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A study of coefficient-related problems for symmetric starlike functions connected with a tan hyperbolic function

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Abstract The article aims to determine the sharp bounds of coefficients, Fekete-Szegö, Zalcman inequalities for the family $\mathcal{SS}^{*}_{\tanh}$ of starlike function with respect to symmetric points linked with tan hyperbolic function. We also estimate determinant of $\left|\mathcal{H}_{2,2}\left(f\right)\right|$ is also obtained for the same class. Further, we study the logarithmic and inverse coefficients for the same class.

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1 Introduction

Let $\mathbb{U}_d = \left\{z \in \mathbb{C} : |z| < 1\right\}$ denotes the open unit disc and the symbol $\mathcal{A}$ indicates the holomorphic (analytic) functions class normalized by $f(0) = f'(0) - 1 = 0$. It signifies that $f \in \mathcal{A}$ has Taylor's series representation

$$f(z) = \sum_{l=1}^{\infty} a_l z^l, \quad (a_1 = 1)\,. \tag{1}$$

Let also $\mathfrak{B}_0$ be the family of Schwarz functions, i.e., holomorphic functions $\mathfrak{w} : \mathbb{U}_d \to \mathbb{U}_d$, $\mathfrak{w}(0) = 0$. We can write the function $\mathfrak{w} \in \mathfrak{B}_0$ as a power series

$$\mathfrak{w}(z) = \sum_{n=1}^{\infty} w_n z^n. \tag{2}$$

For given analytic functions f and g in $\mathbb{U}_d$, we say that f is subordinate to g in $\mathbb{U}_d$ and write $f \prec g$ if there exists $\mathfrak{w} \in \mathfrak{B}_0$ such that

$$f(z) = g(\mathfrak{w}(z)), \quad z \in \mathbb{U}_d.$$

Moreover, if the function g is univalent in $\mathbb{U}_d$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}_d) \subset g(\mathbb{U}_d)$.

Using the concept of subordination, Ma and Minda [18] considered a univalent function ϕ in $\mathbb{U}_d$ with the properties that $\phi'(0) > 0$ and $\mathfrak{Re}\phi > 0$. Also, the region $\phi(\mathbb{U}_d)$ is star-shaped around the fixed point $\phi(0) = 1$ and is symmetric along the real line axis. Applying the familiar idea of subordination, they defined the following unified subfamily of the class $\mathcal{S}$.

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in \mathbb{U}_d) \right\}.$$

They focused on consequences including the covering, growth, and distortion theorems. Over the past few years, a number of collection $\mathcal{S}$ subfamilies have been considered as specific options for the class $\mathcal{S}^*(\phi)$. The following families stand out as being remarkable in the study that has lately been investigated.

(*i*). $\mathcal{S}^*_{\mathcal{L}} \equiv \mathcal{S}^*(\sqrt{1+z})$ [25, 30], $\mathcal{S}^*_{car} \equiv \mathcal{S}^*\left(1 + \frac{2}{3}z + \frac{1}{3}z^2\right)$ [14, 24], $\mathcal{S}^*_{\exp} \equiv \mathcal{S}^*(\exp(z))$ [16, 19],

(*ii*). $\mathcal{S}^*_{\cos} \equiv \mathcal{S}^*(\cos(z))$ [6], $\mathcal{S}^*_{\sin} \equiv \mathcal{S}^*(1 + \sin(z))$ [8], $\mathcal{S}^*_{pet} \equiv \mathcal{S}^*\left(1 + \sinh^{-1} z\right)$ [4],

(*iii*). $\mathcal{S}^*_{\cosh} \equiv \mathcal{S}^*(\cosh(z))$ [1], $\mathcal{S}^*_{\tanh} \equiv \mathcal{S}^*(1 + \tanh(z))$ [31], $\mathcal{S}^*_{c} \equiv \mathcal{S}^*(1 + z + \frac{1}{2}z^2)$ [11].

In [23] Sakaguchi introduced and studied the class $\mathcal{S}^*_s$ of starlike functions justifying the condition with respect respect to the symmetric points. A function f is in $\mathcal{S}^*_s$ if

$$\mathcal{S}^*_s = \left\{ f \in \mathcal{S}, \mathfrak{Re}\left\{ \frac{2zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in \mathbb{U}_d \right\}.$$

As a nature generalization, the class $\mathcal{S}^*_s(\psi)$ defined by

$$\mathcal{S}^*_s(\psi) = \left\{ f \in \mathcal{S}, \mathfrak{Re}\left\{ \frac{2zf'(z)}{f(z) - f(-z)} \right\} \prec \psi(z), \quad z \in \mathbb{U}_d \right\},$$

where proposed in [22], where $\psi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$ be a univalent starlike function with respect to 1 which maps the unit disk $\mathbb{U}_d$ onto a region in the right half plane which is symmetric with respect to the real axis and $B_1 > 0$. For Some recent investigations on univalent functions associated with the symmetric points, we refer to [20, 32, 33].

Recently, Khalil et al. [27, 29] introduced and studied a new subclass of starlike functions $\mathcal{S}^*_{\tanh}$ define by

$$\mathcal{S}^*_{\tanh} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \tanh(z) \qquad (z \in \mathbb{U}_d) \right\}.$$

Motivated by the above works, we defined and study the new subclass of analytic univalent functions $\mathcal{SS}^*_{\tanh}$ associated with the tan hyperbolic function define by

$$\mathcal{SS}^*_{\tanh} = \left\{ f \in \mathcal{S} : \frac{2zf'(z)}{f(z) - f(-z)} \prec 1 + \tanh(z) \qquad (z \in \mathbb{U}_d) \right\}.$$

In this paper, we propose a new approach that depends on the connection between the coefficients of functions belonging to a particular family and the coefficients of corresponding Schwarz functions. In many cases, it is easy to determine the exact bounds of the functional and do the required calculations. Our aim is to calculate the sharp estimates of coefficients, Fekete-Szegö, Zalcman inequalities for the family $\mathcal{SS}^{*}_{\tanh}$ of starlike function with respect to symmetric points connected with tan hyperbolic functions. We also estimate the determinant of $\left|\mathcal{H}_{2,2}(f)\right|$ is also obtained for the same class. Further, we study the logarithmic and inverse coefficients for the same class.
To prove our results, we need the following lemmas for Schwarz functions.

Lemma 1. *([21])Let $\mathfrak{w}(z) = \sum_{n=1}^{\infty} w_n z^n$ be a Schwarz function. Then for any real numbers α and β with $(\alpha,\beta) \in (\alpha,\beta) \in \mathbb{D}_1 \cup \mathbb{D}_2 \cup \mathbb{D}_3 \cup \mathbb{D}_4$, we have the following sharp estimate given by*

$$\left|w_3 + \alpha w_1 w_2 + \beta w_1^3\right| \leq 1.$$

where

$$\begin{aligned}
\mathbb{D}_1 &= \left\{|\alpha| \leq \frac{1}{2},\ -1 \leq \beta \leq 1\right\},\\
\mathbb{D}_2 &= \left\{\frac{1}{2} \leq |\alpha| \leq 2,\ \frac{4}{27}(1+|\alpha|)^3 - (1+|\alpha|) \leq \beta \leq 1\right\},\\
\mathbb{D}_3 &= \left\{|\alpha| \geq 4,\ \beta \geq \frac{2}{3}(|\alpha|-1)\right\},\\
\mathbb{D}_4 &= \left\{2 \leq |\alpha| \leq 4,\ \frac{2(1+|\alpha|)|\alpha|}{4+\alpha^2+2|\alpha|} \leq \beta \leq \frac{1}{12}\left(\alpha^2+2\right)\right\}.
\end{aligned}$$

Lemma 2. *([34])If $\mathfrak{w} \in \mathfrak{B}_0$ is in the form of (2), then*

$$|w_2| \leq 1 - |w_1|^2, \tag{3}$$
$$|w_n| \leq 1,\ n \geq 1. \tag{4}$$

Furthermore, the inequality of (3) can be improved in the manner of

$$\left|w_2 + \eta w_1^2\right| \leq \max\{1, |\eta|\}, \quad \eta \in \mathbb{C}. \tag{5}$$

Lemma 3. *([7])Let $\mathfrak{w}(z) = w_1 z + w_2 z^2 + \ldots$ be a Schwarz function. Then*

$$|w_3| \leq 1 - |w_1|^2 - \frac{|w_2|^2}{1+|w_1|}, \tag{6}$$
$$|w_4| \leq 1 - |w_1|^2 - |w_2|^2. \tag{7}$$

Lemma 4. *Let frakw$(z) = w_1 z + w_2 z^2 + \ldots$be a Schwarz function. Then*

$$\left|w_1 w_3 - w_2^2\right| \leq 1 - |w_1|^2.$$

Proof. Using Lemma 4 and triangle inequality we have

$$\begin{aligned}
\left|w_1 w_3 - w_2^2\right| &\leq |w_1||w_3| + |w_2|^2\\
&\leq |w_1|\left(1 - |w_1|^2 - |w_2|^2\right) + |w_2|^2\\
&= (1-|w_1|)\left(|w_1| + |w_1|^2 + |w_2|^2\right)\\
&\leq (1-|w_1|)(|w_1|+1)\\
&= 1 - |w_1|^2.
\end{aligned}$$

□

2 Coefficient Problems for Functions

We start with the estimates on some initial coefficients of $f \in \mathcal{SS}^*_{\tanh}$.

Theorem 1. *Let $f \in \mathcal{SS}^*_{\tanh}$. Then*

$$\begin{aligned} |a_2| &\leq \frac{1}{2}, \\ |a_3| &\leq \frac{1}{2}, \\ |a_4| &\leq \frac{1}{4}, \\ |a_5| &\leq \frac{1}{4}. \end{aligned}$$

All these bounds are sharp.

Proof. Assume that $f \in \mathcal{SS}^*_{\tanh}$. From the definition, we know there exist a Schwarz function $\mathfrak{w}$ such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = 1 + \tanh \mathfrak{w}(z). \tag{8}$$

Utilizing (1), we get

$$\frac{2zf'(z)}{f(z) - f(-z)} := 1 + 2a_2 z + 2a_3 z^2 + (-2a_2 a_3 + 4a_4) z^3 + \left(-2a_3^2 + 4a_5\right) z^4 + \cdots. \tag{9}$$

Let

$$\mathfrak{w}(z) = w_1 z + w_2 z^2 + w_3 z^3 + w_4 z^4 + \cdots. \tag{10}$$

By some easy computation and utilizing the series representation of (10), we achieve

$$1 + \tanh(\mathfrak{w}(z)) = 1 + w_1 z + w_2 z^2 + \left(-\frac{1}{3}w_1^3 + w_3\right) z^3 + \left(-w_1^2 w_2 + w_4\right) z^4 + \cdots. \tag{11}$$

Now, by comparing (9) and (11), we obtain

$$a_2 = \frac{1}{2}w_1, \tag{12}$$

$$a_3 = \frac{1}{2}w_2, \tag{13}$$

$$a_4 = \frac{1}{4}w_3 - \frac{1}{12}w_1^3 + \frac{1}{8}w_1 w_2, \tag{14}$$

$$a_5 = \frac{1}{8}w_2^2 - \frac{1}{4}w_1^2 w_2 + \frac{1}{4}w_4. \tag{15}$$

From (12), (13) applying Lemma 2 and triangle inequality, we get

$$|a_2| \leq \frac{1}{2} \quad \text{and} \quad |a_3| \leq \frac{1}{2}.$$

By rearranging (14), we have

$$|a_4| = \frac{1}{4}\left|w_3 + \frac{1}{2}w_1 w_2 - \frac{1}{3}w_1^3\right|.$$

By using Lemma 1 with $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{3}$ and triangle inequality, we obtain

$$|a_4| \leq \frac{1}{4}.$$

Reshuffling (15), we have

$$\begin{aligned} |a_5| &= \frac{1}{4}\left|w_4 + \frac{1}{2}w_2^2 - w_1^2 w_2\right|, \\ |a_5| &\leq \frac{1}{4}\left\{|w_4| + \frac{1}{2}|w_2|^2 + |w_1|^2 |w_2|\right\}. \end{aligned}$$

By using Lemma 3 and some simple calculation, we obtain

$$\begin{aligned} |a_5| &\leq \frac{1}{4}\left\{1 - \frac{|w_2|^2}{2}\right\}, \\ |a_5| &\leq \frac{1}{4}. \end{aligned}$$

The bounds on the estimation of $|a_2|$, $|a_3|$, $|a_4|$ and $|a_5|$ are sharp with the extremal functions given respectively by

$$\frac{2zf'(z)}{f(z) - f(-z)} = 1 + z - \frac{1}{3}z^3 + \ldots, \tag{16}$$

$$\frac{2zf'(z)}{f(z) - f(-z)} = 1 + z^2 - \frac{1}{3}z^6 + \ldots, \tag{17}$$

$$\frac{2zf'(z)}{f(z) - f(-z)} = 1 + z^3 - \frac{1}{3}z^9 + \ldots, \tag{18}$$

$$\frac{2zf'(z)}{f(z) - f(-z)} = 1 + z^4 - \frac{1}{3}z^{12} + \ldots. \tag{19}$$

□

Theorem 2. *Let $f \in \mathcal{SS}^*_{\tanh}$. Then*

$$\left|a_3 - \eta a_2^2\right| \leq \max\left\{\frac{1}{2}, \left|-\frac{\eta}{4}\right|\right\}.$$

This result is sharp.

Proof. From (12) and (13), we get

$$\begin{aligned} \left|a_3 - \eta a_2^2\right| &= \frac{1}{2}\left|w_2 - \frac{1}{2}\eta w_1^2\right|, \\ &= \frac{1}{2}\left|w_2 + \left(-\frac{\eta}{2}\right) w_1^2\right|. \end{aligned}$$

Using triangle inequality and Lemma 2, we obtain

$$\left|a_3 - \eta a_2^2\right| \leq \max\left\{\frac{1}{2}, \left|-\frac{\eta}{4}\right|\right\}.$$

□

Putting $\eta = 1$, we establish the below inequality.

Corollary 1. *If $f \in \mathcal{SS}^*_{\tanh}$ is of the form* (1)*, then*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{2}.$$

The result is sharp with the extremal function given by (17)*.*

Now we consider the zalcman functionals for $f \in \mathcal{SS}^*_{\tanh}$.

Theorem 3. *Suppose that $f \in \mathcal{SS}^*_{\tanh}$ be the form of* (1)*, then*

$$|a_4 - a_2 a_3| \leq \frac{1}{4}, \tag{20}$$

and

$$\left|a_5 - a_3^2\right| \leq \frac{1}{4}. \tag{21}$$

These inequalities (20) *and* (21) *are sharp for the extremal function given by* (18) *and* (19)*.*

Proof. In virtue of

$$|a_4 - a_2 a_3| = \frac{1}{4}\left|w_3 - \frac{1}{2}w_1 w_2 - \frac{1}{3}w_1^3\right|,$$

so taking $\alpha = -\frac{1}{2}$ and $\beta = -\frac{1}{3}$ in Lemma 1 yields

$$|a_4 - a_2 a_3| \leq \frac{1}{4}.$$

For $\left|a_5 - a_3^2\right|$, we have

$$\begin{aligned}
\left|a_5 - a_3^2\right| &= \frac{1}{4}\left|w_4 - \frac{1}{2}w_2^2 - w_1^2 w_2\right|, \\
\left|a_5 - a_3^2\right| &\leq \frac{1}{4}\left\{|w_4| + \frac{1}{2}|w_2|^2 + |w_1|^2|w_2|\right\}.
\end{aligned}$$

By using Lemma 3 and some easy computation, we get

$$\begin{aligned}
\left|a_5 - a_3^2\right| &\leq \frac{1}{4}\left\{1 - \frac{9}{8}|w_2|^2\right\}, \\
\left|a_5 - a_3^2\right| &\leq \frac{1}{4}.
\end{aligned}$$

The assertion of Theorem 3 is thus proved. □

Theorem 4. *Let $f \in \mathcal{SS}^*_{\tanh}$ be of the form* (1)*, then*

$$\left|\mathcal{H}_{2,2}\right| = \left|a_2 a_4 - a_3^2\right| \leq \frac{1}{4}.$$

This inequality is sharp with the extremal function given by (17)*.*

Proof. From (12), (13) and (14), we have

$$\mathcal{H}_{2,2} = a_2 a_4 - a_3^2 = \frac{1}{4}\left(w_2^2 - \frac{1}{2}w_1 w_3 + \frac{1}{6}w_1^4 - \frac{1}{4}w_1^2 w_2\right).$$

It is noted that

$$\begin{aligned} |\mathcal{H}_{2,2}| &= \frac{1}{4}\left|\frac{1}{2}\left(w_2^2 - w_1 w_3\right) + \frac{1}{2}\left(\frac{1}{3}w_1^4 - \frac{1}{2}w_1^2 w_2 + w_2^2\right)\right| \\ &\le \frac{1}{8}\left|w_2^2 - w_1 w_3\right| + \frac{1}{8}\left|\frac{1}{3}w_1^4 - \frac{1}{2}w_1^2 w_2 + w_2^2\right| \\ &= \frac{1}{8}T_1 + \frac{1}{8}T_2, \end{aligned}$$

where

$$T_1 = \left|w_2^2 - w_1 w_3\right|$$

and

$$T_2 = \left|\frac{1}{3}w_1^4 - \frac{1}{2}w_1^2 w_2 + w_2^2\right|.$$

Using Lemma 4, we get $T_1 \le 1$. Since

$$\begin{aligned} \left|\frac{1}{3}w_1^4 - \frac{1}{2}w_1^2 w_2 + w_2^2\right| &\le \frac{1}{3}|w_1|^4 + \frac{1}{2}|w_1|^2\left(1 - |w_1|^2\right) + +\left(1 - |w_1|^2\right)^2 \\ &= 1 - \frac{5}{2}|w_1|^2 + \frac{11}{6}|w_1|^4 =: \xi\left(|w_1|^2\right), \end{aligned}$$

where

$$\xi(t) = 1 - \frac{5}{3}t + \frac{11}{6}t^2.$$

It is easy to be observe that ξ is a decreasing function $t \in [0,1]$, thus we have $\xi(t) \le \xi(0) = 1$. As $|w_1|^2 \in [0,1]$, we get $T_2 \le 1$. Hence, we obtain

$$|\mathcal{H}_{2,2}| \le \frac{1}{8}T_1 + \frac{1}{8}T_2 \le \frac{1}{4}.$$

The assertion of Theorem 4 is thus proved. □

3 Logarithmic Coefficient

The logarithmic coefficients of a given function f, represented by $\gamma_n := \gamma_n(f)$, are described as

$$\frac{1}{2}\log\left(\frac{f(z)}{z}\right) = \sum_{n=1}^{\infty}\gamma_n z^n. \tag{22}$$

It seems a natural idea to generalized the Hankel determinant with logarithmic coefficients as entry. In [12, 13], Kowalczyk et al. first introduced the Hankel determinant utilizing logarithmic coefficients as the element, we have

$$\mathcal{H}_{q,n}\left(F_f/2\right) := \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2q-2} \end{vmatrix}. \tag{23}$$

In particular, it is noted that

$$\mathcal{H}_{2,1}\left(F_f/2\right) = \begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{vmatrix} = \left|\gamma_1\gamma_3 - \gamma_2^2\right|.$$

If f is given by (1), then its logarithmic coefficients are given as follows

$$\gamma_1 = \frac{1}{2}a_2 \tag{24}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right) \tag{25}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right). \tag{26}$$

For further investigations of the Hankel determinant on logarithmic coefficients, see [15, 26]

Theorem 5. *Let $f \in \mathcal{SS}^*_{\tanh}$. Then*

$$|\gamma_1| \leq \frac{1}{4},$$

$$|\gamma_2| \leq \frac{1}{4},$$

$$|\gamma_3| \leq \frac{1}{8}.$$

All these bounds are sharp.

Proof. Applying (12), (13), (14) in (24), (25), and (26), we get

$$\gamma_1 = \frac{1}{4}w_1, \tag{27}$$

$$\gamma_2 = \frac{1}{4}w_2 - \frac{1}{16}w_1^2, \tag{28}$$

$$\gamma_3 = \frac{1}{8}w_3 - \frac{1}{16}w_1w_2 - \frac{1}{48}w_1^3. \tag{29}$$

The bounds of γ_1 and γ_2 are clear. For γ_3 rearranging, we get

$$|\gamma_3| = \frac{1}{8}\left|w_3 - \frac{1}{2}w_1w_2 - \frac{1}{6}w_1^3\right|.$$

By using Lemma 1 with $\alpha = -\frac{1}{2}$ and $\beta = -\frac{1}{6}$ and triangle inequality, we obtain

$$|\gamma_3| \leq \frac{1}{8}.$$

The equalities holds for the function given by (16), (17), (18) and using (24), (25), (26). □

Theorem 6. *Let $f \in \mathcal{SS}^*_{\tanh}$ be of the form (1). Then*

$$\left|\gamma_2 - \eta\gamma_1^2\right| \leq \max\left\{\frac{1}{4}, \left|\frac{-(1+\eta)}{16}\right|\right\}.$$

This inequality is sharp.

Proof. From (27) and (28), we have

$$\left|\gamma_2 - \eta\gamma_1^2\right| = \frac{1}{4}\left|w_2 - \frac{1}{4}w_1^2 - \frac{\eta}{4}w_1^2\right|,$$

$$= \frac{1}{4}\left|w_2 + \left(\frac{-(1+\eta)}{4}\right)w_1^2\right|.$$

Using Lemma 2 and triangle inequality, we obtain

$$\left|\gamma_2 - \eta\gamma_1^2\right| \leq \max\left\{\frac{1}{4}, \left|\frac{-(1+\eta)}{16}\right|\right\}.$$

□

Putting $\eta = 1$, we get the below corollary.

Corollary 2. *Let $f \in \mathcal{SS}^*_{\tanh}$ be of the form (1), then*

$$\left|\gamma_2 - \gamma_1^2\right| \leq \frac{1}{4}.$$

Equality is determined by using (24), (25) and (17).

Theorem 7. *Let $f \in \mathcal{SS}^*_{\tanh}$. Then*

$$|\gamma_3 - \gamma_1\gamma_2| \leq \frac{1}{8}.$$

This result is sharp. Equality is determined by using (24), (25), (26) and (18).

Proof. From (27), (28), (29), we get

$$|\gamma_3 - \gamma_1\gamma_2| = \frac{1}{8}\left|w_3 - w_1w_2 - \frac{1}{24}w_1^3\right|.$$

By using Lemma 1 with $\alpha = -1$ and $\beta = -\frac{1}{24}$ and triangle inequality, we obtain

$$|\gamma_3 - \gamma_1\gamma_2| \leq \frac{1}{8}.$$

Which complete the proof. □

Theorem 8. *If $f \in \mathcal{SS}^*_{\tanh}$ is of the form (1), then*

$$\left|\mathcal{H}_{2,1}\left(F_f/2\right)\right| \leq \frac{1}{16}.$$

This result is sharp. Equality is determined by using (24), (25), (26) and (17).

Proof. From (27), (28), (29), we have

$$\mathcal{H}_{2,1}\left(F_f/2\right) = \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{16}\left(w_2^2 - \frac{1}{4}w_1^2w_2 - \frac{1}{2}w_1w_3 + \frac{7}{48}w_1^4\right).$$

It is noted that

$$\begin{aligned}\left|\mathcal{H}_{2,1}\left(F_f/2\right)\right| &= \frac{1}{16}\left|\frac{1}{2}\left(w_2^2 - w_1w_3\right) + \frac{1}{2}\left(w_2^2 + \frac{7}{24}w_1^4 - \frac{1}{2}w_1^2w_2\right)\right| \\ &\leq \frac{1}{32}\left|w_2^2 - w_1w_3\right| + \frac{1}{32}\left|\frac{1}{2}w_2^2 + \frac{7}{24}w_1^4 - \frac{1}{2}w_1^2w_2\right| \\ &= \frac{1}{32}\Phi_1 + \frac{1}{32}\Phi_2,\end{aligned}$$

where

$$\Phi_1 = \left|w_2^2 - w_1w_3\right|$$

and

$$\Phi_2 = \left|\frac{1}{2}w_2^2 + \frac{7}{24}w_1^4 - \frac{1}{2}w_1^2 w_2\right|.$$

Using Lemma 4, we obtain $\Phi_1 \leq 1$. Since

$$\begin{aligned}\left|w_2^2 + \frac{7}{24}w_1^4 - \frac{1}{2}w_1^2 w_2\right| &\leq \left(1-|w_1|^2\right)^2 + \frac{7}{24}|w_1|^4 + \frac{1}{2}|w_1|^2\left(1-|w_1|^2\right) \\ &= 1 - \frac{3}{2}|w_1|^2 + \frac{19}{24}|w_1|^4 = \varrho\left(|w_1|^2\right),\end{aligned}$$

where

$$\varrho(t) = 1 - \frac{3}{2}t + \frac{19}{24}t^2.$$

It is easy to be observe that ϱ is a decreasing function $t \in [0,1]$, thus we have $\varrho(t) \leq \varrho(0) = 1$. As $|w_1|^2 \in [0,1]$, we get $\Phi_2 \leq 1$. Hence, we obtain

$$\left|\mathcal{H}_{2,1}\left(F_f/2\right)\right| \leq \frac{1}{32}\Phi_1 + \frac{1}{32}\Phi_2 \leq \frac{1}{16}.$$

The assertion of Theorem 8 is thus proved. □

4 Inverse Coefficient

The renowned Köebe 1/4-theorem ensures that, for each univalent function f defined in $\mathbb{U}_d$, its inverse f^{-1} exists at least on a disc of radius 1/4 with Taylor's series of the form representation

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n, \quad \left(|w| < \frac{1}{4}\right). \tag{30}$$

Using the representation $f\left(f^{-1}(w)\right) = w$, we get

$$A_2 = -a_2 \tag{31}$$

$$A_3 = -a_3 + 2a_2^2 \tag{32}$$

$$A_4 = -a_4 + 5a_2a_3 - 5a_2^3. \tag{33}$$

Many authors studied Hankel determinants for the inverse functions see [2, 17].

Theorem 9. *Let $f \in \mathcal{SS}^*_{\tanh}$. Then*

$$\begin{aligned}|A_2| &\leq \frac{1}{2}, \\ |A_3| &\leq \frac{1}{2}, \\ |A_4| &\leq \frac{1}{4}.\end{aligned}$$

These bounds are sharp.

Proof. Applying (12), (13), (14) in (31), (32), and (33), we get

$$A_2 = -\frac{1}{2}w_1, \tag{34}$$
$$A_3 = \frac{1}{2}w_1^2 - \frac{1}{2}w_2, \tag{35}$$
$$A_4 = -\frac{1}{4}w_3 + \frac{9}{8}w_1w_2 - \frac{11}{18}w_1^3. \tag{36}$$

The bounds of A_2 and A_3 are clear. For A_4 rearranging, we get

$$|A_4| = \frac{1}{4}\left|w_3 - \frac{9}{2}w_1w_2 + \frac{22}{9}w_1^3\right|.$$

By using Lemma 1 with $\alpha = -\frac{9}{2}$ and $\beta = \frac{22}{9}$ and triangle inequality, we obtain

$$|A_4| \leq \frac{1}{4}.$$

The equalities holds for the function given by (16), (17), (18) and using (31), (32), (33). □

Theorem 10. *Let $f \in \mathcal{SS}^*_{\tanh}$ be of the form (1). Then*

$$\left|A_3 - \eta A_2^2\right| \leq \max\left\{\frac{1}{2}, \left|\frac{\eta - 2}{4}\right|\right\}.$$

This result is sharp.

Proof. From (34) and (35), we have

$$\begin{aligned}\left|A_3 - \eta A_2^2\right| &= \frac{1}{2}\left|w_2 - w_1^2 + \frac{\eta}{2}w_1^2\right|,\\ &= \frac{1}{2}\left|w_2 + \left(\frac{\eta - 2}{2}\right)w_1^2\right|.\end{aligned}$$

Applying triangle inequality and Lemma 2, we obtain

$$\left|A_3 - \eta A_2^2\right| \leq \max\left\{\frac{1}{2}, \left|\frac{\eta - 2}{4}\right|\right\}.$$

□

Putting $\eta = 1$, we achieve the below corollary.

Corollary 3. *If $f \in \mathcal{SS}^*_{\tanh}$ is of the form (1), then*

$$\left|A_3 - A_2^2\right| \leq \frac{1}{2}.$$

Equality is determined by using (31), (32) and (17).

Theorem 11. *If $f \in \mathcal{SS}^*_{\tanh}$ is of the form (1), then*

$$|A_4 - A_2A_3| \leq \frac{1}{4}.$$

This result is sharp. Equality is determined by using (31), (32), (33) and (18).

Proof. From (34), (35), (36), we get

$$|A_4 - A_2A_3| = \frac{1}{4}\left|w_3 - \frac{7}{2}w_1w_2 + \frac{13}{9}w_1^3\right|.$$

By using Lemma 1 with $\alpha = -\frac{7}{2}$ and $\beta = \frac{13}{9}$ and triangle inequality, we obtain

$$|A_4 - A_2A_3| \le \frac{1}{4}.$$

Which complete the proof. □

Theorem 12. *If $f \in \mathcal{SS}^*_{\tanh}$ is of the form (1), then*

$$\left|\mathcal{H}_{2,2}\left(f^{-1}\right)\right| \le \frac{1}{4}.$$

This result is sharp. Equality is determined by using (31), (32), (33) and (17).

Proof. From (34), (35), (36), we have

$$\mathcal{H}_{2,2}\left(f^{-1}\right) = A_2A_4 - A_3^2 = \frac{1}{4}\left(w_2^2 + \frac{1}{4}w_1^2w_2 - \frac{1}{3}w_1w_3 - \frac{2}{9}w_1^4\right).$$

It is noted that

$$\begin{aligned}
\left|\mathcal{H}_{2,2}\left(f^{-1}\right)\right| &= \frac{1}{4}\left|\frac{1}{2}\left(w_2^2 - w_1w_3\right) + \frac{1}{2}\left(w_2^2 - \frac{4}{9}w_1^4 + \frac{1}{2}w_1^2w_2\right)\right| \\
&\le \frac{1}{8}\left|w_2^2 - w_1w_3\right| + \frac{1}{8}\left|w_2^2 - \frac{4}{9}w_1^4 + \frac{1}{2}w_1^2w_2\right| \\
&= \frac{1}{8}\Phi_1 + \frac{1}{8}\Phi_2,
\end{aligned}$$

where

$$\Phi_1 = \left|w_2^2 - w_1w_3\right|$$

and

$$\Phi_2 = \left|w_2^2 - \frac{4}{9}w_1^4 + \frac{1}{2}w_1^2w_2\right|.$$

Using Lemma 4, we obtain $\Phi_1 \le 1$. Since

$$\begin{aligned}
\left|w_2^2 - \frac{4}{9}w_1^4 + \frac{1}{2}w_1^2w_2\right| &\le \left(1 - |w_1|^2\right)^2 + \frac{4}{9}|w_1|^4 + \frac{1}{2}|w_1|^2\left(1 - |w_1|^2\right) \\
&= 1 - \frac{3}{2}|w_1|^2 + \frac{17}{18}|w_1|^4 = \varrho\left(|w_1|^2\right),
\end{aligned}$$

where

$$\varrho(t) = 1 - \frac{3}{2}t + \frac{17}{18}t^2.$$

It is easy to be observe that ϱ is a decreasing function $t \in [0,1]$, thus we have $\varrho(t) \le \varrho(0) = 1$. As $|w_1|^2 \in [0,1]$, we get $\Phi_2 \le 1$. Hence, we obtain

$$\left|\mathcal{H}_{2,2}\left(f^{-1}\right)\right| \le \frac{1}{8}\Phi_1 + \frac{1}{8}\Phi_2 \le \frac{1}{4}.$$

The assertion of Theorem 12 is thus proved. □

5 Conclusion

In this article, we determine the estimates of the problems containing coefficients for functions belonging to the family $\mathcal{SS}^*_{\tanh}$ of function, which are starlike with respect to symmetric points associated with hyperbolic tan function, respectively. In proof of the main results, we use the Lemmas derived by Prokhorov and Szynal, Libera and Zlotkiwicz, Carlson's inequality and bounds on Schwarz function obtained by Eframidis. The approach is focused on the relationship between the coefficients of functions in the given family and the coefficients of corresponding Schwarz functions. All the bounds are proved to be sharp. This work may inspire more investigations on the sharp bounds of analytic functions connected with symmetric points. For future consideration, forth and fifth Hankel determinant may be considered as [3, 5].

Author Contributions

Siraj Osman Omer: Conceptualization, Methodology, Software **Muhammad Aamir**: Data curation, Writing- Original draft preparation. **Muhammad Bilal**: Visualization, Investigation. **Muhammad Aamir**: Supervision.: **Abbas Qadir**: Software, Validation. **Khalil Ullah**: Writing- Reviewing and Editing

Compliance with Ethical Standards

It is declare that all authors don't have any conflict of interest. It is also declare that this article does not contain any studies with human participants or animals performed by any of the authors. Furthermore, informed consent was obtained from all individual participants included in the study.

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