646
13	ϖ_{p4}	15816.23	146.0142	90017.31	180.4991
14	ϖ_{p5}	15859.52	145.6157	89884.57	180.7656
15	ϖ_{p6}	15883.90	145.3922	89947.62	180.6389

Results and Discussion

This study proposes a class of exponential ratio-cum-regression estimators for estimating the population mean under Stratified Random Sampling (STRS) using dual auxiliary variables. The efficiency of both existing and proposed estimator families is evaluated using two real datasets based on the MSE and PRE criteria.

The newly proposed estimators demonstrate varying performance depending on the specific known parameters utilized, such as ρ_{yxh}, C_{yh}, C_{xh}, and others. Table 2 clearly shows that all proposed estimators yield smaller MSE values and higher PRE values compared to the conventional estimators, confirming their superior performance.

Furthermore, the theoretical efficiency conditions are validated numerically, as all computed differences remain positive. The results consistently indicate that the proposed estimator family outperforms existing estimators across both datasets.

Conclusion

The empirical findings confirm that the proposed class of exponential ratio-cum-regression estimators achieves the lowest Mean Squared Errors and the highest Percentage Relative Efficiencies across two real datasets. In particular, for Data Set 1, the estimator ϖ_{p1} attains an MSE of 15484.41 with a PRE of 149.1432, while for Data Set 2, it achieves an MSE of 88478.17 with a PRE of 183.639. These results clearly demonstrate the robustness and efficiency of the proposed estimators, making them strong alternatives to conventional estimators when auxiliary information is available.

Conflicts of Interest

The authors declare that they have no competing interest.

Author Contributions

Shumaila Wazir: Conceptualization, Methodology, Writing – original draft preparation, Data analysis, Numerical implementation, Proofreading. **Muhammad Atif**: Formal analysis, software, validation, investigation, data curation, and visualization. **Mujeeb Hussain**: Supervision, methodology, writing—review and editing, and theoretical derivations.

Compliance with Ethical Standards

It is declare that all authors don't have any conflict of interest. It is also declare that this article does not contain any studies with human participants or animals performed by any of the authors. Furthermore, informed consent was obtained from all individual participants included in the study.

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