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Several topological indices and entropies for certain families of commutative graphs over Quaternion groups

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Abstract

A group graph is a type of graph formed by combining a group, usually a finite group, with a generating set for that group. Group graphs are employed in various mathematical situations, including algebraic and computational group theory. A graph G is known as a commutative graph if the vertex set of G is a group and two elements are adjacent to each other if they are commuting to each other. In this work, we consider the family of commutative graphs over Quaternion groups. The edge partition mappings related to the degree of each vertex of the graph G are computed. Further, we established many results on various kinds of topological indices and entropies by using M-polynomials. The numerical comparison among computed topological indices has been proposed.

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1 Introduction

Among the most extraordinary and unique fields of mathematics that make it possible for any structure to be seen is graph theory. It receives a lot of attention from scientists these days because of its wide range of applications in computer science, electrical systems, biological networks, linked systems, and chemistry, among other fields. Among scientists, the chemical graph theory is a rapidly emerging field. It supports understanding the essential characteristics of a molecular graph. A significant amount of molecular compounds exist with a variety of uses in everyday life, business, commercial, industrial, medicinal chemistry, and research facilities.

Researchers have been increasingly drawn to the developing field of chem informatics the interaction of information science, chemistry, mathematics, and statistics in recent years. The integration of these is one of the primary factors contributing to the significance of chemical informatics. Fundamental fields of science. Graph theory rose to prominence in mathematical chemistry, and this new field was given the moniker "chemical graph theory. " Researchers are using this innovative field to study and forecast the physicochemical properties of chemical compounds by calculating different topological indices based on the molecular graph of the compound [16].

One kind of topological depiction of a molecule that shows its connections and structure is a molecular graph. These molecular graphs describe a wide range of molecular features and contain organic chemical and physical equitability. They are essential in applications like the link between quantitative structure and activity (QSAR) and computational drug design, digital screens, and quantitative structure-property relationship (QSPR) research [39]. Molecular graphs have been characterized by a variety of topological indices, many of which are effective graph descriptors [20, 24]. Moreover, it has been found that a number of these indices closely match the physical, chemical, or organic properties of molecules [3–5, 18, 23, 25, 27, 32–34, 38, 48, 53].

They play a crucial part in comprehending and forecasting molecular behavior and properties in a range of chemical and pharmacological contexts. In mathematics, atoms are characterized with the vertices of graphs and electrostatic bonds are characterized with the edges. A graph that depicts a molecule's connectivity and formation to serve as a topological representation of it is called a molecular graph. Numerous topological indices, such as degree- and distance-based topological indices as well as other derived indices, are used to investigate these molecular graphs. Chemical graph theory, in particular, heavily relies on distance-based topological indices, especially in chemistry [15, 21].

M-polynomials and associated topological indices for Nanostar dendrimer have been recently computed in [47].

The number of vertices adjacent to a vertex in a graph is known as its neighborhood degree. This feature would allow a topological index to be based on the graph's vertex degree giving, which may be used to forecast differences in the graph's properties.

An extremely useful tool for degree-based topological indices is the M-polynomial [28]. The current authors presented neighborhood degree sum-based topological indices, which simplify the computation of these indices. M-polynomial [46], which has a similar character to the M-polynomial for degree-based indices for neighborhood degree sum-based indices. In [37, 44], M different nanostructures and composite graphs' polynomial and topological indices are calculated. The various classes of graphs and their properties in different aspects of graph theory have been discussed in [1, 6–14, 40–43, 51, 52].

The number of vertices adjacent to a vertex in a graph is known as its neighborhood degree. This feature would allow a topological index to be based on the graph's vertex degree distribution, which may be used to forecast different predictions of the graph's properties. A few topological indices that depend on the vertex's neighborhood degree include the Randic, Zagreb, and Gutman indices. These indexes have been employed in chemical graph theory to forecast toxicity, solubility, and boiling temperatures of molecules [26, 29–31, 35, 36, 45].

The main objectives of this work are;

- To compute the partition mappings on the edge set concerning each vertex degree for a proposed commutative graph.
- To find the topological indices of commutative graphs over Quaternion groups.
- To establish the three kinds of entropies for a proposed algebraic structure.

The advantages of the proposed work are various commutative properties of non-abelian groups like Quaternion can be observed using computing edge partition mappings, M-polynomials, and topological indices. The rest of the article is organized as; In section 2, the M-Polynomials and topological indices has been studied. In section 3, the topological indices have been computed over a commutative graph. The three kinds of Zagreb entropies have been computed for a proposed family of graphs in section 4. In the last section 5, the concluding remarks regarding our work have been given.

2 The M-Polynomials and topological indices

In this section, we will discuss M-polynomial and some well-known topological indices which are elaborated in [2, 16, 22]. The M-Polynomial can be defined as

$$M(G,x,y) = \sum_{\delta\leq i\leq j\leq \triangle} m_{ij}(G)x^i y^j \tag{1}$$

Several degree-based topological indices that communicate the chemical effects of the substance over study can be produced in closed form by M-Polynomial. One algebraic polynomial that is helpful in theoretical chemistry is M Polynomials. Computing the precise expressions of topological functions from multiple degrees is important.

The first Zagreb index $N_1(G)$ and the second Zagreb index $N_2(G)$ are defined as follows

$$N_1(G) = \sum_{\upsilon\omega\in E(G)} (d_\upsilon + d_\omega) \tag{2}$$

$$N_1(G) = \sum_{\upsilon\omega\in E(G)} (d_\upsilon d_\omega) \tag{3}$$

The qualify Zagreb index ${}^nN_2(G)$ is explained to be:

$${}^nN_2(G) = \sum_{\upsilon\omega\in E(G)} \frac{1}{d_\upsilon d_\omega} \tag{4}$$

The general Randic index is specified to be:

$$R_\gamma(G) = \sum_{\upsilon\omega\in E(G)} (d_\upsilon d_\omega)^\gamma \tag{5}$$

The general inverse Randic index is specified to be:

$$RR_\gamma(G) = \sum_{\upsilon\omega\in E(G)} \frac{1}{(d_\upsilon d_\omega)^\gamma} \tag{6}$$

The symmetric division index, SDD(G), can purify to be:

$$SDD(G) = \sum_{\upsilon\omega \in E(G)} \frac{d_{\upsilon}^2 + d_{\omega}^2}{d_v d_{\omega}} \tag{7}$$

The degree-based topological indices are derived to build an M- polynomial.

Table 1. The relationship between M-polynomial and topological indices.

Topological indices	$f(\alpha, \beta)$	Relation between TI and M-polynomial
$M_1(G)$	$\alpha + \beta$	$(D_\alpha + D_\beta(M(G, \alpha, \beta))_{\alpha=\beta=1}$
$M_2(G)$	$\alpha\beta$	$[(D_\alpha D_\beta)(M(G, \alpha, \beta))]_{\alpha=\beta=1}$
${}^mM_2(G)$	$\frac{1}{\alpha\beta}$	$[(S_\alpha S_\beta)(M(G, \alpha, \beta))]_{\alpha=\beta=1}$
$R_\gamma(G)$	$(\alpha\beta)^\gamma$	$[(D_\alpha^\gamma D_\beta^\gamma)(M(G, \alpha, \beta))]_{\alpha=\beta=1}$
$RR_\gamma(G)$	$\frac{1}{(\alpha\beta)^\gamma}$	$[(S_\alpha^\gamma S_\beta^\gamma)(M(G, \alpha, \beta))]_{\alpha=\beta=1}$
$SDD(G)$	$\frac{\alpha^2+\beta^2}{\alpha\beta}$	$[(S_\alpha D_\beta + S_\beta D_\alpha)(M(G, \alpha, \beta))]_{\alpha=\beta=1}$

2.1 Some well-known entropies

Entropy measures can be used in a variety of graph applications, including network analysis, data compression, and information theory. They can also be used to study the properties of random graphs, such as their phase transitions and critical behavior.

The historical background of entropies was first introduced by a German physicist named Rudolf Clausius in the mid-19th century within the branch of thermodynamics, delivered as a measuring instrument of disorder or randomness within a system. In the study of heat engines, entropy becomes a versatile concept applicable to a variety of disciplines.

In the branch of chemistry, entropy is connected to the distribution of matter and energy in a system. Entropy is used in biological systems, statistical interpretation, information theory, and thermodynamic perspective. Ranjini et. al n 2013 shows the first, 2nd, and 3rd clarified introduction of the Zagreb indices. There are some well-known entropies which are discussed in [19, 49].

$$ReZG_1(G) = \sum_{x_i,y_j \varepsilon E(G)} \{\frac{\mu_{x_i} + \mu_{y_j}}{\mu_{x_i}\mu_{y_j}}\} \tag{8}$$

$$ReZG_2(G) = \sum_{x_i,y_j \varepsilon E(G)} \{\frac{\mu_{x_i}\mu_{y_j}}{\mu_{x_i} + \mu_{y_j}}\} \tag{9}$$

$$ReZG_3(G) = \sum_{x_i,y_j \varepsilon E(G)} \{(\mu_{x_i}\mu_{y_j})(\mu_{x_i} + \mu_{y_j})\} \tag{10}$$

In 2014 chen et al., introduced the basic idea of entropy, giving its definition

$$ENT_{\phi(G)} = \sum_{x_iy_j \varepsilon E(G)} \frac{\phi(x_iy_j)}{\sum_{x_iy_j \varepsilon E(G)} \phi(x_iy_j)} \log\{\frac{\phi(x_iy_j)}{\sum_{x_iy_j \varepsilon E(G)} \phi(x_iy_j)}\} \tag{11}$$

First zagreb redefined is

$$\phi(x_iy_j) = \frac{\mu_{x_i} + \mu_{y_j}}{\mu_{x_i}\mu_{y_j}} \tag{12}$$

then

$$RezG_1(G) = \sum_{x_iy_j \varepsilon E(G)} \phi(x_iy_j \tag{13}$$

Then by using above relation

$$ENT_{ReZG_1} = \log(ReZG_1) - \frac{1}{ReZG_1} \log\{ \prod_{x_iy_j \varepsilon E(G)} [\frac{\mu_{x_i} + \mu_{y_j}}{\mu_{x_i}\mu_{y_j}}]^{[\mu_{x_i}+\mu_{y_j}/\mu_{x_i}\mu_{y_j}]}\} \tag{14}$$

2Second zagreb entropy redefined if:

$$\phi(x_iy_j) = \frac{\mu_{x_i}\mu_{y_j}}{\mu_{x_i} + \mu_{y_j}} \tag{15}$$

$$ReZG_2(G) = \sum_{x_iy_j \varepsilon E(G)} \phi(x_iy_j \tag{16}$$

Now by using relation second zagreb entropy is

$$ReZG_2(G) = \log(ReZG_2) - \frac{1}{ReZG_2} \log\{ \prod_{x_iy_j \in E(G)} [\frac{\mu_{x_i}\mu_{y_j}}{\mu_{x_i} + \mu_{y_j}}]^{[\mu_{x_i}\mu_{y_j}/\mu_{x_i}+\mu_{y_j}]}\} \tag{17}$$

3:Third zagreb entropy redefined if:

$$\phi(x_iy_j) = \{(\mu_{x_i}\mu_{y_j})(\mu_{x_i} + \mu_{y_j})\} \tag{18}$$

$$ReZG_3(G) = \sum_{x_iy_j \in E(G)} \phi(x_iy_j \tag{19}$$

Now by using relation third zagreb entropy is

$$\begin{aligned} ReZG_3(G) &= \log(ReZG_3) \\ &- \frac{1}{ReZG_3} \log\{ \prod_{x_iy_j \in E(G)} [(\mu_{x_i}\mu_{y_j})(\mu_{x_i} + \mu_{y_j})]^{[(\mu_{x_i}\mu_{y_j})(\mu_{x_i}+\mu_{y_j})]}\} \end{aligned} \tag{20}$$

3 Commutative graphs over Quaternion groups and their topological indices

In this section, we will discuss some commutative graphs over a family of Quaternion groups.

Definition 1. *[50] The general abstract form of Quaternion group is*

$$Q_{2n} = \langle \alpha, \beta \mid \alpha^{2n} = e, \alpha^n = \beta^2, \beta\alpha = \alpha^{2n-1}\beta \rangle.$$

The abstract form of Q_8 *is*

$$Q_8 = \langle \alpha, \beta \mid \alpha^4 = e\alpha^2 = \beta^2, \beta\alpha = \alpha^3\beta \rangle.$$

Definition 2. *[17] A graph* G *is said to be a commutative graph over a group* G^* *if the vertices of* G *are the element of the group* G^* *and two elements of group* G^* *will be adjacent to each other if they are commute to each other.*

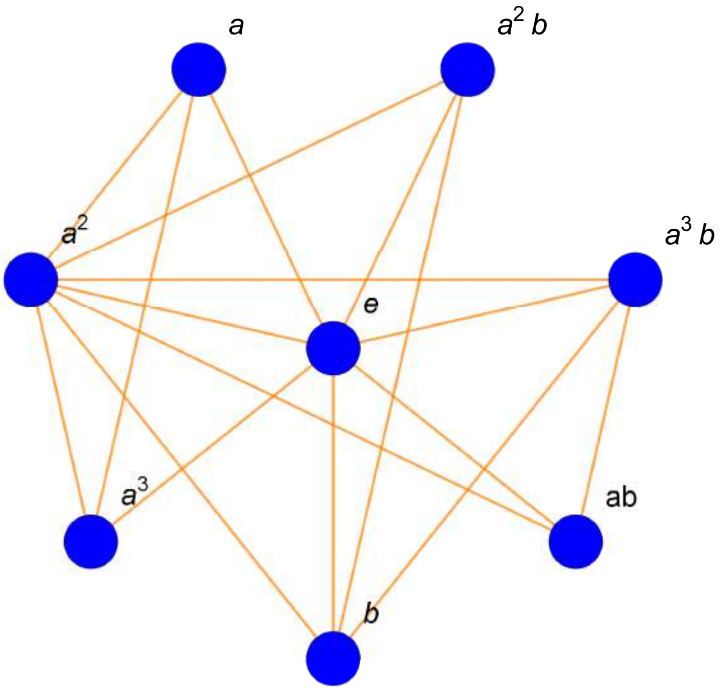

Figure 1. A commutative graph over Q_8.

The commutative graph over a Quaternion group $\mathbb{Q}(\mathbb{G}_8)$ is shown in Figure 1.

Theorem 1. *Let $\mathbb{Q}(\mathbb{G}_\digamma)$ be a commutative graph over Quaternion groups, then*

$$\begin{aligned} M(D_1[\digamma], u, v) &= u^{4\digamma-1}v_2^{4\digamma-1} + nu^3v^3 + 4nu^3v^{4\digamma-1} + (4\digamma-1)u^{2\digamma-1}v^{4\digamma-1} \\ &+ (2\digamma^2 - 5\digamma + 3)u^{2\digamma-1}v^{2\digamma-1}. \end{aligned} \tag{21}$$

Proof. Let $\mathbb{Q}(\mathbb{G}_\digamma)$ be a commutative graph over Quaternion groups. The edge partition mapping with respect degree of each vertices.

$$|m_{i,j}| = \begin{cases} (4\digamma-4), & i=3, j=4\digamma-1 \\ 2\digamma-4, & i=2\digamma-1, j=4\digamma-1 \\ 1, & i=4\digamma-1, j=4\digamma-1 \\ \digamma, & i=3, j=3 \\ 2\digamma^2-5\digamma+3, & i=2\digamma-1, j=2\digamma-1. \end{cases} \tag{22}$$

$$\begin{aligned} M(G; u, v) &= \sum_{i \leq j} |m_{(i,j)}| u^i v^j \\ &= \sum_{3 \leq 4\digamma-1} |m_{(3,4\digamma-1)}| u^3 v^{4\digamma-1} + \sum_{2\digamma-1 \leq 4\digamma-1} |m_{(2\digamma-1,4\digamma-1)}| u^{2\digamma-1} v^{4\digamma-1} \\ &= \sum_{4\digamma-1 \leq 4\digamma-1} |m_{(4\digamma-1,4\digamma-1)}| u^{4\digamma-1} v^{4\digamma-1} + \sum_{3 \leq 3} |m_{(3,3)}| u^3 v^3 \\ &= \sum_{2\digamma-1 \leq 2\digamma-1} |m_{(2\digamma-1,2\digamma-1)}| u^{2\digamma-1} v^{2\digamma-1}. \end{aligned} \tag{23}$$

By using the relation M-polynomial.

$$\begin{aligned} M(D_1[\digamma], u, v) &= u^{4\digamma-1}v_2^{4\digamma-1} + nu^3v^3 + 4nu^3v^{4\digamma-1} + (4\digamma-1)u^{2\digamma-1}v^{4\digamma-1} \\ &+ (2\digamma^2 - 5\digamma + 3)u^{2\digamma-1}v^{2\digamma-1}. \end{aligned} \tag{24}$$

□

Theorem 2. *Let* $\mathbb{Q}(\mathbb{G}_\digamma)$ *be a commutative graph over Quaternion groups, then*

$$M_1(\mathbb{Q}(\mathbb{G}_\digamma)) = 4\digamma(2\digamma^2 + 2\digamma - 5) \tag{25}$$

$$M_2(\mathbb{Q}(\mathbb{G}_\digamma)) = 2\digamma^2(4\digamma^2 + 2\digamma + 21) \tag{26}$$

$$\begin{aligned} {}^mM_2(\mathbb{Q}(\mathbb{G}_\digamma)) &= \frac{1}{9(4\digamma-1)^2(2\digamma-1)^2}[64\digamma^5 + 5\digamma^4 - 811\digamma^3 \\ &+ 303\digamma^2 - 56\digamma - 25] \end{aligned} \tag{27}$$

$$\begin{aligned} R_\alpha(\mathbb{Q}(\mathbb{G}_\digamma)) &= (4\digamma-1)^{2\alpha-1} + 3^{2\alpha}\digamma + 3^\alpha(4\digamma-1)^\alpha 4\digamma \\ &+ (2\digamma-1)(4\digamma-1)^\alpha(4\digamma-4) + (2\digamma-1)^{2\alpha}(2\digamma^2 - 5\digamma + 3) \end{aligned} \tag{28}$$

$$\begin{aligned} RR_r(\mathbb{Q}(\mathbb{G}_\digamma)) &= \frac{1}{(4\digamma-1)^{2r-1}} + \frac{\digamma}{3^{2r}} + \frac{4\digamma}{3^r(4\digamma-1)^r} \\ &+ \frac{4\digamma-4}{(2\digamma-1)^r(4\digamma-1)^r} + \frac{2\digamma^2 - 5\digamma + 3}{(2\digamma-1)^{2r}} \end{aligned} \tag{29}$$

$$SDD(\mathbb{Q}(\mathbb{G}_\digamma)) = \frac{(24\digamma^3 166\digamma^2 - 66\digamma + 5}{3(2\digamma-1)} \tag{30}$$

Proof. Let $\mathbb{Q}(\mathbb{G}_\digamma)$ be a commutative graph over Quaternion groups. The edge partition mapping with respect to degree of each end point of the edge of $\mathbb{Q}(\mathbb{G}_\digamma)$.

$$\begin{aligned} M(ND_1[\digamma], u, v) &= (1)u^{4\digamma-1}v^{4\digamma-1} + nu^3v^3 + 4nu^3v^{4\digamma-1} \\ &+ (4\digamma-4)u^{2\digamma-1}v^{4\digamma-1} + (2\digamma^2 - 5\digamma + 3)u^{2\digamma-1}v^{2\digamma--1}. \end{aligned} \tag{31}$$

$$\begin{aligned} D_u(h(u,v)) &= (4\digamma-1)u^{4\digamma-1}v^{4\digamma-1} + 3nu^3v^3 + 12nu^3v^{4\digamma-1} \\ &+ (4\digamma-4)(2\digamma-1)u^{2\digamma-1}v^{4\digamma-1} \\ &+ (2\digamma^2 - 5\digamma + 3)(2\digamma-1)u^{2\digamma-1}v^{2\digamma-1}. \end{aligned} \tag{32}$$

$$\begin{aligned}
D_v(h(u,v)) &= (4F-1)u^{4F-1}v^{4F-1}+3nu^3v^3+4F(4F-1)u^3v^{4F-1}\\
&+ (4F-1)(4F-4)u^{2F-1}v^{4F-1}\\
&+ (2F-1)(2F^2-5F+3)u^{2F-1}v^{2F-1}. \qquad (33)
\end{aligned}$$

$$\begin{aligned}
\delta_u(h(u,v)) &= \frac{1}{4F-1}u^{4F-1}v^{4F-1}+\frac{F}{3}u^3v^3+\frac{4F}{3}u^3v^{4F-1}\\
& \frac{4F-4}{2F-1}u^{2F-1}v^{4F-1}+\frac{2F^2-5F+3}{2F-1}u^{2F-1}v^{2F-1} \qquad (34)
\end{aligned}$$

$$\begin{aligned}
\delta_v(h(u,v)) &= \frac{1}{4F-1}u^{4F-1}v^{4F-1}+\frac{F}{3}u^3v^3+\frac{4F}{4F-1}u^3v^{4F-1}\\
&+ \frac{4F-4}{4F-1}u^{2F-1}v^{4F-1}+\frac{2F^2-5F+3}{2F-1}u^{2F-1}v^{2F-1} \qquad (35)
\end{aligned}$$

$$\begin{aligned}
D_uD_v(h(u,v)) &= (4F-1)^2u^{4F-1}v^{4F-1}+9nu^3v^3+12F(4F-1)u^3v^{4F-1}\\
&+ (4F-4)(2F-1)(4F-1)u^{2F-1}v^{4F-1}\\
&+ (2F-1)^2(2F^2-5F+3)u^{2F-1}v^{2F-1} \qquad (36)
\end{aligned}$$

$$\begin{aligned}
\delta_u\delta_v(h(u,v)) &= \frac{1}{(4F-1)^2}u^{4F-1}v^{4F-1}+\frac{F}{9}u^3v^3\\
&+ \frac{4F}{3(4F-1)}u^3v^{4F-1}+\frac{4F-4}{(2F-1)(4F-1)}u^{2F-1}v^{4F-1}\\
&+ \frac{2F^2-5F+3}{(2F-1)^2}u^{2F-1}v^{2F-1} \qquad (37)
\end{aligned}$$

$$\begin{aligned}
D_u\delta_v(h(u,v)) &= u^{4F-1}v^{4F-1}+nu^3v^3+\frac{4F(4F-1)}{3}u^3v^{4F-1}\\
&+ \frac{(4F-4)(4F-1)}{2F-1}u^{2F-1}v^{4F-1}+(2F^2-5F+4)u^{2F-1}v^{2F-1} \qquad (38)
\end{aligned}$$

$$\begin{aligned}
D_v\delta_u(h(u,v)) &= u^{4F-1}v^{4F-1}+nu^3v^3+12nu^3v^{4F-1}\\
&+ (4F-4)u^{2F-1}v^{4F-1}+(2F^2-5F+3)u^{2F-1}v^{2F-1} \qquad (39)
\end{aligned}$$

$$\begin{aligned}
D_v^{\alpha}(h(u,v)) &= (4F-1)^{\alpha}u^{4F-1}v^{4F-1}\\
&+ 3^{\alpha}nu^3v^3+(4F-1)^{\alpha}4nu^3v^{4F-1}\\
&+ (4F-1)^{\alpha}(4F-4)u^{2F-1}v^{4F-1}\\
&+ (2F-1)^{\alpha}(2F^2-5F+3)u^{2F-1}v^{2F-1} \qquad (40)
\end{aligned}$$

$$\begin{aligned}
\delta_v^{\alpha}(h(u,v)) &= \frac{1}{(4F-1)^{\alpha}}u^{4F-1}v^{4F-1}\\
&+ \frac{F}{3^{\alpha}}u^3v^3+\frac{4F}{(4F-1)^{\alpha}}u^3v^{4F-1}\ nonimber \qquad (41)\\
&+ \frac{4F-4}{(4F-1)^{\alpha}}u^{2F-1}v^{4F-1}+\frac{(2F^2-5F+3}{(2F-1)^{\alpha}}u^{2F-1}v^{2F-1}. \qquad (42)
\end{aligned}$$

$$\begin{aligned}
D_u^\alpha D_v^\alpha(h(u,v)) &= (4\digamma-1)^{2\alpha-1}u^{4\digamma-1}v^{4\digamma-1}+3^{2\alpha}u^3v^3 \\
&+ 3^\alpha(4\digamma-1)^\alpha 4nu^3v^{4\digamma-1}+(2\digamma-1)^\alpha(4\digamma-1)^\alpha(4\digamma-4)u^{2\digamma-1}v^{4\digamma-1} \\
&+ (2\digamma-1)^{2\alpha}(2\digamma^2-5\digamma+3)u^{2\digamma-1}v^{2\digamma-1} \qquad (43) \\
\delta_u^\alpha\delta_v^\alpha(h(u,v)) &= \frac{1}{(4\digamma-1)^{2\alpha}}u^{4\digamma-1}v^{4\digamma-1}+\frac{\digamma}{3^{2\alpha}}u^3v^3 \\
&+ \frac{4\digamma}{3^\alpha(4\digamma-1)^\alpha}u^3v^{4\digamma-1}+\frac{4\digamma-4}{(2\digamma-1)^\alpha(4\digamma-1)^\alpha}u^{2\digamma-1}v^{4\digamma-1} \\
&+ \frac{2\digamma^2-5\digamma+3}{(2\digamma-1)^{2\alpha}}u^{2\digamma-1}v^{2\digamma-1} \qquad (44)
\end{aligned}$$

$$\begin{aligned}
M_1(\mathbb{Q}(\mathbb{G}_\digamma)) &= [(D_u+D_{v)}(h(u,v))]_{(u,v)=(1,1)} \\
&= 4\digamma(2\digamma^2+2\digamma-5) \qquad (45) \\
M_2(\mathbb{Q}(\mathbb{G}_\digamma)) &= [(D_uD_v)(h(u,v)))]_{(u,v)=(1,1)} \\
&= 2\digamma^2(4\digamma^2+2\digamma+21) \qquad (46) \\
{}^mM_2(\mathbb{Q}(\mathbb{G}_\digamma)) &= [(\delta_u\delta_v)(h(u,v))]_{(u,v)=(1,1)} \\
&= \frac{1}{9(4\digamma-1)^2(2\digamma-1)^2}[64\digamma^5+5\digamma^4-811\digamma^3 \\
&+ 303\digamma^2-56\digamma-25] \qquad (47) \\
R_\alpha(\mathbb{Q}(\mathbb{G}_\digamma)) &= [(D_u^\alpha D_v^\alpha)(h(u,v))]_{(u,v)=(1,1)} \\
&= (4\digamma-1)^{2\alpha-1}+3^{2\alpha}\digamma+3^\alpha(4\digamma-1)^\alpha 4\digamma \\
&+ (2\digamma-1)(4\digamma-1)^\alpha(4\digamma-4)+(2\digamma-1)^{2\alpha}(2\digamma^2-5\digamma+3) \qquad (48) \\
RR_r(\mathbb{Q}(\mathbb{G}_\digamma)) &= [(\delta_u^\alpha\delta_v^\alpha)(h(u,v))]_{(u,v)=(1,1)} \\
&= \frac{1}{(4\digamma-1)^{2r-1}}+\frac{\digamma}{3^{2r}}+\frac{4\digamma}{3^r(4\digamma-1)^r} \\
&+ \frac{4\digamma-4}{(2\digamma-1)^r(4\digamma-1)^r}+\frac{2\digamma^2-5\digamma+3}{(2\digamma-1)^{2r}} \qquad (49) \\
SDD(\mathbb{Q}(\mathbb{G}_\digamma)) &= [(D_u\delta_v+D_v\delta_u)(h(u,v))]_{(u,v)=(1,1)} \\
&= \frac{(24\digamma^3 166\digamma^2-66\digamma+5}{3(2\digamma-1)} \qquad (50)
\end{aligned}$$

□

The numerically comparison of $M_1, M_2,^m M_2, SSD$ are given in Table 2.

Table 2. The numerically comparison of $M_1, M_2,^m M_2, SSD$.

$\digamma$	M_1	M_2	${}^{m}M_2$	SSD
4.	560.	2976.	0.196634	$2.421276985111925 \times 10^{1907}$
5.	1100.	6550.	0.414259	$1.931173147614155 \times 10^{2214}$
6.	1896.	12744.	0.589353	$1.108813531416056 \times 10^{2465}$
7.	2996.	22638.	0.743059	$1.147459589706650 \times 10^{2677}$
8.	4448.	37504.	0.884531	$5.200888203294627 \times 10^{2860}$
9.	6300.	58806.	1.01836	$5.163105144858650 \times 10^{3022}$
10.	8600.	88200.	1.14711	$4.210526315789493 \times 10^{3167}$
11.	11396.	127534.	1.27231	$5.162942324108966 \times 10^{3298}$
12.	14736.	178848.	1.39492	$2.440897990487771 \times 10^{3418}$
13.	18668.	244374.	1.5156	$3.003931924318481 \times 10^{3528}$
14.	23240.	326536.	1.6348	$2.542981797243119 \times 10^{3630}$
15.	28500.	427950.	1.75284	$1.985183594262906 \times 10^{3725}$
16.	34496.	551424.	1.86995	$1.158332781020169 \times 10^{3814}$
17.	41276.	699958.	1.9863	$2.797428360595647 \times 10^{3897}$
18.	48888.	876744.	2.10203	$1.154298443168719 \times 10^{3976}$
19.	57380.	1.08517×10^{6}	2.21725	$2.668677422833932 \times 10^{4050}$
20.	66800.	1.3288×10^{6}	2.33203	$9.44173144502600 \times 10^{4120}$
21.	77196.	1.61141×10^{6}	2.44644	$1.205124310323865 \times 10^{4188}$
22.	88616.	1.93697×10^{6}	2.56053	$1.160578113233533 \times 10^{4252}$
23.	101108.	2.30961×10^{6}	2.67435	$1.598313534052897 \times 10^{4313}$
24.	114720.	2.7337×10^{6}	2.78794	$5.498014380283905 \times 10^{4371}$
25.	129500.	3.21375×10^{6}	2.90132	$7.706219587023967 \times 10^{4427}$

4 Entropies for commutative graph over Quaternion groups

In this section, we will discuss some entropies for commutative graphs over Quaternion groups.

Lemma 1. *Let $\mathbb{Q}(\mathbb{G}_{\digamma})$ be a commutative graph over Quaternion groups, then*

$$\begin{aligned}
\mathbb{RZG}_1(\mathbb{Q}(\mathbb{G}_{\digamma})) &= \frac{8\digamma^3}{(2\digamma-2)(2\digamma-1)} - \frac{26\digamma^2}{(2\digamma-2)(2\digamma-1)} + \frac{16\digamma^2}{3(4\digamma-1)} + \frac{12\digamma^2}{(2\digamma-1)(4\digamma-1)} \\
&+ \frac{2\digamma}{3} + \frac{27\digamma}{(2\digamma-2)(2\digamma-1)} - \frac{8\digamma}{3(4\digamma-1)} - \frac{28\digamma}{(2\digamma-1)(4\digamma-1)} + \frac{8\digamma}{(4\digamma-1)^2} \\
&- \frac{9}{(2\digamma-2)(2\digamma-1)} - \frac{8}{3(4\digamma-1)} + \frac{8}{(2\digamma-1)(4\digamma-1)} - \frac{2}{(4\digamma-1)^2}. \qquad (51)
\end{aligned}$$

$$\begin{aligned}
\mathbb{RZG}_2(\mathbb{Q}(\mathbb{G}_{\digamma})) &= \frac{8\digamma^4}{4\digamma-3} - \frac{32\digamma^3}{4\digamma-3} + \frac{16\digamma^3}{6\digamma-2} + \frac{46\digamma^2}{4\digamma-3} + \frac{48\digamma^2}{4\digamma+2} \\
&- \frac{44\digamma^2}{6\digamma-2} + \frac{16\digamma^2}{8\digamma-2} + \frac{3\digamma}{2} - \frac{28\digamma}{4\digamma-3} - \frac{60\digamma}{4\digamma+2} + \frac{26\digamma}{6\digamma-2} \\
&- \frac{8\digamma}{8\digamma-2} + \frac{6}{4\digamma-3} + \frac{12}{4\digamma+2} - \frac{4}{6\digamma-2} + \frac{1}{8\digamma-2}. \qquad (52)
\end{aligned}$$

$$\mathbb{RZG}_3(\mathbb{Q}(\mathbb{G}_{\digamma})) = 32\digamma^5 - 56\digamma^4 + 304\digamma^3 - 246\digamma^2 + 38\digamma + 12. \qquad (53)$$

Proof. Let $\mathbb{Q}(\mathbb{G}_{\digamma})$ be a commutative graph over dihedral group for even number $\digamma$. The degree sequence for edge partition of $\mathbb{Q}(\mathbb{G}_{\digamma})$,

$$|\gamma_{(\alpha,\beta)}| = \begin{cases} 4\digamma-4, & \text{if } \alpha=3,\ \beta=4\digamma-1, \\ 2\digamma-4, & \text{if } \alpha=2\digamma-1,\ \beta=4\digamma-1, \\ 1, & \text{if } \alpha=4\digamma-1,\ \beta=4\digamma-1, \\ \digamma, & \text{if } \alpha=3,\ \beta=3, \\ \digamma^2-5\digamma+3, & \text{if } \alpha=2\digamma-1,\ \beta=2\digamma-1. \end{cases} \tag{54}$$

$$\begin{aligned} \mathbb{RZG}_1(\mathbb{Q}(\mathbb{G}_{\digamma})) &= \sum_{\alpha_i\beta_j\in E}\frac{d_{\alpha_i}+d_{\beta_j}}{d_{\alpha_i}d_{\beta_j}} \\ &= (4\digamma-4)\left(\frac{4\digamma+2}{12\digamma-3}\right)+(2\digamma-4)\frac{6\digamma-2}{(2\digamma-1)(4\digamma-1)}+\frac{8\digamma-2}{(4\digamma-1)(4\digamma-1)}+(\digamma)\frac{6}{9} \\ &+ (\digamma^2-5\digamma+3)\frac{4\digamma-2}{(2\digamma-1)(2\digamma-1)} \\ &= \frac{8\digamma^3}{(2\digamma-2)(2\digamma-1)}-\frac{26\digamma^2}{(2\digamma-2)(2\digamma-1)}+\frac{16\digamma^2}{3(4\digamma-1)}+\frac{12\digamma^2}{(2\digamma-1)(4\digamma-1)} \\ &+ \frac{2\digamma}{3}+\frac{27\digamma}{(2\digamma-2)(2\digamma-1)}-\frac{8\digamma}{3(4\digamma-1)}-\frac{28\digamma}{(2\digamma-1)(4\digamma-1)}+\frac{8\digamma}{(4\digamma-1)^2} \\ &- \frac{9}{(2\digamma-2)(2\digamma-1)}-\frac{8}{3(4\digamma-1)}+\frac{8}{(2\digamma-1)(4\digamma-1)}-\frac{2}{(4\digamma-1)^2}. \end{aligned} \tag{55}$$

$$\begin{aligned} \mathbb{RZG}_2(\mathbb{Q}(\mathbb{G}_{\digamma})) &= \sum_{\alpha_i\beta_j\in E}\frac{d_{\alpha_i}d_{\beta_j}}{d_{\alpha_i}+d_{\beta_j}} \\ &= (4\digamma-4)\left(\frac{12\digamma-3}{4\digamma+2}\right)+(2\digamma-4)\frac{2\digamma-1}{(6\digamma-2)(4\digamma-1)}+\frac{(4\digamma-1)(4\digamma-1)}{8\digamma-2}+(\digamma)\frac{9}{6} \\ &+ (\digamma^2-5\digamma+3)\frac{(2\digamma-1)(2\digamma-1)}{4\digamma-2} \\ &= \frac{8\digamma^4}{4\digamma-3}-\frac{32\digamma^3}{4\digamma-3}+\frac{16\digamma^3}{6\digamma-2}+\frac{46\digamma^2}{4\digamma-3}+\frac{48\digamma^2}{4\digamma+2} \\ &- \frac{44\digamma^2}{6\digamma-2}+\frac{16\digamma^2}{8\digamma-2}+\frac{3\digamma}{2}-\frac{28\digamma}{4\digamma-3}-\frac{60\digamma}{4\digamma+2}+\frac{26\digamma}{6\digamma-2} \\ &- \frac{8\digamma}{8\digamma-2}+\frac{6}{4\digamma-3}+\frac{12}{4\digamma+2}-\frac{4}{6\digamma-2}+\frac{1}{8\digamma-2}. \end{aligned} \tag{56}$$

$$\begin{aligned} \mathbb{RZG}_3(\mathbb{Q}(\mathbb{G}_{\digamma})) &= \sum_{\alpha_i\beta_j\in E}(d_{\alpha_i}d_{\beta_j})(d_{\alpha_i}+d_{\beta_j}) \\ &= (4\digamma-4)(12\digamma-3)(4\digamma+2)+(2\digamma-4)(2\digamma-1)((6\digamma-2)(4\digamma-1) \\ &+ (4\digamma-1)(4\digamma-1)(8\digamma-2)+54\digamma \\ &+ (\digamma^2-5\digamma+3)(2\digamma-1)(2\digamma-1)(4\digamma-2) \\ &= 32\digamma^5-56\digamma^4+304\digamma^3-246\digamma^2+38\digamma+12. \end{aligned} \tag{57}$$

□

The numerically comparison of $\mathbb{RZG}_3(\mathbb{Q}(\mathbb{G}_{\mathcal{F}}))$, $\mathbb{RZG}_3(\mathbb{Q}(\mathbb{G}_{\mathcal{F}}))$, $\mathbb{RZG}_3(\mathbb{Q}(\mathbb{G}_{\mathcal{F}}))$ are given in Table 3.

Table 3. The numerically comparison of $\mathbb{RZG}_3(\mathbb{Q}(\mathbb{G}_{\mathcal{F}}))$, $\mathbb{RZG}_3(\mathbb{Q}(\mathbb{G}_{\mathcal{F}}))$, $\mathbb{RZG}_3(\mathbb{Q}(\mathbb{G}_{\mathcal{F}}))$.

$\mathcal{F}$	$ReZG_1$	$ReZG_2$	$ReZG_3$
3.	8.85758	49.0655	9360.
4.	13.081	111.052	34116.
5.	17.2076	213.686	97052.
6.	21.2892	368.821	233304.
7.	25.3462	588.39	495864.
8.	29.3882	884.358	959420.
9.	33.4204	1268.7	1.7242×10^6
10.	37.446	1753.42	2.91979×10^6
11.	41.4668	2350.48	4.70902×10^6
12.	45.484	3071.9	7.29176×10^6
13.	49.4984	3929.67	1.09088×10^7
14.	53.5108	4935.78	1.58456×10^7
15.	57.5214	6102.24	2.24362×10^7
16.	61.5307	7441.03	3.10672×10^7
17.	65.5389	8964.16	4.21814×10^7
18.	69.5461	10683.6	5.62814×10^7
19.	73.5526	12611.4	7.39343×10^7
20.	77.5584	14759.6	9.57744×10^7

Theorem 3. *Let $\mathbb{Q}(\mathbb{G}_{\mathcal{F}})$ be a commutative graph over Quaternion groups, then*

$$\begin{aligned}
\mathbb{ENT}_{\mathbb{RZG}_1}(\mathbb{Q}(\mathbb{G}_{\mathcal{F}})) &= \log\left(\frac{192\mathcal{F}^3-256\mathcal{F}^2+6\mathcal{F}+25}{48\mathcal{F}^2-36\mathcal{F}+6}\right) \\
&- \frac{6(2\mathcal{F}-1)(4\mathcal{F}-1)}{192\mathcal{F}^3-256\mathcal{F}^2+6\mathcal{F}+25}\log\Bigg(2^{\frac{2\left(56\mathcal{F}^3-18\mathcal{F}^2-89\mathcal{F}+33\right)}{3\left(8\mathcal{F}^2-6\mathcal{F}+1\right)}} \\
&\times 3^{\frac{-24\mathcal{F}^2+10\mathcal{F}+8}{12\mathcal{F}-3}}\mathcal{F}^{\frac{2\mathcal{F}}{3}}\left(\frac{1}{4\mathcal{F}-1}\right)^{\frac{2}{4\mathcal{F}-1}}\left(\frac{-2\mathcal{F}^2+\mathcal{F}+1}{1-4\mathcal{F}}\right)^{\frac{8(\mathcal{F}-1)(2\mathcal{F}+1)}{12\mathcal{F}-3}} \\
&\times \left(\frac{8\mathcal{F}^2-18\mathcal{F}+9}{4\mathcal{F}-2}\right)^{\frac{8\mathcal{F}^2-18\mathcal{F}+9}{4\mathcal{F}-2}}\left(\frac{(\mathcal{F}-2)(3\mathcal{F}-1)}{8\mathcal{F}^2-6\mathcal{F}+1}\right)^{\frac{4(\mathcal{F}-2)(3\mathcal{F}-1)}{8\mathcal{F}^2-6\mathcal{F}+1}}\Bigg). \qquad (58)
\end{aligned}$$

Theorem 4. *Let $\mathbb{Q}(\mathbb{G}_{\digamma})$ be a commutative graph over Quaternion groups, then*

$$
\begin{aligned}
\mathbb{ENT}_{\mathbb{RZG}_2}(\mathbb{Q}(\mathbb{G}_{\digamma})) &= \log\left(\frac{96\digamma^6-240\digamma^5+832\digamma^4-1398\digamma^3+1005\digamma^2-310\digamma+33}{48\digamma^3-28\digamma^2-14\digamma+6}\right.\\
&- \frac{2(2\digamma+1)(3\digamma-1)(4\digamma-3)}{96\digamma^6-240\digamma^5+832\digamma^4-1398\digamma^3+1005\digamma^2-310\digamma+33}\log\left(2^{-\frac{3\digamma}{2}}\right.\\
&\times 3^{\frac{3\digamma}{2}+\frac{3(4\digamma-4)(4\digamma-1)}{4\digamma+2}}\digamma^{\frac{3\digamma}{2}}\left(\frac{(4\digamma-4)(4\digamma-1)}{4\digamma+2}\right)^{\frac{3(4\digamma-4)(4\digamma-1)}{4\digamma+2}}\\
&\times \left(\frac{(2\digamma-4)(2\digamma-1)(4\digamma-1)}{6\digamma-2}\right)^{\frac{(2\digamma-4)(2\digamma-1)(4\digamma-1)}{6\digamma-2}}\left(\frac{(4\digamma-1)^2}{8\digamma-2}\right)^{\frac{(4\digamma-1)^2}{8\digamma-2}}\\
&\times \left.\left(\frac{(2\digamma-2)(2\digamma-1)\left(2\digamma^2-5\digamma+3\right)}{4\digamma-3}\right)^{\frac{(2\digamma-2)(2\digamma-1)\left(2\digamma^2-5\digamma+3\right)}{4\digamma-3}}\right). \qquad (59)
\end{aligned}
$$

Theorem 5. *Let $\mathbb{Q}(\mathbb{G}_{\digamma})$ be a commutative graph over Quaternion groups, then*

$$
\begin{aligned}
\mathbb{ENT}_{\mathbb{RZG}_3}(\mathbb{Q}(\mathbb{G}_{\digamma})) &= \log\left(32\digamma^5-56\digamma^4+304\digamma^3-246\digamma^2+38\digamma+12\right)\\
&- \frac{1}{32\digamma^5-56\digamma^4+304\digamma^3-246\digamma^2+38\digamma+12}\log\left(2^{54\digamma}3^{162\digamma+3(4\digamma-4)(4\digamma-1)(4\digamma+2)}\right.\\
&\times \digamma^{54\digamma}((4\digamma-4)(4\digamma-1)(4\digamma+2))^{3(4\digamma-4)(4\digamma-1)(4\digamma+2)}\\
&\times ((2\digamma-4)(2\digamma-1)(4\digamma-1)(6\digamma-2))^{(2\digamma-4)(2\digamma-1)(4\digamma-1)(6\digamma-2)}\\
&\times \left((4\digamma-1)^2(8\digamma-2)\right)^{(4\digamma-1)^2(8\digamma-2)}\\
&\times \left.\left((2\digamma-2)(2\digamma-1)(4\digamma-3)\left(2\digamma^2-5\digamma+3\right)\right)^{(2\digamma-2)(2\digamma-1)(4\digamma-3)\left(2\digamma^2-5\digamma+3\right)}\right). \qquad (60)
\end{aligned}
$$

5 Conclusion

A group graph is a type of graph created by joining a group, typically a finite group, and its generating set. Group graphs are used in many mathematical contexts, including algebraic and computational group theory. A graph $G(V,E)$ is considered commutative if two constituents of the group communicate with one another. This work focuses on a commutative graph built with Quaternion groups. Edge partition mappings for each vertex in the graph G are generated based on their degree. Furthermore, we develop numerous results for various types of topological indices and entropies connected with the graph. These findings add to our understanding of the structural features of commutative graphs constructed from Quaternion groups. In future work, anyone can find the eigenvalues, eigenvectors, and energy of the proposed commutative graph structure.

Author Contributions

All authors equally contribute to this manuscript. The authors have read and agreed to the published version of the manuscript.

Data Statement

This work comprises numerical data that is not from any data site.

Conflict of Interest

The authors declare no conflict of interest.

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