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A Novel Hesitant Cubical Dombi Fuzzy Aggregation Operators for Selecting Green Supplier Chain Managements

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Abstract A hesitant fuzzy (HF) set enhances the concept of fuzzy sets by addressing disagreements among decision-makers about the membership degree of an element. Similarly, the Cubical Fuzzy Set (CFS) is useful for managing uncertainty in decision-making problems. However, existing methods often lack integration of hesitation and cubical uncertainty, and there is limited exploration of their combined effects on aggregation processes. In this paper, we introduce the Hesitant Cubical Fuzzy Set (HCFS), which integrates the principles of HF sets and CFS to address these limitations. We define several set-theoretical operations for HCFSs and develop Dombi operations for them. Furthermore, we present a range of aggregation operators based on Dombi operations, including the Hesitant Cubical Dombi Fuzzy Weighted Arithmetic Averaging (HCDFWAA) Operator, the Hesitant Cubical Dombi Fuzzy Weighted Geometric Averaging (HCDFWGA) Operator, the Hesitant Cubical Dombi Fuzzy Ordered Weighted Arithmetic Averaging (HCDFOWAA) Operator, and the Hesitant Cubical Dombi Fuzzy Ordered Weighted Geometric Averaging (HCDFOWGA) Operator, and examine their properties. Additionally, we propose a multi-criteria group decision-making method and algorithm within the Hesitant Cubical Fuzzy framework. To address gaps in practical application, we provide an example of the selection of green suppliers in supply chain management. We also perform a comparative analysis with existing operators to highlight the advantages and effectiveness of our approach, emphasizing how the integration of hesitation and cubical uncertainty can enhance decision-making processes.

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1 Introduction

1.1 Brief History

The concept of fuzzy sets (FS), introduced by Zadeh [1] in 1965, addresses problems involving uncertainty and has been applied across numerous fields, including robotics, computer science, clustering, optimization, data mining, and medical science. Fuzzy set theory has significantly impacted these fields by offering a mathematical framework to handle imprecise and ambiguous data. Notable examples include the work of Gadekallu and Gao [2], who proposed a model using rough sets for attribute reduction and a fuzzy logic system for classifying and predicting heart and diabetes diseases. Similarly, Sankaran et al. [3] developed a method to ensure multi-path and multi-constraint Quality of Service through Reliable Fuzzy and Heuristic Concurrent Ant Colony Optimization.

Over the years, fuzzy logic has evolved with extensions such as intuitionistic fuzzy sets (IFS), Pythagorean fuzzy sets (PyFS), and picture fuzzy sets (PFS), each addressing specific limitations of the original concept and expanding its applicability. For example, IFS introduced by Atanassov extended FS by incorporating membership and non-membership degrees, constrained by $\mu + \upsilon \leq 1$. To overcome the limitations of IFS, Yager introduced PyFS with the condition $\mu^2 + \upsilon^2 \leq 1$, further refining the approach to uncertainty.

A more recent development in fuzzy set theory is the cubical fuzzy set (CFS), introduced by Asghar Khan et al. CFS extends the traditional FS by requiring that the cubed sum of the degrees of membership (μ), neutral (γ), and non-membership (υ) be less than or equal to 1. This extension generalizes FS, IFS, PyFS, PFS, qROFS, and SFS and is applicable in decision-making problems. The evolution of these fuzzy set extensions highlights the dynamic nature of fuzzy set theory and its critical role in advancing decision-making processes in complex and uncertain environments.

1.2 Fuzzy Set Theory and Their Extensions

Zadeh [1] defined an FS using a membership function with values between 0 and 1. In this system, if the membership degree (MD) of an element is μ, then its non-membership degree (NMD) is υ, leading to a hesitation degree of zero. This approach, however, has limitations. To address these, Atanassov [4] introduced the intuitionistic fuzzy set (IFS), which extends FS by assigning two values within the range $[0, 1]$: the MD (μ) and the NMD (υ), constrained by $\mu + \upsilon \leq 1$. When this sum exceeds $\mu + \upsilon \leq 1$, the IFS model becomes impractical. To solve this, Yager [5, 6] developed the Pythagorean fuzzy set (PyFS), an extension of IFS, with the condition $\mu^2 + \upsilon^2 \leq 1$. Another extension is the Picture fuzzy set (PFS) by Cuong [7, 8], effective in representing human opinions by modeling judgments with degrees of yes, abstention, no, and rejection. PFS identifies three degrees: MD (μ), abstinence degree (AD) or neutral degree (γ), and NMD (υ), with the condition $0 \leq \mu+\gamma+\upsilon \leq 1$. While PFS is widely used in decision-making [9, 10], similarity measures, [11–15] correlation coefficients [16, 17], and clustering, it becomes insufficient when $\mu+\gamma+\upsilon > 1$. To address this, Gungogdu and Kahraman introduced the Spherical fuzzy set (SFS), which extends PFS by satisfying $0 \leq \mu^2 + \gamma^2 + \upsilon^2 \leq 1$. They explored SFS operations and their applications in decision-making problems. Kahraman et al. proposed a decision-making method integrating SFS with the TOPSIS method, demonstrating its use in selecting a green supplier chain management.

1.3 Aggregation Operators

The hesitant fuzzy set (HFS), introduced by Torra and Narukawa [23, 24], extends FS to model scenarios where decision-makers have different opinions about an element. For example, if one decision-maker assigns a membership grade of 0.7 and another assigns 0.3 to the same element, reaching a consensus is challenging. HFS effectively handles such situations. Due to its advantages, numerous researchers have developed multiple decision-making methods within the hesitant fuzzy environment [25, 26]. Xia et al. [27] described various HF aggregation operator (Op) and developed a group decision-making method, while Chen et al. [28] introduced interval-valued HF sets (IvHFSs), a generalization of HFS. Peng et al. [29] explored continuous HF aggregation Ops using the continuous ordered weighted averaging (OWA) Op. They defined the Continuous HF Aggregation Ordered Weighted Averaging (CHFOWA) Op and Continuous HF Aggregation Ordered Weighted Geometric (CHFOWG) Op, and extended these to interval-valued HFS (IvHFS). Mu et al. [30] proposed a new aggregation principle for HF Elements (HFE). Amin et al. [31] defined several aggregation Ops for triangular cubic linguistic HFS, and Fahmi et al. [32] introduced new operational laws for trapezoidal cubic HF (TrCHF) numbers, along with new aggregation Ops. Jiang et al. [33] defined interval-valued dual HFS and described aggregation Ops based on Hamacher t-norm and t-conorm. Liu et al. [34] introduced Dombi aggregation Ops for interval-valued HFS based on Dombi t-norm and t-conorm. Further studies related to aggregation Ops of HFS, extensions of HFS, and decision-making can be found in the literature [35–37].

1.4 Decision Making Techniques

HFSs are crucial for modeling problems involving multiple decision-makers by capturing their varying opinions. In an HFS, the HFEs are subsets of the interval [0, 1], representing degrees of membership. However, this framework does not adequately represent non-membership and neutral status. The Cubical Fuzzy Set (CFS), introduced by Asghar Khan et al. [40], extends FS by requiring that the cubed sum of the degrees of membership (μ), neutral (γ) and non-membership (υ) be less than or equal to 1. CFS generalizes FS, IFS, PyFS, PFS, qROFS, and SFS and is applicable in decision-making problems. Several generalizations of HFSs exist. To combine the benefits of both HFSs and CFSs, we introduce the Hesitant CFS (HCFS). In HCFS, multiple Cubical fuzzy values can be assigned to elements under evaluation, whereas CFS theory allows only a single Cubical fuzzy value per element. CFS is inadequate for modeling problems involving disagreements among decision-makers. Conversely, HCFS accommodates multiple opinions about an element. In recent years, the study of fuzzy sets has evolved to address more complex and nuanced decision-making scenarios. Picture fuzzy sets (PFS) have been extended into spherical fuzzy sets (SFS) to better capture uncertainty and hesitancy in expert opinions. Further extending these concepts, cubical fuzzy sets (CFS) provide an even more robust framework by incorporating additional dimensions of hesitancy and uncertainty. The hierarchical relationship between these fuzzy sets is depicted in Figure 1, illustrating how each set builds upon the previous one to enhance the modeling of uncertainty.

This figure demonstrates the extension from picture fuzzy sets to spherical fuzzy sets and finally to cubical fuzzy sets, highlighting the progression and increased complexity in modeling uncertainty and hesitancy.

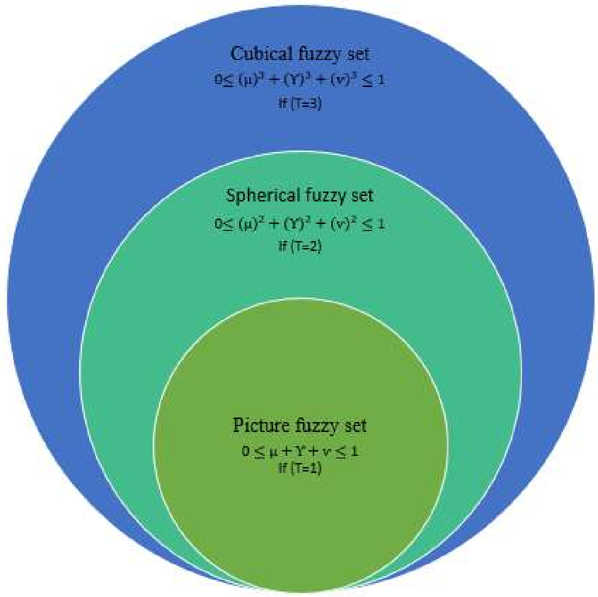

Figure 1. Hierarchical relationship among fuzzy sets

1.5 Motivation of Paper

In the field of fuzzy logic and decision-making, aggregation operators are essential for integrating information from diverse sources, especially when faced with uncertainty and varying opinions. Traditional fuzzy sets and their extensions, such as HFS and CFS, provide effective tools for handling different types of uncertainty. However, they often fall short in fully combining hesitation and cubical uncertainty, leading to limitations in addressing complex decision-making challenges.

The motivation for this paper stems from the need to bridge the gap between HFS and CFS by developing a comprehensive framework that integrates the benefits of both approaches. Our goal is to enhance the flexibility and applicability of fuzzy logic in decision-making processes, particularly in scenarios involving multiple decision-makers with varying opinions. The proposed HCFS framework aims to provide a more robust and versatile tool for capturing and aggregating diverse opinions, thereby improving the accuracy and reliability of decision-making outcomes. This research addresses the shortcomings of existing methods and presents innovative solutions to better manage the complexities of real-world decision-making scenarios.

1.6 Contribution of Study

This paper presents the HCFS, an innovative approach that merges the concepts of HF sets and CFS to overcome these limitations. We introduce a detailed framework for HCFSs, outlining various set-theoretical operations and developing Dombi operations tailored for them. Our research advances current methodologies by proposing several aggregation operators based on Dombi operations, including the HCDFWAA Op, the HCDFWGA Op, the HCDFOWAA Op and the HCDFOWGA Op. We provide an in-depth analysis of their properties to offer a clearer understanding of their effectiveness.

A significant aspect of this research is the application of the HCFS framework to a multi-criteria group decision-making (MCGDM) process, illustrated through a case study on selecting green suppliers in supply chain management. This practical example highlights the value of our proposed methods and addresses gaps in existing research by demonstrating how hesitation and cubical uncertainty can be integrated in real-world scenarios. Additionally, we conduct a comparative analysis with existing aggregation operators, showcasing the enhanced performance and flexibility of our approach. By addressing the shortcomings of traditional methods and presenting innovative solutions, this study makes substantial contributions to both the theoretical and practical aspects of fuzzy logic and decision-making.

2 Preliminaries

This section provides some basic definitions and operations that will be needed in the following sections.

Definition 1. *[22] An CFS $\mathbb{C}$ on a universe χ is represented as follows:*

$$\mathbb{C} = \{(x, c(x), i(x), d(x)) : x \in \chi\}, \tag{1}$$

where $c(x), i(x), d(x) \in [0, 1], 0 \leq c^3(x), i^3(x), d^3(x) \leq 1$ for all $x \in \chi$. We consider the triplet (c, i, d) as CF number (CFN). Here, s, i and d are the MD, AD and NMD of $x \in \mathbb{C}$, respectively. Further $\pi_{\mathbb{C}}(x) = \sqrt{1 - (c^3(x) + i^3(x) + d^3(x))}$ is the hesitancy degree of x in $\mathbb{C}$.

Definition 2. *[23, 24] Let χ be a fixed set, a HFS on χ is in terms of a function that applied to χ returns of $[0, 1]$. The matimatical symbol of HFS*

$$A = \{x, h_A(x) >: x \in \chi\}, \tag{2}$$

where $h_A(x)$ is a set of some values in $[0, 1]$, denoting the possible MDs of the element $x \in \chi$ to the set A. $h = h_A(x)$ is called an HF element (HFE).

From now on, set of all CFN is denoted by γ.

2.1 Hesistant Cubical Fuzzy Sets

In this section, we define the concept of HCFS and their set-theoretical operations.

Definition 3. *Let χ be a nonempty set. A HCFS over χ denoted by T_H is defined as follows:*

$$T_H = \{(x, \hbar(x)) : \hbar(x) \subseteq \gamma, x \in \chi\}. \tag{3}$$

Here $\hbar(x) = \hbar$ is collection of CFNs and $\hbar$ is called HCF element (HCFE). The number of elements of an HCFE is called length of HCFE $\hbar$ and denoted by $\varrho_\hbar$.

In other words, an *HCFS* is collection of *HCFEs*.

Example 1. *Let us consider a set $\chi = \{x_1, x_2, x_3, x_4\}$. Then, we can write an HCFS T as follows:*

$$T = \left\{ \begin{array}{c} (x_1, \{(0.8, 0.6, 0.4), (0.5, 0.4, 0.9), (0.2, 0.7, 0.6)\}), \\ (x_2, \{(0.9, 0.3, 0.5), (0.4, 0.2, 0.7)\}), \\ (x_3, \{(0.7, 0.5, 0.2), (0.7, 0.7, 0.7), (0.3, 0.4, 0.8)\}), \\ (x_4, \{(0.2, 0.7, 0.2)\}) \end{array} \right\}.$$

Definition 4. *Let $\hbar$ be an HCFE. Then, SV of HCFE $\hbar$ denoted by $\backepsilon\vartheta(\hbar)$ is defined as*

$$\backepsilon\vartheta(\hbar) = \frac{1}{\varrho_\hbar}\sum_{k=1}^{\varrho_\hbar}(c_k^3 - d_k^3) \tag{4}$$

for some positive integer n. Here, $\backepsilon\vartheta(\hbar) \in [-1, 1]$.

Definition 5. *Let $\hbar$ be an HCFE. Then, AV of HCFE $\hbar$ denoted by $AV(\hbar)$ is defined as*

$$AV(\hbar) = \frac{1}{\varrho_\hbar}\sum_{k=1}^{\varrho_\hbar}\left(c_k^3 + i_k^3 + d_k^3\right) \tag{5}$$

for some positive integer n. Here, $AV(\hbar) \in [0, 1]$.

Definition 6. *Let $\hbar_1$ and $\hbar_2$ be two HCFEs, $\backepsilon\vartheta(\hbar_1)$ are the Score values of $\hbar_1$ and $\backepsilon\vartheta(\hbar_2)$ are the score values of $\hbar_2$, and $AV(\hbar_1)$ are the accuracy values of $\hbar_1$ and $AV(\hbar_2)$ are the accuracy values of $\hbar_2$,. Then*

1. *If $\backepsilon\vartheta(\hbar_1) < \backepsilon\vartheta(\hbar_2)$ then $\hbar_1 < \hbar_2$*
2. *If $\backepsilon\vartheta(\hbar_1) > \backepsilon\vartheta(\hbar_2)$ then $\hbar_1 > \hbar_2$*
3. *If $\backepsilon\vartheta(\hbar_1) = \backepsilon\vartheta(\hbar_2)$, there are three cases*
   (a) *If $AV(\hbar_1) < AV(\hbar_2)$ then $\hbar_1 < \hbar_2$*
   (b) *If $AV(\hbar_1) > AV(\hbar_2)$ then $\hbar_1 > \hbar_2$*
   (c) *If $AV(\hbar_1) = AV(\hbar_2)$, then $\hbar_1 = \hbar_2$.*

Example 2. *Let us consider HCFEs $\hbar_1 = \{(0.8, 0.6, 0.4), (0.5, 0.4, 0.9), (0.2, 0.7, 0.6)\}$ and $\hbar_2 = \{(0.9, 0.3, 0.5), (0.4, 0.2, 0.7)\}$ of HCFEs T given in example 1, then*

$$\begin{aligned}
\backepsilon\vartheta(\hbar_1) &= \frac{1}{\varrho_\hbar}[(0.8^3 - 0.4^3) + (0.5^3 - 0.9^3) + (0.2^3 - 0.6^3)] \\
\backepsilon\vartheta(\hbar_1) &= -0.1213 \\
\backepsilon\vartheta(\hbar_2) &= \frac{1}{\varrho_\hbar}[(0.9^3 - 0.5^3) + (0.4^3 - 0.7^3)] \\
\backepsilon\vartheta(\hbar_2) &= 0.1083
\end{aligned}$$

Definition 7. *Let $T_1 = \{(x, \hbar_1(x)) : x \in X\}$ and $T_2 = \{(x, \hbar_2(x)) : x \in X\}$ be two HCFSs over a common universe χ. If, for all $x \in X, \backepsilon\vartheta(\hbar_1(x)) \le \backepsilon\vartheta(\hbar_2(x))$, then it is said that T_1 is an HCF subset of T_2, and denoted by $T_1 \lhd T_2$.*

Example 3. *Let T_1 and T_2 be two HCFSs over $\chi = \{x_1, x_2, x_3\}$ given as follows:*

$$T_1 = \left\{\begin{array}{c}
(x_1, \{(.4, .5, .3), (.5, .6, .4), (.2, .7, .6)\}) \\
(x_2, \{(.2, .6, .4), (.1, .7, .6), (.2, .5, .5)\}) \\
\left(x_3, \{(.3, .7, .3), (.2, .9, .2), (.1, .5, .3)\}\right)
\end{array}\right\}$$

$$T_2 = \left\{\begin{array}{c}
(x_1, \{(.6, .7, .5), (.5, .7, .5), (.3, .8, .6)\}) \\
(x_2, \{(.3, .8, .3), (.2, .6, .2), (.7, .5, .1)\}) \\
\left(x_3, \{(.6, .9, .3), (.4, .4, .5), (.2, .6, .3)\}\right)
\end{array}\right\}$$

Thus equation (1), for $x_i \in X(i = 1, 2, 3)$, SVs of HCFEs are obtained as follows:

From the table, it is clear that $T_1 \lhd T_2$.

Table: 01

	x_1	x_2	x_3
SV($\hbar_1$)	-0.0367	-0.1293	-0.0087
SV($\hbar_2$)	-0.0326	0.114	0.0363

2.2 Set-theoretical Operations of HCFSs

In this section, we define the union, intersection and complement of a HCFS along with their examples.

Definition 8. *Let $\hbar = \{(s_t, i_t, d_t) : 1 \leq t \leq \varrho_\hbar\}$ be a CFE over χ. Then, lower and uper bounds of $\hbar$ are defined as follows:*

$$\hbar^- = \min_t (s_t^3 - d_t^3) \tag{6}$$

$$\hbar^+ = \max_t (s_t^3 - d_t^3), \tag{7}$$

respectively.

The following example illustrates the lower and upper bounds of an $\hbar$.
Let $\hbar = \{(.4, .5, .3), (.5, .6, .4), (.2, .7, .6)\}$ be an *CFS* element. $\varrho_\hbar = 3$

$$\begin{aligned} \hbar^- &= \min\left\{\left(0.4^3 - 0.3^3\right), (0.5^3 - 0.4^3), (0.2^3 - 0.6^3)\right\} \\ &= \min\{0.037, 0.061, -0.208\} \\ &= -0.208, \end{aligned}$$

$$\begin{aligned} \hbar^+ &= \max\left\{\left(0.4^3 - 0.3^3\right), (0.5^3 - 0.4^3), (0.2^3 - 0.6^3)\right\} \\ &= \max\{0.037, 0.061, -0.208\} \\ &= 0.061. \end{aligned}$$

Definition 9. *Let T_1 and T_2 be two HCFSs over χ and let $\hbar_1$ and $\hbar_2$ be HCFEs of T_1 and T_2 for all $x \in \chi$. Then, based on HCFEs, set-theoretical operations between T_1 and T_2 are defined as follows:*

1. Union:

$$T_1 \cup T_2 = \bigcup_{x \in \chi} \left\{ \left(x, \bigcup_{\substack{(c_1, i_1, d_1) \in \hbar_1 \\ (c_2, i_2, d_2) \in \hbar_2}} \{(c_k, i_k, d_k) : c_k^3 - d_k^3 = \max\{c_1^3 - d_1^3, c_2^3 - d_2^3\}, k = 1, 2\} \right) \right\} \tag{8}$$

2. Intersection:

$$T_1 \cap T_2 = \bigcup_{x \in \chi} \left\{ \left(x, \bigcup_{\substack{(c_1, i_1, d_1) \in \hbar_1 \\ (c_2, i_2, d_2) \in \hbar_2}} \{(c_k, i_k, d_k) : c_k^3 - d_k^3 = \min\{c_1^3 - d_1^3, c_2^3 - d_2^3\}, k = 1, 2\} \right) \right\} \tag{9}$$

3. Complement:

$$T_1^c = \bigcup_{x\in\chi}\left\{\left(x, \bigcup_{(c_1,i_1,d_1)\in\hbar_1}\{(c_1,i_1,d_1)\right)\right\} \tag{10}$$

Example 4. *Let T_1 and T_2 be two HCFSs over $\chi = \{x_1, x_2, x_3\}$ for $n = 4$ given as follows:*

$$T_1 = \left\{\begin{array}{c}(x_1, \{(.5, .7, .9), (.3, .2, .7), (.6, .5, .4)\}),\\ (x_2, \{(.4, .6, .3), (.2, .5, .9)\}),\\ (x_3, \{(.8, .4, .3)\})\end{array}\right\}$$

$$T_2 = \left\{\begin{array}{c}(x_1, \{(.4, .8, .5), (.5, .7, .6)\}),\\ (x_2, \{(.2, .7, .6), (.4, .3, .8)\}),\\ (x_3, \{(.5, .4, .8), (.6, .7, .2), (.9, .2, .4)\})\end{array}\right\}$$

Then, using above Definition , union, intersection and complement of HCFEs are T_1 and T_2 are obtained as follows:

$$\begin{aligned}
T_1 \cup T_2 &= \left\{\begin{array}{c}(x_1, \{(.4, .8, .9), (.5, .7, .6), (.6, .5, .4)\}),\\ (x_2, \{(.4, .6, .3), (.2, .5, .9), (.4, .3, .8)\}),\\ (x_3, \{(.8, .4, .3), (.9, .2, .4)\})\end{array}\right\},\\
T_1 \cap T_2 &= \left\{\begin{array}{c}(x_1, \{(.5, .7, .9), (.3, .2, .7), (.4, .8, .5), (.5, .7, .9)\}),\\ (x_2, \{(.2, .7, .6), (.4, .3, .8), (.2, .5, .9)\}),\\ (x_3, \{(.5, .4, .8), (.6, .7, .2), (.8, .4, .3)\})\end{array}\right\},\\
T_1^c &= \left\{\begin{array}{c}(x_1, \{(.9, .7, .5), (.7, .2, .3), (.4, .5, .6)\}),\\ (x_2, \{(.3, .6, .4), (.9, .5, .2)\}),\\ (x_3, \{(.3, .4, .8)\})\end{array}\right\}
\end{aligned}$$

3 Hesitant Cubical Dombi Fuzzy Aggregation Operators

Aggregation Ops are essential for consolidating multiple values into a single value. In this section, we first define Dombi Ops for two HCFEs using the Dombi t-norm and t-conorm. An HCFE comprises CFEs with three components: MD, AD and NMD. For the summation ($\oplus$) operation between HCFEs, we use the Dombi t-conorm for MDs, the Dombi t-norm for AD and NMD. For the product ($\oplus$) operation between HCFEs, we apply the Dombi t-conorm for MDs and the Dombi t-norm for AD and NMD.

We then introduce the HCDFWAA and HCDFWGA Ops as generalizations of the Dombi Ops for two HCFEs. In these Ops, only the HCFEs are weighted, without considering the significance of their ordered positions. To address this limitation, we propose hesitant Cubical Dombi fuzzy ordered weighted arithmetic (HCDFOWAA) and geometric (HCDFOWGA) averaging Ops. These Ops account for both the weights of the elements and the importance degrees of the HFE using score functions. Specifically, we first sort the HCFEs based on their SVs using a defined score function. Subsequently, we apply the weight vectors (WVs) in the defined order, preserving the sequence. The underlying concept of ordered weighted averaging (OWA) is detailed in [41].

3.1 Dombi t-norm and t-conorm

The Dombi product and Dombi sum, which are specific types of t-norms and t-conorms, are described in [42] as follows:

Definition 10. *[42] Let f and g be two real numbers in the interval* $[0,1]$. *Then, Dombi t-norm id defined by*

$$f \otimes g = \frac{1}{1 + \left(\left(\frac{1-f}{f}\right)^{\gamma} + \left(\frac{1-g}{g}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \gamma > 0. \tag{11}$$

Dombi t-conorm is given by

$$f \oplus g = \frac{1}{1 + \left(\left(\frac{f}{1-f}\right)^{\gamma} + \left(\frac{g}{1-g}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \gamma > 0. \tag{12}$$

respectively.

3.2 Dombi Operations of HCFEs

In this subsection, we introduce various Dombi operations for HCFEs.

Definition 11. *Let $\hbar_1 = \{(c_{1t}, i_{1t}, d_{1t}) : 1 \leq t \leq \varrho_{\hbar 1}\}$ and $\hbar_2 = \{(c_{2t}, i_{2t}, d_{2t}) : 1 \leq t \leq \varrho_{\hbar 2}\}$ be two HCFEs and* $\gamma > 0$, *and the Dombi operations for HCF elements are defined as given below:*

1.

$$\hbar_1 \oplus \hbar_2 = \bigcup_{\substack{(c_{1t}, i_{1t}, d_{1t}) \in \hbar_1 \\ (c_{2r}, i_{2r}, d_{2r}) \in \hbar_2}} \left\{ \left(\begin{array}{c} \sqrt[3]{1 - \frac{1}{1 + \left\{ \left(\frac{c_{1t}^3}{1-c_{1t}^3}\right)^{\gamma} + \left(\frac{c_{2r}^3}{1-c_{2r}^3}\right)^{\gamma} \right\}^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1 + \left\{ \left(\frac{1-i_{1t}^3}{i_{1t}^3}\right)^{\gamma} + \left(\frac{1-i_{2r}^3}{i_{2r}^3}\right)^{\gamma} \right\}^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1 + \left\{ \left(\frac{1-d_{1t}^3}{d_{1t}^3}\right)^{\gamma} + \left(\frac{1-d_{2r}^3}{d_{2r}^3}\right)^{\gamma} \right\}^{\frac{1}{\gamma}}}} \end{array} \right) \right\} \tag{13}$$

2.

$$\hbar_1 \otimes \hbar_2 = \bigcup_{\substack{(c_{1t}, i_{1t}, d_{1t}) \in \hbar_1 \\ (c_{2r}, i_{2r}, d_{2r}) \in \hbar_2}} \left\{ \left(\begin{array}{c} \sqrt[3]{\frac{1}{1 + \left\{ \left(\frac{1-c_{1t}^3}{c_{1t}^3}\right)^{\gamma} + \left(\frac{1-c_{2r}^3}{c_{2r}^3}\right)^{\gamma} \right\}^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1 + \left\{ \left(\frac{1-i_{1t}^3}{i_{1t}^3}\right)^{\gamma} + \left(\frac{1-i_{2r}^3}{i_{2r}^3}\right)^{\gamma} \right\}^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{1 - \frac{1}{1 + \left\{ \left(\frac{d_{1t}^3}{1-d_{1t}^3}\right)^{\gamma} + \left(\frac{d_{2r}^3}{1-d_{2r}^3}\right)^{\gamma} \right\}^{\frac{1}{\gamma}}}} \end{array} \right) \right\} \tag{14}$$

3.

$$\lambda \hbar_1 = \bigcup_{(c_{1r}, i_{1r}, d_{1r}) \in \hbar_1} \left\{ \left(\begin{array}{c} \sqrt[3]{1 - \frac{1}{1 + \left\{ \lambda \left(\frac{c_{1t}^3}{c_{1t}^3}\right)^{\gamma} \right\}^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1 + \left\{ \lambda \left(\frac{1-i_{1t}^3}{i_{1t}^3}\right)^{\gamma} \right\}^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1 + \left\{ \lambda \left(\frac{1-d_{1t}^3}{d_{1t}^3}\right)^{\gamma} \right\}^{\frac{1}{\gamma}}}} \end{array} \right) \right\} \tag{15}$$

4.

$$\hbar_1^{\lambda} = \bigcup_{(c_{1r},i_{1r},d_{1r})\in\hbar_1} \left\{ \left(\begin{array}{c} \sqrt[3]{\frac{1}{1+\left\{\lambda\left(\frac{1-c_{1t}^3}{c_{1t}^3}\right)^{\gamma}\right\}^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left\{\lambda\left(\frac{1-i_{1t}^3}{i_{1t}^3}\right)^{\gamma}\right\}^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{1-\frac{1}{1+\left\{\lambda\left(\frac{d_{1t}^3}{1-d_{1t}^3}\right)^{\gamma}\right\}^{\frac{1}{\gamma}}}} \end{array} \right) \right\} \tag{16}$$

Example 5. *Let us consider* $\hbar(x_1) = \hbar_1 = \{(0.8, 0.6, 0.4), (0.5, 0.4, 0.9), (0.2, 0.7, 0.6)\}$ *and* $\hbar(x_2) = \hbar_2 = \{(0.9, 0.3, 0.5), (0.4, 0.2, 0.7)\}$ *in Example (1). For* $\gamma = 1$ *and* $\lambda = 2$.

$$\begin{aligned} & \hbar_1 \oplus \hbar_2 \\ = & \{(0.9240, 0.2908, 0.3536), \\ & (0.8081, 0.1981, 0.3849), (0.9041, 0.2684, 0.4925), \\ & (0.5587, 0.1928, 0.6726), (0.9002, 0.2950, 0.5346), \\ & (0.4141, 0.1990, 0.5346)\} \end{aligned}$$

$$\begin{aligned} & \hbar_1 \otimes \hbar_2 \\ = & \{(0.7549, 0.2908, 0.9381), \\ & (0.3922, 0.1981, 0.8567), (0.4925, 0.2684, 0.6390), \\ & (0.3536, 0.1928, 0.6192), (0.1998, 0.2950, 0.8900), \\ & (0.1928, 0.1990, 0.8224)\} \end{aligned}$$

$$\begin{aligned} 2\hbar_1 = & \{(0.8781, 0.9578, 0.9889), \\ & (0.6057, 0.9889, 0.7527), (0.2513, 0.9256, 0.9579)\} \end{aligned}$$

$$\begin{aligned} \hbar_1^2 = & \{(0.8689, 0.9578, 0.4937), \\ & (0.9772, 0.9889, 0.9448), (0.9987, 0.9256, 0.7082)\} \end{aligned}$$

3.3 Hesitant Cubical Dombi Fuzzy Weighted Arthemtic Averaging Operator

Definition 12. *Let* $\hat{H}^m = \hbar_k = \{(s_{kj}, i_{kj}, d_{kj}) : 1 \leq j \leq \varrho_{\hbar_k}, k = 1, 2, ..., m\}$ *be an m dimensional collection of HCFEs. A HCDFWAA Op is defined by a function HCDFWAA:* $\hat{H}^m \to \hat{H}$ *as given below:*

$$\begin{aligned} & HCDFWAA(\hbar_1, \hbar_2, ..., \hbar_m) \\ = & \bigoplus_{z=1}^{m} w_z \hbar_z \\ = & (w_1 \hbar_1) \oplus (w_2 \hbar_2) \oplus ... \oplus (w_m \hbar_m) \end{aligned} \tag{17}$$

where w_z *is weight of* $\hbar_z (z = 1, 2, ..., m)$, $0 \leq w_z \leq 1$ *and* $\sum_{z=1}^{m} w_z = 1$.

We establish the following theorem based on the application of Dombi operations to HCFEs.

Theorem 1. *Let $\hbar_k \in \hat{H}^m$. Then*

$$\begin{aligned}
&HCDFWAA(\hbar_1, \hbar_2, ..., \hbar_m) \\
&= \bigoplus_{z=1}^{m} w_z \hbar_z \\
&= \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1 \\ (c_2,i_2,d_2)\in\hbar_2 \\ \cdots \\ (c_m,i_m,d_m)\in\hbar_m}} \left(\begin{array}{c} \sqrt[3]{1 - \frac{1}{1+\left(\sum_{z=1}^{m} w_z \left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{m} w_z \left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{m} w_z \left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \end{array} \right)
\end{aligned}$$

where $w = (w_1, w_2, ..., w_m)$ be the m WV of $\hbar_k (k = 1, 2, ..., m)$, such that $w_k > 0$ and $\sum_{z=1}^{m} w_k = 1$.

Proof. The theorem can be demonstrated using mathematical induction as follows:

1. For $m = 2$, applying Dombi operations to HCFEs yields the following results:

$$\begin{aligned}
&w_1 \hbar_1 \\
&= \bigcup_{(c_1,i_1,d_1)\in\hbar_1} \left(\begin{array}{c} \sqrt[3]{1 - \frac{1}{1+\left(w_1 \left(\frac{c_1^3}{1-c_1^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(w_1 \left(\frac{1-i_1^3}{i_1^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(w_1 \left(\frac{1-d_1^3}{d_1^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \end{array} \right)
\end{aligned}$$

$$\begin{aligned}
&w_2 \hbar_2 \\
&= \bigcup_{(c_2,i_2,d_2)\in\hbar_2} \left(\begin{array}{c} \sqrt[3]{1 - \frac{1}{1+\left(w_2 \left(\frac{c_2^3}{1-c_2^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(w_2 \left(\frac{1-i_2^3}{i_2^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(w_2 \left(\frac{1-d_2^3}{d_2^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \end{array} \right)
\end{aligned}$$

$$w_1\hbar_1 \oplus w_2\hbar_2 = \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1\\(c_2,i_2,d_2)\in\hbar_2}} \left(\sqrt[3]{1-\frac{1}{1+\left(w_1\left(\frac{c_1^3}{1-c_1^3}\right)^{\gamma}+w_2\left(\frac{c_2^3}{1-c_2^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(w_1\left(\frac{1-i_1^3}{i_1^3}\right)^{\gamma}+w_2\left(\frac{1-i_2^3}{i_2^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(w_1\left(\frac{1-d_1^3}{d_1^3}\right)^{\gamma}+w_2\left(\frac{1-d_2^3}{d_2^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \right)$$

$$w_1\hbar_1 \oplus w_2\hbar_2 = \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1\\(c_2,i_2,d_2)\in\hbar_2}} \left(\sqrt[3]{1-\frac{1}{1+\left(\sum_{z=1}^{2} w_z\left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{2} w_z\left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{2} w_z\left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \right)$$

Then, the theorem valid for $m = 2$

2. Assume the theorem is valid when $z = k$ which is

$$\oplus_{z=1}^{k}(w_z\hbar_z) = \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1\\(c_2,i_2,d_2)\in\hbar_2\\ \cdots \\(c_k,i_k,d_k)\in\hbar_k}} \left(\sqrt[3]{1-\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \right)$$

When $z = k + 1$

$$\oplus_{z=1}^{k}(w_z\hbar_z) \oplus w_{k+1}\hbar_{k+1} = \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1\\(c_2,i_2,d_2)\in\hbar_2\\ \cdots \\(c_k,i_k,d_k)\in\hbar_k}} \left(\sqrt[3]{1-\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \right)$$

$$
\begin{aligned}
& \oplus w_{k+1}\hbar_{k+1} \oplus_{z=1}^{k} (w_z\hbar_z) \oplus w_{k+1}\hbar_{k+1} \\
= & \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1 \\ (c_2,i_2,d_2)\in\hbar_2 \\ \cdots \\ (c_k,i_k,d_k)\in\hbar_k}} \left(\begin{array}{c} \sqrt[3]{1-\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \end{array} \right) \\
= & \bigcup_{(c_{k+1},i_{k+1},d_{k+1})\in\hbar_{k+1}} \left(\begin{array}{c} \sqrt[3]{1-\frac{1}{1+\left(w_{k+1}\left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(w_{k+1}\left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(w_{k+1}\left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \end{array} \right)
\end{aligned}
$$

$$
\begin{aligned}
& \oplus_{z=1}^{k+1}(w_z\hbar_z) \\
= & \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1 \\ (c_2,i_2,d_2)\in\hbar_2 \\ \cdots \\ (c_{k+1},i_{k+1},d_{k+1})\in\hbar_{k+1}}} \left(\begin{array}{c} \sqrt[3]{1-\frac{1}{1+\left(\sum_{z=1}^{k+1} w_z\left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k+1} w_z\left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k+1} w_z\left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \end{array} \right)
\end{aligned}
$$

Then, the theorem is valid for $z = k + 1$. Consequently, the theorem is established for all $z = \acute{N}$.

□

Example 6. *Let us consider* $\hbar(x_1) = \hbar_1 = \{(.8, .6, .4), (.5, .4, .9), (.2, .7, .6)\}$, $\hbar(x_2) = \hbar_2 = \{(.9, .3, .5), (.4, .2, .7)\}$ *and* $\hbar(x_3) = \hbar_3 = \{(.8, .4, .3)\}$ *when* $\gamma = 1$, *with weighted vector* $w = (.3, .2, .5)$, *we get*

$$
\begin{aligned}
& HCDFWAA(\hbar_1, \hbar_2, \hbar_3) \\
= & \oplus_{z=1}^{3}(w_z\hbar_z) \\
= & \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1 \\ (c_2,i_2,d_2)\in\hbar_2 \\ (c_3,i_3,d_3)\in\hbar_3}} \left(\begin{array}{c} \sqrt[3]{1-\frac{1}{1+\left(\sum_{z=1}^{3} w_z\left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{3} w_z\left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{3} w_z\left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \end{array} \right)
\end{aligned}
\tag{18}
$$

$$HCDFWAA(\hbar_1, \hbar_2, \hbar_3) = \left\{ \begin{array}{c} (.8337, .3920, .3429), (.7722, .3081, .3477), \\ (.8068, .3690, .3652), (.7163, .2988, .3714), \\ (.8020, .3961, .3596), (.7054, .3097, .3655) \end{array} \right\}$$

Theorem 2. *(Idempotency) Let $\hbar_k (k = 1, 2, ..., m)$ be a number of HCFNs. Then $\hbar_k = (c_k, i_k, d_k)(k = 1, 2, ..., m)$ be a number of HCFEs are all equal, i.e., $\hbar_k = h$ for all k, then HCDFWAA $(\hbar_1, \hbar_2, ..., \hbar_m) = \hbar$.*

3.4 Hesitant Cubical Dombi Fuzzy Weighted Geometric Averaging Operator

Definition 13. *Let $\hat{H}^m = \hbar_k = \{(s_{kj}, i_{kj}, d_{kj}) : 1 \leq j \leq \varrho_{\hbar_k}, k = 1, 2, ..., m\}$ be an m dimensional collection of HCFEs. A HCDFWGA Op is defined by a function HCDFWGA: $\hat{H}^m \to \hat{H}$ as given below:*

$$\begin{aligned} & HCDFWGA(\hbar_1, \hbar_2, ..., \hbar_m) \\ = & \bigotimes_{z=1}^{m} (\hbar_z^{w_z}) \\ = & (\hbar_1^{w_1}) \otimes (\hbar_2^{w_2}) \otimes ... \otimes (\hbar_m^{w_m}) \end{aligned} \tag{19}$$

where w_z is weight of $\hbar_z (z = 1, 2, ..., m), 0 \leq w_z \leq 1$ and $\sum_{z=1}^{m} w_z = 1$.

We derive the following theorem based on the application of Dombi operations to HCFEs:

Theorem 3. *Let $\hbar_k \in \hat{H}^m$. Then*

$$\begin{aligned} & HCDFWGA(\hbar_1, \hbar_2, ..., \hbar_m) \\ = & \bigotimes_{z=1}^{m} (\hbar_z^{w_z}) \\ = & \bigcup_{\substack{(c_1, i_1, d_1) \in \hbar_1 \\ (c_2, i_2, d_2) \in \hbar_2 \\ ... \\ (c_m, i_m, d_m) \in \hbar_m}} \left(\begin{array}{c} \sqrt[3]{\dfrac{1}{1 + \left(\sum_{z=1}^{m} w_z \left(\frac{1 - c_z^3}{c_z^3} \right)^{\gamma} \right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\dfrac{1}{1 + \left(\sum_{z=1}^{m} w_z \left(\frac{i_z^3}{1 - i_z^3} \right)^{\gamma} \right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{1 - \dfrac{1}{1 + \left(\sum_{z=1}^{m} w_z \left(\frac{d_z^3}{1 - d_z^3} \right)^{\gamma} \right)^{\frac{1}{\gamma}}}} \end{array} \right) \end{aligned}$$

where $w = (w_1, w_2, ..., w_m)$ be the m WV of $\hbar_k (k = 1, 2, ..., m)$, such that $w_k > 0$ and $\sum_{z=1}^{m} w_k = 1$.

Proof. The theorem can be proven using mathematical induction as given below:

1.

$$\hbar_1^{w_1} = \bigcup_{(c_1,i_1,d_1)\in\hbar_1} \left(\sqrt[3]{\frac{1}{1+\left(w_1\left(\frac{c_1^3}{1-c_1^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(w_1\left(\frac{1-i_1^3}{i_1^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{1-\frac{1}{1+\left(w_1\left(\frac{1-d_1^3}{d_1^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \right)$$

$$\hbar_2^{w_2} = \bigcup_{(c_2,i_2,d_2)\in\hbar_2} \left(\sqrt[3]{\frac{1}{1+\left(w_2\left(\frac{c_2^3}{1-c_2^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(w_2\left(\frac{1-i_2^3}{i_2^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{1-\frac{1}{1+\left(w_2\left(\frac{1-d_2^3}{d_2^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \right)$$

$$\hbar_1^{w_1} \otimes \hbar_2^{w_2} = \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1\\(c_2,i_2,d_2)\in\hbar_2}} \left(\sqrt[3]{\frac{1}{1+\left(w_1\left(\frac{c_1^3}{1-c_1^3}\right)^{\gamma}+w_2\left(\frac{c_2^3}{1-c_2^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(w_1\left(\frac{1-i_1^3}{i_1^3}\right)^{\gamma}+w_2\left(\frac{1-i_2^3}{i_2^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{1-\frac{1}{1+\left(w_1\left(\frac{1-d_1^3}{d_1^3}\right)^{\gamma}+w_2\left(\frac{1-d_2^3}{d_2^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \right)$$

$$\hbar_1^{w_1} \otimes \hbar_2^{w_2} = \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1\\(c_2,i_2,d_2)\in\hbar_2}} \left(\sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{2} w_z\left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{2} w_z\left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{1-\frac{1}{1+\left(\sum_{z=1}^{2} w_z\left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \right)$$

Therefore, the theorem is true for $m = 2$

2. Assume the theorem is valid for $z = k$, which means

$$\otimes_{z=1}^{k}(\hbar_z^{w_z}) = \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1\\(c_2,i_2,d_2)\in\hbar_2\\ \cdots \\(c_k,i_k,d_k)\in\hbar_k}} \left(\sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{1-\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \right)$$

When $z = k + 1$

$$\otimes_{z=1}^{k}(\hbar_z^{w_z}) \otimes \hbar_{k+1}^{w_{k+1}} = \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1\\(c_2,i_2,d_2)\in\hbar_2\\ \cdots \\(c_k,i_k,d_k)\in\hbar_k}} \left(\sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{1-\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \right)$$

$$\otimes\hbar_{k+1}^{w_{k+1}}\ \otimes_{z=1}^{k}(\hbar_z^{w_z}) \otimes \hbar_{k+1}^{w_{k+1}} = \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1\\(c_2,i_2,d_2)\in\hbar_2\\ \cdots \\(c_k,i_k,d_k)\in\hbar_k}} \left(\sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{1-\frac{1}{1+\left(\sum_{z=1}^{k} w_z\left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \right)$$

$$= \bigcup_{(c_{k+1},i_{k+1},d_{k+1})\in\hbar_{k+1}} \left(\sqrt[3]{\frac{1}{1+\left(w_{k+1}\left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{\frac{1}{1+\left(w_{k+1}\left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \sqrt[3]{1-\frac{1}{1+\left(w_{k+1}\left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \right)$$

$$\otimes_{z=1}^{k+1}(\hbar_z^{w_z})$$

$$= \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1\\(c_2,i_2,d_2)\in\hbar_2\\ \dots \\(c_{k+1},i_{k+1},d_{k+1})\in\hbar_{k+1}}} \left(\begin{array}{c} \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k+1} w_z\left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{k+1} w_z\left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{1-\frac{1}{1+\left(\sum_{z=1}^{k+1} w_z\left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \end{array} \right)$$

Hence, the theorem is proved for $z = k + 1$.

□

Example 7. *Let us consider* $\hbar(x_1) = \hbar_1 = \{(.8, .6, .4), (.5, .4, .9), (.2, .7, .6)\}$, $\hbar(x_2) = \hbar_2 = \{(.9, .3, .5), (.4, .2, .7)\}$ *and* $\hbar(x_3) = \hbar_3 = \{(.8, .4, .3)\}$ *when* $\gamma = 1$, *with weighted vector* $w = (.3, .2, .5)$, *we get*

$$HCDFWGA(\hbar_1, \hbar_2, \hbar_3) \tag{20}$$

$$= \otimes_{z=1}^{3}(\hbar_z^{w_z})$$

$$= \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1\\(c_2,i_2,d_2)\in\hbar_2\\(c_3,i_3,d_3)\in\hbar_3}} \left(\begin{array}{c} \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{3} w_z\left(\frac{c_z^3}{1-c_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{3} w_z\left(\frac{1-i_z^3}{i_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{1-\frac{1}{1+\left(\sum_{z=1}^{3} w_z\left(\frac{1-d_z^3}{d_z^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \end{array} \right)$$

$$HCDFWGA(\hbar_1, \hbar_2, \hbar_3)$$

$$= \left\{ \begin{array}{c} (.8166, .3920, .3899), (.5976, .3081, .4958), \\ (.6495, .3690, .7716, .5358), (.2988, .7833), \\ (.2956, .3961, .4809), (.2886, .3097, .5511) \end{array} \right\}$$

Theorem 4. *(Idempotency) Let* $\hbar_k (k = 1, 2, ..., m)$. *If* $(k = 1, 2, ..., m)$ *then HCDFWGA* $(\hbar_1, \hbar_2, ..., \hbar_m) = \hbar$.

Proof. Straightforward, thus the proof is omitted. □

3.5 Hesitant Cubical Dombi Fuzzy Ordered Weighted Arithmetic Averaging Operator

Definition 14. *Let* $\hat{H}^m = \hbar_k = \{(s_{kj}, i_{kj}, d_{kj}) : 1 \leq j \leq \varrho_{\hbar_k}, k = 1, 2, ..., m\}$ *be an m dimensional collection of HCFEs. A HCDFOWAA Op is defined by a function HCDFOWAA:* $\hat{H}^m \to \hat{H}$ *as given below:*

$$HCDFOWAA(\hbar_1, \hbar_2, ..., \hbar_m) \tag{21}$$

$$= \bigoplus_{z=1}^{m}(w_z\hbar_{\sigma(z)})$$

$$= (w_1\hbar_{\sigma(1)}) \oplus (w_2\hbar_{\sigma(2)}) \oplus ... \oplus (w_m\hbar_{\sigma(m)})$$

where $\hbar_z$ is the zth largest of $\hbar_z$ and w_z is WV of $\hbar_z(z = 1, 2, ..., m), 0 \leq w_z \leq 1$ and $\sum_{z=1}^{m} w_z = 1$.

Theorem 5. *Let $\hbar_k \in \hat{H}^m(k = 1, 2, ..., m)$. Then*

$$\begin{aligned} & HCDFOWAA(\hbar_1, \hbar_2, ..., \hbar_m) \\ = & \bigoplus_{z=1}^{m} w_z \hbar_{\sigma(z)} \\ = & \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1 \\ (c_2,i_2,d_2)\in\hbar_2 \\ \dots \\ (c_m,i_m,d_m)\in\hbar_m}} \left(\begin{array}{c} \sqrt[3]{1 - \frac{1}{1+\left(\sum_{z=1}^{m} w_z \left(\frac{c_{\sigma(z)}^3}{1-c_{\sigma(z)}^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{m} w_z \left(\frac{1-i_{\sigma(z)}^3}{i_{\sigma(z)}^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{m} w_z \left(\frac{1-d_{\sigma(z)}^3}{d_{\sigma(z)}^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \end{array} \right) \end{aligned}$$

where $\hbar_{\sigma(z)}$ is the zth largest of $\hbar_z$ and $w = (w_1, w_2, ..., w_m)$ be the m WV of $\hbar_k(k = 1, 2, ..., m)$, such that $0 < w_k < 1$ and $\sum_{z=1}^{m} w_k = 1$.

Proof. The proof is conducted in a manner similar to that of Theorem (1). □

3.6 Hesitant Cubical Dombi Fuzzy Ordered Weighted Geometric Averaging Operator

Definition 15. *Let $\hat{H}^m = \hbar_k = \{(s_{kj}, i_{kj}, d_{kj}) : 1 \leq j \leq \varrho_{\hbar_k}, k = 1, 2, ..., m\}$ be an m dimensional collection of HCFEs. A HCDFOWGA Op is defined by a function HCDFOWGA: $\hat{H}^m \to \hat{H}$ as given below:*

$$\begin{aligned} & HCDFOWGA(\hbar_1, \hbar_2, ..., \hbar_m) \\ = & \bigotimes_{z=1}^{m} (\hbar_{\sigma(z)}^{w_z}) \\ = & (\hbar_{\sigma(1)}^{w_1}) \otimes (\hbar_{\sigma(2)}^{w_2}) \otimes ... \otimes (\hbar_{\sigma(m)}^{w_m}) \end{aligned} \tag{22}$$

where w_z is WV of $\hbar_z(z = 1, 2, ..., m), 0 \leq w_z \leq 1$ and $\sum_{z=1}^{m} w_z = 1$.

Theorem 6. *Let $\hbar_k \in \hat{H}^m (k = 1, 2, ..., m)$. Then*

$$
\begin{aligned}
& HCDFOWGA(\hbar_1, \hbar_2, ..., \hbar_m) \\
= & \bigotimes_{z=1}^{m} (\hbar_{\sigma(z)}^{w_z}) \\
= & \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1 \\ (c_2,i_2,d_2)\in\hbar_2 \\ \dots \\ (c_m,i_m,d_m)\in\hbar_m}} \left(\begin{array}{c} \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{m} w_z \left(\frac{1-c_{\sigma(z)}^3}{c_{\sigma(z)}^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{m} w_z \left(\frac{1-i_{\sigma(z)}^3}{i_{\sigma(z)}^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{1-\frac{1}{1+\left(\sum_{z=1}^{m} w_z \left(\frac{d_{\sigma(z)}^3}{1-d_{\sigma(z)}^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \end{array} \right)
\end{aligned}
$$

where $\hbar_{\sigma(z)}$ is the zth largest of $\hbar_z$ and $w = (w_1, w_2, ..., w_m)$ be the m WV of $\hbar_k (k = 1, 2, ..., m)$, such that $0 < w_k < 1$ and $\sum_{z=1}^{m} w_k = 1$.

Example 8. *Let us consider $\hbar(x_1) = \hbar_1 = \{(.8, .6, .4), (.5, .4, .9), (.2, .7, .6)\}$, $\hbar(x_2) = \hbar_2 = \{(.9, .3, .5), (.4, .2, .7)\}$ and $\hbar(x_3) = \hbar_3 = \{(.8, .4, .3)\}$ when $\gamma = 1$, with weighted vector $w = (.3, .2, .5)$, we get*

$$
\begin{aligned}
& HCDFOWGA(\hbar_1, \hbar_2, ..., \hbar_m) \\
= & \bigotimes_{z=1}^{m} (\hbar_{\sigma(z)}^{w_z}) \\
= & \bigcup_{\substack{(c_1,i_1,d_1)\in\hbar_1 \\ (c_2,i_2,d_2)\in\hbar_2 \\ \dots \\ (c_m,i_m,d_m)\in\hbar_m}} \left(\begin{array}{c} \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{m} w_z \left(\frac{1-c_{\sigma(z)}^3}{c_{\sigma(z)}^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{\frac{1}{1+\left(\sum_{z=1}^{m} w_z \left(\frac{1-i_{\sigma(z)}^3}{i_{\sigma(z)}^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}}, \\ \sqrt[3]{1-\frac{1}{1+\left(\sum_{z=1}^{m} w_z \left(\frac{d_{\sigma(z)}^3}{1-d_{\sigma(z)}^3}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}} \end{array} \right)
\end{aligned}
\tag{23}
$$

Using (4), SVs of HCFEs are obtained as follows:

$SV(\hbar_1) = -0.1213$
$SV(\hbar_2) = 0.1625$
$SV(\hbar_3) = 0.485$
Here, $SV(\hbar_3) > SV(\hbar_2) > SV(\hbar_1)$ and
$\hbar_{\sigma(1)} = \hbar_3 = \{(.8, .4, .3)\}$
$\hbar_{\sigma(2)} = \hbar_2 = \{(.9, .3, .5), (.4, .2, .7)\}$
$\hbar_{\sigma(3)} = \hbar_1 = \{(.8, .6, .4), (.5, .4, .9), (.2, .7, .6)\}$

Then

$$
\begin{aligned}
& HCDFOWGA(\hbar_1, \hbar_2, \hbar_3) \\
= & \left\{ \begin{array}{l} (0.8166, 0.4110, 0.4049), (0.5904, 0.3690, 0.8341) \\ (0.2509, 0.4195, 0.5298), (0.5976, 0.3150, 0.5041) \\ (0.5062, 0.2990, 0.8402), (0.2472, 0.3179, 0.5851) \end{array} \right\}
\end{aligned}
$$

Theorem 7. *(Idempotency property) Let* $\hbar_k \in \hat{H}^m(k = 1, 2, ..., m)$. *If* $\hbar_k = \hbar$ *for* $(k = 1, 2, ..., m)$ *then* $HCDFOWGA(\hbar_1, \hbar_2, ..., \hbar_m) = \hbar$

Proof. Straightforward. □

4 Multiple Criteria Group Decision-Making Method Under HCF Information

In this section, we develop a MCGDM method. For convenience, we first provide a table of frequently used notations in Table 2.

Table 02: Notations in the "MCGDM" Method under HCF Information

$k = \{k_1, k_2, ..., k_l\}$	Set of alternatives
$\epsilon = \{\epsilon_1, \epsilon_2, ..., \epsilon_c\}$	Set of Criteria
$\partial = \{\partial_1, \partial_2, ..., \partial_t\}$	Set of decision-makers
D_{k_i}	For alternative k_i decision matrix
HCF_i	Collection of i column elements of D_{k_i} matrix
ζ_{yj}	Elements D_{k_i} corresponding to y row and j column
$\Im_i$	HCFE corresponding to alternative k_i

Let $k = \{k_1, k_2, ..., k_l\}$ be set of alternatives, $\epsilon = \{\epsilon_1, \epsilon_2, ..., \epsilon_c\}$ be a set of criteria and $\partial = \{\partial_1, \partial_2, ..., \partial_t\}$ be a set of decision-makers. Let us consider $w = (w_1, w_2, ..., w_3)$, such that $w_j \in (0, 1]$ and $\sum_{j=1}^{c} w_j = 1$ as the WV of the criteria which is determined by decision-makers.

The steps of the MCGDM method are as given below:

Step 1 The evaluation of the alternative k_i according to criteria j performed by decision-makers $\partial_y(y = 1, 2, ..., t)$ can be written as $\zeta_{yj}(i = 1, 2, ..., l; j = 1, 2, ..., c; y = 1, 2, ..., t)$. Hence, HCF decision matrix $DM_{k_i} = [\zeta_{yj}]_{t\times c}$ can be construted as follows:

$$
D_{k_i} = [\zeta_{yj}]_{t\times c} = \begin{pmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1c} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{t1} & \zeta_{t2} & \cdots & \zeta_{tc} \end{pmatrix} \tag{24}
$$

Step 2 For all $i = 1, 2, ..., l$. HCFS denoted by HCF_i is obtained as follows:

$$
HCF_i = \left\{ (\epsilon_j, \hbar_j) : j = 1, 2, ..., s \right\}. \tag{25}
$$

Here $\hbar_{\epsilon j} = \cup_{y=1}^{t} \{\zeta_{yj}\}$.

Step 3 For $k_i, i = 1, 2, ..., l$ HCF element related to k_i denoted by $\Im_i$, is defined as follows

$$\Im_i = \bigoplus_{j=1}^{c} w_j \hbar_{\epsilon j} \tag{26}$$

Step 4 Find SVs of $\Im_i (i = 1, 2, ..., l)$.
Step 5 Order SV $\Im_i (i = 1, 2, ..., l)$.
Step 6 Choose the alternative with the maximum SV.

5 A Real Life Case Study on the Selection of Green Supplier Chain Managmnet

Several government and non-government organizations have recently focused on promoting eco-friendly resources. As environmental concerns increase, many companies are working to develop green products or select green suppliers to enhance corporate performance while reducing pollution, emissions, and energy consumption. Decision-makers face multiple alternative providers when choosing the best green supplier, based on various criteria. In this section, we analyze a case study of a green supplier selection problem using several of the proposed methods.

In this problem, a manufacturer wants to select the best supplier from a list of five alternatives: $\{\Im_1, \Im_2, \Im_3, \Im_4, \Im_5\}$. Following that, a senior member of the organization assembled a group of three experts to oversee the supplier selection process. The alternatives are evaluated based on three criteria:

1. ϵ_1 : Represents the management system.
2. ϵ_2 : Represents the commitment of the manager to green supplier chain management (GSCM).
3. ϵ_3 : Represents the use of green technology.

Based on the information provided, we need to determine the best optimal solution in green supplier chain management. We have utilized three experts' evaluations under the HCFS framework, with three alternatives and five criteria. To aggregate the expert information, we employed the proposed aggregation Ops, such as HCDFWAA and HCDFWGA, using the WV $w = (0.3, 0.2, 0.5)^T$. Additionally, we applied all steps of our proposed algorithms to the green supplier chain management problem to achieve the optimal results. The computational results of all steps in our developed approach are as follows.

Table 03: Linguistic Variable Table for Evaluation of the Candidates

Grades	HCFNs
Extremely Lacking (EL)	(0.232, 0.721, 0.543)
Inadequate (I)	(0.121, 0.932, 0.071)
Below Average (BA)	(0.332, 0.398, 0.738)
Satisfactory (S)	(0.876, 0.324, 0.561)
Above Average (AA)	(0.032, 0.853, 0.223)
Competent (C)	(0.675, 0.542, 0.238)
Excellent (E)	(0.456, 0.233, 0.901)

Step 1 Experts evaluate the alternatives using HCFNs corresponding to the linguistic variables listed in Table 3 for each criterion.

Table 04

		ϵ_1	ϵ_2	ϵ_3
D_{k1}	∂_1	(0.232, 0.721, 0.543)	(0.876, 0.324, 0.561)	(0.456, 0.233, 0.901)
	∂_2	(0.121, 0.932, 0.071)	$*$	(0.232, 0.721, 0.543)
	∂_3	$*$	(0.121, 0.932, 0.071)	(0.456, 0.233, 0.901)
D_{k2}	∂_1	(0.032, 0.853, 0.223)	(0.876, 0.324, 0.561)	$*$
	∂_2	(0.232, 0.721, 0.543)	(0.456, 0.233, 0.901)	(0.032, 0.853, 0.223)
	∂_3	(0.876, 0.324, 0.561)	$*$	(0.332, 0.398, 0.738)
D_{k3}	∂_1	(0.232, 0.721, 0.543)	(0.456, 0.233, 0.901)	$*$
	∂_2	$*$	(0.121, 0.932, 0.071)	(0.876, 0.324, 0.561)
	∂_3	(0.332, 0.398, 0.738)	(0.121, 0.932, 0.071)	(0.876, 0.324, 0.561)
D_{k4}	∂_1	(0.121, 0.932, 0.071)	(0.675, 0.542, 0.238)	(0.332, 0.398, 0.738)
	∂_2	(0.232, 0.721, 0.543)	$*$	(0.121, 0.932, 0.071)
	∂_3	(0.121, 0.932, 0.071)	$*$	(0.456, 0.233, 0.901)
D_{k5}	∂_1	$*$	(0.232, 0.721, 0.543)	(0.675, 0.542, 0.238)
	∂_2	(0.032, 0.853, 0.223)	$*$	(0.232, 0.721, 0.543)
	∂_3	(0.675, 0.542, 0.238)	(0.456, 0.233, 0.901)	(0.232, 0.721, 0.543)

Step 2 Using HCF decision matrices given in Step 1, HCF$_i$ ($i = 1, 2, ..., 5$)

$$HCF_1 = \left\{ \begin{array}{c} (\epsilon_1, \{(0.232, 0.721, 0.543), (0.121, 0.932, 0.071)\}), \\ (\epsilon_2, \{(0.876, 0.324, 0.561), (0.121, 0.932, 0.071)\}), \\ (\epsilon_3, \{(0.456, 0.233, 0.901), (0.232, 0.721, 0.543), (0.456, 0.233, 0.901)\}) \end{array} \right\},$$

$$HCF_2 = \left\{ \begin{array}{c} (\epsilon_1, \{(0.032, 0.853, 0.223), (0.232, 0.721, 0.543), (0.876, 0.324, 0.561)\}), \\ (\epsilon_2, \{(0.876, 0.324, 0.561), (0.456, 0.233, 0.901)\}), \\ (\epsilon_3, \{(0.032, 0.853, 0.223), (0.332, 0.398, 0.738)\}) \end{array} \right\},$$

$$HCF_3 = \left\{ \begin{array}{c} (\epsilon_1, \{(0.232, 0.721, 0.543), (0.332, 0.398, 0.738)\}), \\ (\epsilon_2, \{(0.456, 0.233, 0.901), (0.121, 0.932, 0.071), (0.121, 0.932, 0.071)\}), \\ (\epsilon_3, \{(0.876, 0.324, 0.561), (0.876, 0.324, 0.561)\}) \end{array} \right\},$$

$$HCF_4 = \left\{ \begin{array}{c} (\epsilon_1, \{(0.121, 0.932, 0.071), (0.232, 0.721, 0.543), (0.121, 0.932, 0.071)\}), \\ (\epsilon_2, \{(0.675, 0.542, 0.238)\}), \\ (\epsilon_3, \{(0.332, 0.398, 0.738), (0.121, 0.932, 0.071), (0.456, 0.233, 0.901)\}) \end{array} \right\},$$

$$HCF_5 = \left\{ \begin{array}{c} (\epsilon_1, \{(0.032, 0.853, 0.223), (0.675, 0.542, 0.238)\}), \\ (\epsilon_2, \{(0.232, 0.721, 0.543), (0.456, 0.233, 0.901)\}), \\ (\epsilon_3, \{(0.675, 0.542, 0.238), (0.232, 0.721, 0.543), (0.232, 0.721, 0.543)\}) \end{array} \right\}.$$

Step 3 For $\gamma = 1$, HCDFWAA values of HCF$_i$ ($i = 1, 2, ..., 5$) and HCDFWGA values of HCF$_i$ ($i = 1, 2, ..., 5$) are obtained as in Table 05 and Table 06 respectively.

Table 05; HCDFWAA values of HCF_i (i = 1, 2, ..., 5)

	HCDFWAA
$\Im_1$	$\left\{\begin{matrix} (0.6826, 0.2787, 0.6471), (0.6664, 0.4997, 0.5464), (0.6826, 0.2787, 0.6471), \\ (0.3768, 0.2910, 0.12124), (0.2180, 0.7489, 0.1210), (0.3768, 0.2910, 0.1212), \\ (0.6815, 0.2796, 0.1060), (0.6652, 0.5090, 0.1059), (0.6815, 0.2796, 0.1060), \\ (0.3698, 0.2921, 0.0895), (0.1928, 0.8002, 0.0894), (0.3698, 0.2921, 0.0895) \end{matrix}\right\}$
$\Im_2$	$\left\{\begin{matrix} (0.6626, 0.5186, 0.2390), (0.6697, 0.4121, 0.324), (0.2739, 0.3882, 0.2400), \\ (0.3375, 0.3456, 0.3271), (0.6640, 0.5112, 0.2750), (0.6710, 0.4091, 0.6173), \\ (0.2891, 0.3858, 0.0403), (0.3474, 0.3441, 0.6656), (0.7970, 0.4011, 0.2754), \\ (0.7994, 0.3536, 0.6261), (0.7299, 0.3401, 0.0403), (0.7343, 0.3132, 0.6775) \end{matrix}\right\}$
$\Im_3$	$\left\{\begin{matrix} (0.8001, 0.3173, 0.5856), (0.8001, 0.3173, 0.5856), (0.7976, 0.3990, 0.1211), \\ (0.7976, 0.3990, 0.1211), (0.7976, 0.3990, 0.1211), (0.7976, 0.3990, 0.1211), \\ (0.8011, 0.3050, 0.6380), (0.8011, 0.3050, 0.6380), (0.7985, 0.3703, 0.1213), \\ (0.7985, 0.3703, 0.1212), (0.7985, 0.3703, 0.1212), (0.7985, 0.3703, 0.1212) \end{matrix}\right\}$
$\Im_4$	$\left\{\begin{matrix} (0.4607, 0.4712, 0.1054), (0.4359, 0.7638, 0.0763), (0.4989, 0.2897, 0.1054), \\ (0.4648, 0.4644, 0.3820), (0.4406, 0.7205, 0.0891), (0.5022, 0.2887, 0.3860), \\ (0.4607, 0.4712, 0.1054), (0.4359, 0.7638, 0.0763), (0.4989, 0.2897, 0.1054) \end{matrix}\right\}$
$\Im_5$	$\left\{\begin{matrix} (0.5682, 0.6219, 0.2483), (0.2064, 0.7521, 0.3169), (0.2064, 0.7521, 0.3169), \\ (0.5804, 0.3720, 0.2495), (0.2984, 0.3843, 0.3203), (0.2984, 0.3843, 0.3203), \\ (0.6411, 0.5646, 0.2546), (0.4993, 0.6436, 0.3349), (0.4993, 0.6436, 0.3349), \\ (0.6491, 0.3634, 0.2560), (0.5172, 0.3746, 0.3391), (0.5172, 0.3746, 0.3391) \end{matrix}\right\}$

Table 06; HCDFWGA values of HCF$_i$ (i = 1, 2, ..., 5)	
	HCDFWGA
$\Im_1$	$\left\{\begin{array}{c}(0.3233, 0.2787, 0.8405), (0.2496, 0.4997, 0.5468), (0.3233, 0.2787, 0.8405),\\ (0.19162, 0.2910, 0.8371), (0.1781, 0.7489, 0.5097), (0.1916, 0.2910, 0.8371),\\ (0.1788, 0.2796, 0.8360), (0.1684, 0.5090, 0.4953), (0.1788, 0.2796, 0.8360),\\ (0.1515, 0.2921, 0.8323), (0.1459, 0.8002, 0.4434), (0.1515, 0.2921, 0.8323)\end{array}\right\}$
$\Im_2$	$\left\{\begin{array}{c}(0.0345, 0.5186, 0.3667), (0.0478, 0.412, 0.6516), (0.0345, 0.388, 0.7090),\\ (0.0478, 0.3456, 0.7771), (0.0403, 0.5112, 0.4572), (0.2975, 0.4091, 0.6722),\\ (0.0403, 0.3858, 0.7230), (0.2929, 0.3441, 0.7852), (0.0403, 0.4011, 0.4663),\\ (0.4110, 0.3536, 0.6747), (0.0403, 0.3401, 0.7248), (0.3951, 0.3132, 0.7862)\end{array}\right\}$
$\Im_3$	$\left\{\begin{array}{c}(0.3339, 0.3173, 0.7459), (0.3339, 0.3173, 0.7459), (0.1937, 0.3990, 0.5208),\\ (0.1937, 0.3990, 0.5208), (0.1937, 0.3990, 0.5208), (0.1937, 0.3990, 0.5208),\\ (0.4490, 0.3050, 0.7722), (0.4490, 0.3050, 0.7722), (0.2017, 0.3703, 0.6180),\\ (0.2017, 0.3703, 0.6180), (0.2017, 0.3703, 0.6180), (0.2017, 0.3703, 0.6180)\end{array}\right\}$
$\Im_4$	$\left\{\begin{array}{c}(0.1760, 0.4712, 0.6326), (0.1303, 0.7638, 0.1444), (0.1787, 0.2897, 0.8326),\\ (0.2966, 0.4644, 0.6571), (0.1483, 0.7205, 0.3842), (0.3221, 0.2887, 0.8373),\\ (0.1760, 0.4712, 0.6326), (0.1303, 0.7638, 0.1444), (0.1787, 0.2897, 0.8326)\end{array}\right\}$
$\Im_5$	$\left\{\begin{array}{c}(0.0478, 0.6219, 0.3586), (0.0477, 0.7521, 0.4937), (0.0477, 0.7521, 0.4937),\\ (0.0479, 0.3720, 0.7093), (0.0477, 0.3843, 0.7316), (0.0477, 0.3843, 0.7316),\\ (0.3773, 0.5646, 0.3603), (0.2598, 0.6436, 0.4945), (0.2598, 0.6436, 0.4945),\\ (0.5966, 0.3634, 0.7095), (0.2852, 0.3746, 0.7317), (0.2852, 0.3746, 0.7317)\end{array}\right\}$

Step 04 SVs of $\Im_1$, (i = 1, 2, ..., 5) under score function are obtained as in Table 07.

Table 07; SV of $\Im_1$ according to HCDFWAA and HCDFWGA values					
	$\partial\vartheta(\Im_1)$	$\partial\vartheta(\Im_2)$	$\partial\vartheta(\Im_3)$	$\partial\vartheta(\Im_4)$	$\partial\vartheta(\Im_5)$
HCDFWAA	0.1140	0.1573	0.4318	0.0892	0.0971
HCDFWGA	−0.42133	−0.2920	−0.2451	−0.2062	−0.2089

Step 05 Using Eq (1), the ordering of the candidates is determined and presented in Table 08.

Table 08; SV of $\Im_1$ obtained using HCDFWAA and HCDFWGA Ops	
	Ranking
HCDFWAA	$\partial\vartheta(\Im_3) > \partial\vartheta(\Im_2) > \partial\vartheta(\Im_1) > \partial\vartheta(\Im_4) > \partial\vartheta(\Im_5)$
HCDFWGA	$\partial\vartheta(\Im_4) > \partial\vartheta(\Im_5) > \partial\vartheta(\Im_3) > \partial\vartheta(\Im_2) > \partial\vartheta(\Im_1)$

Step 06 From the above illustration, although the overall ranking values of the alternatives differ depending on the Op used, the optimal alternatives are $\Im_3$ and $\Im_5$ for the two Ops, respectively.

6 Comparison Analysis with Existing Method

The suggested approaches for MCGDM are validated and compared with current methods, as shown in Table 09.

Considering the weight of criteria as $w = (0.3, 0.2, 0.5)^T$, and applying the TOPSIS method [38] and the GRA method [39] , we obtain the ranking of alternatives. Table 09 shows that the best alternative identified by the proposed approaches matches the top alternative from both the TOPSIS method [38] and the GRA method [39] , demonstrating the viability of the proposed DM methods.

Table 09: Comparative Study and Ranking of the Alternatives

	$\partial\vartheta(\Im_1)$	$\partial\vartheta(\Im_2)$	$\partial\vartheta(\Im_3)$	$\partial\vartheta(\Im_4)$	$\partial\vartheta(\Im_5)$
HCDFWAA	0.1140	0.1573	0.4318	0.0892	0.0971
HCDFWGA	−0.42133	−0.2920	−0.2451	−0.2062	−0.2089
Topsis [38]	0.1020	0.1332	0.3453	0.0437	0.0874
GRA [39]	0.1003	0.1235	0.2437	0.0543	0.0784

	Ranking
HCDFWAA	$\partial\vartheta(\Im_3) > \partial\vartheta(\Im_2) > \partial\vartheta(\Im_1) > \partial\vartheta(\Im_4) > \partial\vartheta(\Im_5)$
HCDFWGA	$\partial\vartheta(\Im_4) > \partial\vartheta(\Im_5) > \partial\vartheta(\Im_3) > \partial\vartheta(\Im_2) > \partial\vartheta(\Im_1)$
Topsis [38]	$\partial\vartheta(\Im_3) > \partial\vartheta(\Im_2) > \partial\vartheta(\Im_1) > \partial\vartheta(\Im_4) > \partial\vartheta(\Im_5)$
GRA [39]	$\partial\vartheta(\Im_3) > \partial\vartheta(\Im_2) > \partial\vartheta(\Im_1) > \partial\vartheta(\Im_4) > \partial\vartheta(\Im_5)$

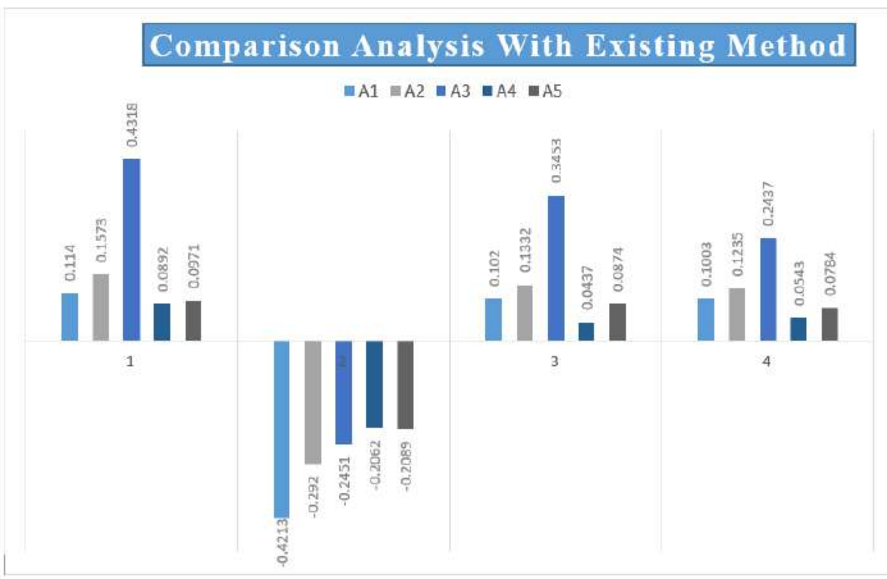

Figure 2. Graphical presentation of proposed method.

7 Analysis of the Effect of Parameter γ on the Results

To demonstrate the effect of the parameter γ in the HCDFWAA and HCDFWGA formulas on MCGDM results, we assign different values to γ ranging from 1 to 10 and rank the candidates based on their scores using HCDFWAA and HCDFWGA. The ranking orders of the candidates according to their scores and their rankings based on the HCDFWAA and HCDFWGA Ops are shown in Table 10. It is evident that, when γ varies in the HCDFWAA formula, the optimal candidate remains the same, and the ordering of candidates ($\partial\vartheta(\Im_3) > \partial\vartheta(\Im_2) > \partial\vartheta(\Im_1) > \partial\vartheta(\Im_5) > \partial\vartheta(\Im_4)$) remains consistent, except when $\gamma = 1$. According to Table 11, when γ is adjusted for the HCDFWGA Op, the ranking orders of the candidates remain consistent,

$(\partial\vartheta(\Im_4) > \partial\vartheta(\Im_5) > \partial\vartheta(\Im_3) > \partial\vartheta(\Im_2) > \partial\vartheta(\Im_1))$ except when $\gamma = 1$. Additionally, the best suitable candidate for $1 \leq \gamma \leq 10$.

The graphical representation of Table 10 is shown in Fig 3.

The graphical representation of Table 11 is shown in Fig 4.

Using the HCDFWAA Op and score function, we identify alternative $\Im_3$ as the optimum element due to its maximum SV. Conversely, using the HCDFWGA Op and the score function of HCFEs, alternative $\Im_4$ has the highest SV. This shows that different aggregation Ops yield different alternatives with the maximum SVs. In Table 07, for $\gamma = 2, 3, 4, ..., 10$, alternative $\Im_3$ consistently has the minimum SV. In the HCDFWGA Op, the third component of a CFE is derived using the Dombi t-conorm, which negatively impacts the SV. Thus, alternative $\Im_3$, which has the minimum score, can be considered the optimum element. Furthermore, this relationship can be represented as $1 - S\Im_i$ for the SV obtained using the HCDFWGA Op.

Table 10; Ranking Order for Different γ Values in the HCDFWAA Op

γ	$\partial\vartheta(\Im_1)$	$\partial\vartheta(\Im_2)$	$\partial\vartheta(\Im_3)$	$\partial\vartheta(\Im_4)$	$\partial\vartheta(\Im_5)$	*Ranking*
1	0.1140	0.1573	0.4318	0.0892	0.0971	$\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)>\partial\vartheta(\Im_5)>\partial\vartheta(\Im_4)$
2	0.2373	0.3179	0.5417	0.1839	0.1707	$\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)>\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)$
3	0.2790	0.3711	0.5762	0.2262	0.1989	$\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)>\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)$
4	0.2994	0.3970	0.5930	0.2497	0.2138	$\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)>\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)$
5	0.3114	0.4124	0.6029	0.2645	0.2230	$\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)>\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)$
6	0.3194	0.4226	0.6094	0.2746	0.2293	$\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)>\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)$
7	0.3250	0.4297	0.6140	0.2820	0.2338	$\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)>\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)$
8	0.3292	0.4351	0.6175	0.2876	0.2372	$\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)>\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)$
9	0.3324	0.4393	0.6201	0.2920	0.2399	$\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)>\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)$
10	0.3350	0.4426	0.6223	0.2956	0.2421	$\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)>\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)$

Table 11; Ranking Order for Different γ Values in the HCDFWGA Op

γ	$\partial\vartheta(\Im_1)$	$\partial\vartheta(\Im_2)$	$\partial\vartheta(\Im_3)$	$\partial\vartheta(\Im_4)$	$\partial\vartheta(\Im_5)$	*Ranking*
1	−0.4213	−0.2920	−0.2451	−0.2062	−0.2089	$\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)>\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)$
2	−0.5026	−0.4265	−0.3729	−0.3409	−0.3442	$\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)>\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)$
3	−0.5298	−0.4733	−0.4164	−0.3625	−0.3913	$\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)>\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)$
4	−0.5435	−0.4967	−0.4381	−0.3733	−0.4149	$\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)>\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)$
5	−0.5517	−0.5106	−0.4510	−0.3799	−0.4291	$\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)>\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)$
6	−0.5572	−0.5199	−0.4596	−0.3843	−0.4384	$\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)>\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)$
7	−0.5611	−0.5265	−0.4657	−0.3875	−0.4451	$\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)>\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)$
8	−0.5640	−0.5314	−0.4703	−0.3899	−0.4501	$\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)>\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)$
9	−0.5663	−0.5352	−0.4739	−0.3917	−0.4540	$\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)>\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)$
10	−0.5682	−0.53823	−0.4767	−0.3932	−0.4571	$\partial\vartheta(\Im_4)>\partial\vartheta(\Im_5)>\partial\vartheta(\Im_3)>\partial\vartheta(\Im_2)>\partial\vartheta(\Im_1)$

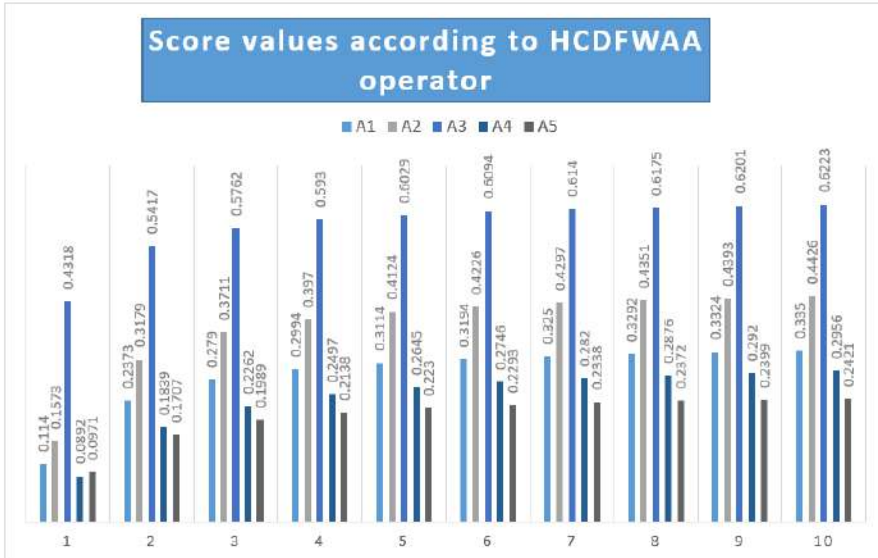

Figure 3. Fig. 3 Ranking Order for Different γ Values in the HCDFWAA Op

8 Comparative Analysis and Discussion

In this section, we compare HCFS with other extensions of the fuzzy sets. Let $T_H = (x, \{(c_k, i_k, d_k) : 1 \leq k \leq lx\}) : x \in \chi$. The comparison table is presented in Table 12.

Table 12: Comparison Table of HCFS with Various Extensions of Fuzzy Sets

	n(degrees of component)			k(length)		
	1	2	>2	1	>1	Condition
Fuzzy set [1]	T	F	F	T	F	$0 \leq c_k + d_k = 1, i_k = 0$
Intuitionistic fuzzy set [4]	T	F	F	T	F	$c_k + i_k + d_k = 1$
Pythagorean fuzzy set [5]	T	T	F	T	F	$0 \leq c_k^2 + d_k^2 \leq 1$
Picture fuzzy set [7]	T	F	F	T	F	$0 \leq c_k + i_k + d_k \leq 1$
Sperical fuzzy set [20]	T	T	F	T	F	$0 \leq c_k^2 + i_k^2 + d_k^2 \leq 1$
q-rung orthopair fuzzy set [6]	T	T	T	T	F	$0 \leq c_k^n + d_k^n \leq 1$
Cubical fuzzy set [40]	T	T	T	T	F	$0 \leq c_k^3 + i_k^3 + d_k^3 \leq 1$
Hesistant fuzzy set [23, 24]	T	F	F	T	T	$0 \leq c_k + d_k = 1, i_k = 1$
Intuitionistic hesitant fuzzy set [43]	T	F	F	T	T	$0 \leq c_k + d_k = 1$
Hesitant Pythagorean fuzzy set [44]	T	T	F	T	T	$0 \leq c_k^2 + d_k^2 \leq 1$
q-rung Orthopair hesitant fuzzy set [45]	T	T	T	T	T	$0 \leq c_k^n + d_k^n \leq 1$
Picture hesitant fuzzy set [36]	T	T	F	T	T	$0 \leq c_k + i_k + d_k \leq 1$
Hesitant Cubical fuzzy set	T	T	T	T	T	$0 \leq c_k^3 + i_k^3 + d_k^3 \leq 1$

Here, we observe that HCFS extends the set structures specified in Table 12. Consequently, the set structure defined in this paper combines the advantages of the other fuzzy set extensions listed in Table 12. It also addresses certain problems that existing set theories cannot model. The following example illustrates the relationship between these sets.

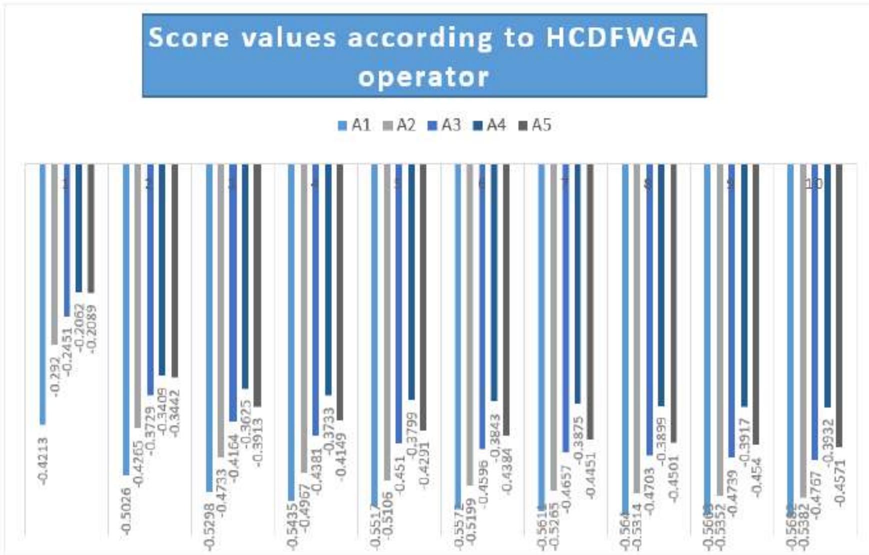

Figure 4. Fig. 4 Ranking Order for Different Different γ Values in the HCDFWGA Op

9 Conclusion

In this paper, we introduced the concept of hesitant cubical sets and defined their set-theoretical operations, such as union, intersection, and complement, with illustrative examples. We explored arithmetic operations between two HCFEs using Dombi t-norm and t-conorm, and presented several aggregation Ops, including HCDFWAA, HCDFWGA, HCDFOWAA, and HCDFOWGA.

We also developed a MCGDM method and applied it to a problem of selecting a candidate for a position in green supplier chain management. The results were analyzed and compared across various parameters. Additionally, we compared our proposed set structure with other fuzzy set extensions and highlighted its advantages.

Future research will focus on investigating additional aggregation Ops, similarity measures, distance measures, and decision-making methods based on AHP, VIKOR, TOPSIS and other techniques.

10 Limitations and Future Directions

The Hesitant Cubical Fuzzy Set (HCFS) framework, while innovative, has several limitations. Its complexity and computational overhead can hinder real-time decision-making. Scalability issues arise as the number of decision-makers and criteria increase, impacting efficiency. The framework's reliance on subjective judgments introduces potential biases, and integrating it into existing systems can be challenging without specialized knowledge. Current validation is limited to a single case study, necessitating broader experimental validation across diverse domains.

Future research should aim to develop more efficient algorithms to reduce computational complexity and investigate scalable solutions for larger datasets. Extensive validation in various fields like healthcare and finance is essential to confirm versatility. Automated decision support systems using machine learning can enhance objectivity, while user-friendly tools and software can promote wider adoption. Integrating real-time data sources can improve responsiveness, and hybrid approaches combining HCFS with other techniques like neural networks can address complex decision-making challenges. Addressing these limitations and exploring these directions will refine the HCFS framework for better decision-making in complex environments.

Author Contributions

Asghar khan: Conceptualization, Methodolog, Data curation, Original draft preparation. , Visualization, Investigation, Supervision.: **Tahir Zaman**: Software, Validation , Writing- Reviewing and Editing.

Compliance with Ethical Standards

It is declare that all authors don't have any conflict of interest. It is also declare that this article does not contain any studies with human participants or animals performed by any of the authors. Furthermore, informed consent was obtained from all individual participants included in the study.

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References

[1] Zadeh, L. A. (1965). Fuzzy sets. Information and control, 8(3), 338-353.

[2] Gadekallu, T. R., & Gao, X. Z. (2021). An efficient attribute reduction and fuzzy logic classifier for heart disease and diabetes prediction. Recent Advances in Computer Science and Communications (Formerly: Recent Patents on Computer Science), 14(1), 158-165.

[3] Sakthidasan, K., Gao, X. Z., Devabalaji, K. R., & Roopa, Y. M. (2021). Energy based random repeat trust computation approach and Reliable Fuzzy and Heuristic Ant Colony mechanism for improving QoS in WSN. Energy Reports, 7, 7967-7976.

[4] Atanassov, K. T., & Atanassov, K. T. (1999). Intuitionistic fuzzy sets (pp. 1-137). Physica-Verlag HD.

[5] Yager, R. R. (2013, June). Pythagorean fuzzy subsets. In 2013 joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS) (pp. 57-61). IEEE.

[6] Yager, R. R. (2013). Pythagorean membership grades in multicriteria decision making. IEEE Transactions on fuzzy systems, 22(4), 958-965.

[7] Cuong, B. C. (2013). Picture fuzzy sets-first results. part 1, seminar neuro-fuzzy systems with applications. Institute of Mathematics, Hanoi.

[8] Cuong, B. C. (2013). Picture fuzzy sets-first results. part 1, seminar neuro-fuzzy systems with applications. Institute of Mathematics, Hanoi.

[9] Garg, H. (2017). Some picture fuzzy aggregation operators and their applications to multicriteria decision-making. Arabian Journal for Science and Engineering, 42(12), 5275-5290.

[10] Peng, X., & Dai, J. (2017). Algorithm for picture fuzzy multiple attribute decision-making based on new distance measure. International Journal for Uncertainty Quantification, 7(2).

[11] Wei, G. (2017). Some cosine similarity measures for picture fuzzy sets and their applications to strategic decision making. Informatica, 28(3), 547-564.

[12] Wei, G., & Gao, H. (2018). The generalized Dice similarity measures for picture fuzzy sets and their applications. Informatica, 29(1), 107-124.

[13] Wei, Guiwu. "Some similarity measures for picture fuzzy sets and their applications." Iranian Journal of Fuzzy Systems 15, no. 1 (2018): 77-89.

[14] Rafiq, M., Ashraf, S., Abdullah, S., Mahmood, T., & Muhammad, S. (2019). The cosine similarity measures of spherical fuzzy sets and their applications in decision making. Journal of Intelligent & Fuzzy Systems, 36(6), 6059-6073.

[15] Thao, N. X. (2020). Similarity measures of picture fuzzy sets based on entropy and their application in MCDM. Pattern analysis and applications, 23, 1203-1213.

[16] Singh, P. (2015). Correlation coefficients for picture fuzzy sets. Journal of Intelligent & Fuzzy Systems, 28(2), 591-604.

[17] Ganie, A. H., Singh, S., & Bhatia, P. K. (2020). Some new correlation coefficients of picture fuzzy sets with applications. Neural Computing and Applications, 32(16), 12609-12625.

[18] Son, L. H. (2016). Generalized picture distance measure and applications to picture fuzzy clustering. Applied Soft Computing, 46(C), 284-295.

[19] HÃ³a, Ä. Ã. (2016). Some improvements of fuzzy clustering algorithms using picture fuzzy sets and applications for geographic data clustering. VNU Journal of Science: Computer Science and Communication Engineering, 32(3).

[20] Kutlu Gündoğdu, F., & Kahraman, C. (2019). Spherical fuzzy sets and spherical fuzzy TOPSIS method. Journal of intelligent & fuzzy systems, 36(1), 337-352.

[21] Kutlu Gündoğdu, F., & Kahraman, C. (2020). Spherical fuzzy sets and decision making applications. In Intelligent and Fuzzy Techniques in Big Data Analytics and Decision Making: Proceedings of the INFUS 2019 Conference, Istanbul, Turkey, July 23-25, 2019 (pp. 979-987). Springer International Publishing.

[22] Khan, A., Jan, A. U., Amin, F., & Zeb, A. (2022). Multiple attribute decision-making based on cubical fuzzy aggregation operators. Granular Computing, 1-18.

[23] Torra, V., & Narukawa, Y. (2009, August). On hesitant fuzzy sets and decision. In 2009 IEEE international conference on fuzzy systems (pp. 1378-1382). IEEE.

[24] Torra, V. (2010). Hesitant fuzzy sets. International journal of intelligent systems, 25(6), 529-539.

[25] Xu, Z., & Xia, M. (2011). On distance and correlation measures of hesitant fuzzy information. International Journal of Intelligent Systems, 26(5), 410-425.

[26] Li, D., Zeng, W., & Li, J. (2015). New distance and similarity measures on hesitant fuzzy sets and their applications in multiple criteria decision making. Engineering applications of artificial intelligence, 40, 11-16.

[27] Xia, M., Xu, Z., & Chen, N. (2013). Some hesitant fuzzy aggregation operators with their application in group decision making. Group Decision and Negotiation, 22, 259-279.

[28] Chen, N., Xu, Z., & Xia, M. (2013). Interval-valued hesitant preference relations and their applications to group decision making. Knowledge-based systems, 37, 528-540.

[29] Peng, D. H., Wang, T. D., Gao, C. Y., & Wang, H. (2014). Continuous Hesitant Fuzzy Aggregation Operators and Their Application to Decision Making under Interval-Valued Hesitant Fuzzy Setting. The Scientific World Journal, 2014(1), 897304.

[30] Mu, Z., Zeng, S., & Baležentis, T. (2015). A novel aggregation principle for hesitant fuzzy elements. Knowledge-Based Systems, 84, 134-143.

[31] Amin, F., Fahmi, A., Abdullah, S., Ali, A., Ahmad, R., & Ghani, F. (2018). Triangular cubic linguistic hesitant fuzzy aggregation operators and their application in group decision making. Journal of intelligent & fuzzy systems, 34(4), 2401-2416.

[32] Fahmi, A., Abdullah, S., Amin, F., Ali, A., Ahmed, R., & Shakeel, M. (2019). Trapezoidal cubic hesitant fuzzy aggregation operators and their application in group decision-making. Journal of Intelligent & Fuzzy Systems, 36(4), 3619-3635.

[33] Jiang, C., Jiang, S., & Chen, J. (2019). Interval-valued dual hesitant fuzzy Hamacher aggregation operators for multiple attribute decision making. Journal of Systems Science and Information, 7(3), 227-256.

[34] Liu, H. B., Liu, Y., & Xu, L. (2020). Dombi Interval-Valued Hesitant Fuzzy Aggregation Operators for Information Security Risk Assessment. Mathematical Problems in Engineering, 2020(1), 3198645.

[35] Zeng, W., Xi, Y., Yin, Q., & Guo, P. (2021). Weighted dual hesitant fuzzy set and its application in group decision making. Neurocomputing, 458, 714-726.

[36] Wang, R., & Li, Y. (2018). Picture hesitant fuzzy set and its application to multiple criteria decision-making. Symmetry, 10(7), 295.

[37] Liang, D., Darko, A. P., Xu, Z., & Wang, M. (2019). Aggregation of dual hesitant fuzzy heterogenous related information with extended Bonferroni mean and its application to MULTIMOORA. Computers & Industrial Engineering, 135, 156-176.

[38] Liu, S., Hu, Y., Zhang, X., Li, Y., & Liu, L. (2020). Blockchain service provider selection based on an integrated BWM-entropy-TOPSIS method under an intuitionistic fuzzy environment. IEEE Access, 8, 104148-104164.

[39] Wei, G. W. (2010). GRA method for multiple attribute decision making with incomplete weight information in intuitionistic fuzzy setting. Knowledge-Based Systems, 23(3), 243-247.

[40] Khan, A., Jan, A. U., Amin, F., & Zeb, A. (2022). Multiple attribute decision-making based on cubical fuzzy aggregation operators. Granular Computing, 1-18.

[41] Yager, R. R. (1988). On ordered weighted averaging aggregation operators in multicriteria decision-making. IEEE Transactions on systems, Man, and Cybernetics, 18(1), 183-190.

[42] Dombi, J. (1982). A general class of fuzzy operators, the DeMorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. Fuzzy sets and systems, 8(2), 149-163.

[43] Chen, X., Li, J., Qian, L., & Hu, X. (2016, January). Distance and similarity measures for intuitionistic hesitant fuzzy sets. In 2016 International Conference on Artificial Intelligence: Technologies and Applications (pp. 182-186). Atlantis Press.

[44] Garg, H. (2018). Hesitant Pythagorean fuzzy sets and their aggregation operators in multiple attribute decision-making. International Journal for Uncertainty Quantification, 8(3).

[45] Yang, W., & Pang, Y. (2020). New q-rung orthopair hesitant fuzzy decision making based on linear programming and TOPSIS. IEEE Access, 8, 221299-221311.