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Mathematical and computational modeling of migrations, population spread, and Location-dependent carrying capacities

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Abstract

In single-variable calculus, differential equation models for population dynamics are of high importance, and by developing these ordinary differential equation models, a practical and intelligible approach can be created to partial differential equations. This paper focuses on partial differential equation models for migrations, population spread and location-dependent carrying capacities. However, the logistic equation from which the partial differential equation models develop includes the diffusion equation, traveling wave equation, and building blocks with location-dependent parameters. This methodology is suitable for multivariable calculus lectures, evaluation tasks, and problem activities. The text is accompanied by interactive examples, and the article is formatted to be used as a cumulative distribution function document in which some of the input can be hidden.

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1 INTRODUCTION

When studying population growth, researchers frequently encounter straightforward differential equations. The equations for exponential and logistic growth are widely useful in understanding population dynamics [20,24].

Thomas Malthus was one of the first people in history to utilize mathematics to comprehend how populations vary over time [8,26]. He discovered that if population growth outpaced food production, it could result in a future filled with hunger and human suffering.

Researchers Pierre-François Verhulst looked at possible explanations for what can impede exponential development [9,27]. Malthus had already brought forth this concept, believing that factors like sickness and hunger may be involved. Verhulst transformed this conversation into a mathematical model by putting it into equations in paper. [11,25]

$$y^{'} = my - \varphi(y) \tag{1}$$

The word "m" here stands for exponential growth, even though the term $-\phi(y)$ represents the effect of challenges on the rate of development that is dependent on the population size y [3,10,12]. For these challenges, the simple expression is $\phi(y) = ky^2$. When we put $m = kA$, we get the factored representation.

$$y^{'} = ky(A - y) \tag{2}$$

Verhulst deduced this equation and demonstrated that it ultimately results in a population size that doesn't increase much [15-17]. He made an educated prediction as to how many people would someday reside in Belgium using facts. He gave the solution's graph the name "logistic curve," which is a translation of the Greek term for "calculating [18,19]. " Verhulst was the first to accurately analyze the impact of factors like obstacles on the expansion of the human population [1,14]

When it comes to explaining ideas about how solutions behave, particularly the concept of a constant balancing point, the logistic equation is highly useful [5,22,23]. The appropriate ordinary differential equation (ODE) will now be expressed as follows:

$$y^{'} = ky(A - y), \quad k, A > 0, \quad y(0) = y_0 \geq 0 \tag{3}$$

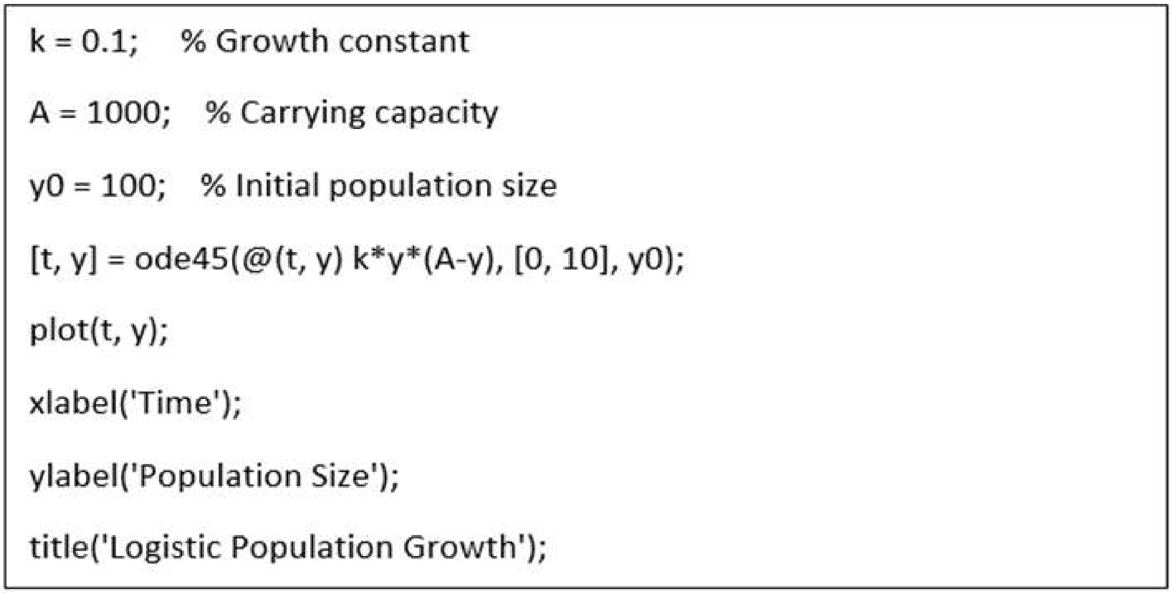
2 Simulation

The logistic model can be used in Octave to examine population changes over time. In this instance, 'y(t)' indicates the population size at a time 't'. We emphasize three main points when we discuss population dynamics: A informs us the greatest population size that can exist, while k shows how quickly the population grows while it is small and y=0 which is where we start. The 'DSolve' function in Octave provides a straightforward method for resolving issues with this paradigm.

$$\frac{Ae^{Akt}y_0}{A - y_0 + e^{Akt}y_0} \tag{4}$$

By dividing the numerator and denominator of the usual mathematical formula for the solution by e^{Akt}, we obtain the expression:

$$y(t) = \frac{Ay_0}{y_0 + (A - y_0)e^{-Akt}}$$



```
k = 0.1;    % Growth constant

A = 1000;   % Carrying capacity

y0 = 100;   % Initial population size

[t, y] = ode45(@(t, y) k*y*(A-y), [0, 10], y0);

plot(t, y);

xlabel('Time');

ylabel('Population Size');

title('Logistic Population Growth');
```

Figure 1. Logistic Population Growth

Using this straightforward technique, we can quickly confirm two essential claims:

$$y(0) = y_0 \quad \text{and} \quad /\lim_{t \longrightarrow \infty} y(t) = A \quad \text{for} \quad y_0 = 0.$$

In Octave, you can experiment with three settings using a tool akin to Octave's Manipulate: k, A, and y0. Seeing how these settings affects a population model in this way might be helpful. In this model:

The population changes very little at two specific locations: at the bottom (y = 0) and at the top (y = A). A maximum limit resembles the carrying capacity. On the graph, it is displayed as a straight line with no slope. The purple line depicts the population shift throughout time. On the up-and-down line, the purple line begins at the beginning value y0.

When you change the growth constant k:

```
%Parameters      growth
Constant = 0.01;
carryingCapacity = 1;
initialPopulation = 0.1;
%Timevalues
t = 0:0.01:600;
%LogisticGrowthequation
yLogistic = carryingCapacity * initialPopulation ./ (initialPopulation +
(carryingCapacity - initialPopulation) *exp (-growthConstant * t));
%Plotting
plot(t, yLogistic);
xlabel('t','FontSize' , 18);%Increasefontsizeforthex-axislabel
ylabel('y','FontSize' , 18);%Increasefontsizeforthey-axislabel
```

```
title('Logistic␣Growth' ,'FontSize'  , 18);%Increasefontsizeforthetitle
gridon;
```

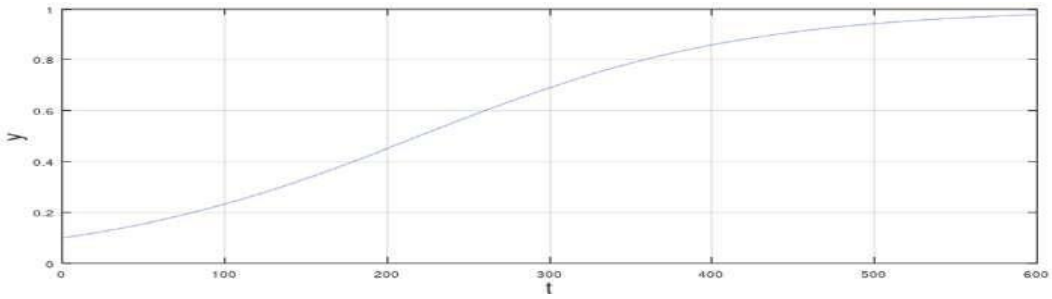


Figure 2. Demonstration of logistic growth

The population expands more quickly if k is increased, especially when it is small. The population expands more slowly when k is reduced, especially when the population is small. Therefore, depending on how big k is, adjusting k causes the population to expand quicker or slower.

However, if you want it to stay at A forever, you must set the initial population to A. When starting with a population other than zero, it often stabilizes at a level A as time progresses to infinity. However, if the population starts out close to zero, it doesn't stay there over time, making the equilibrium at zero unstable. A phase diagram, where the solution curves diverge from zero and converge toward A, can be used to depict these dynamics.

3 A SIGNIFICANT THEORY: PROFILE CURVE OF A SPATIAL TIME EVOLUTION:

Suppose that a population is distributed over a small area of land or ocean that appears to us as a line. At first, we could investigate this population with a function $u_0(x)$. This provides information on the population at each spot x along the strip. However, using a function with two variables, time and location, is more accurate because population size changes over time.

As a result, we express the population size at each place x and time t using $u(t, x)$ with $u(0, x) = u_0(x)$ capturing the preliminary population distribution that we have seen. We use a partial differential equation to represent population variations over time:

$$\frac{\partial u}{\partial t} = ku(A(x) - u) \tag{5}$$

with conditions:

$$k, A(x) > 0, \quad u(0, x) = u_0(x) \geq 0$$

This is similar to a modified logistic equation, but the maximum population A now depends on your location A(x). This modification enables us to take into account various local environmental variables. We are looking for the function u=u(t,x) that solves the partial differential equation (PDE).

In this configuration, we have a starting value for each place, denoted by u(0, x)=u0(x), rather than a single starting value like y0. In essence, we're dealing with distinct initial circumstances at each location.

We will make use of the graph of u(0,x) considered the original profile curve.’ Consider u(t, x) as a ’profile curve’ that depicts. The order of individuals in the line at a particular point at that point. When we look at it for a specific t. This curve reveals the population’s overall distribution. We understand the right side of our partial differential equation (PDE) as the rule that controls how the initial profile curve u(0, x) evolves over time, while the left side of our PDE defines how this distribution varies over time. Understanding this idea of how a profile curve changes over time is essential for what comes next.

Here is an illustration of this viewpoint. The carrying capacity, which varies by location, is defined using the function

$$A(x) = 300 + 100\cos\frac{2\pi}{50}x\ ,$$

and we set the growth constant as $k = 0.04$. Below, plots of $A(x)$ and the original population distribution $u_0(x)$ are visible.

The following MATLAB code plots the carrying capacity $A(x)$ and the initial population $u_0(x)$:

```
%Parameters
A = 80;
x = 0:0.01:100;
y1 = 3 +cos(2 *pi* x / 50) * 2 *pi* 50;
y2 = 1 +exp(3) +cos(2 *pi* x / 50) * 2 *pi* 50;
%Plotting
subplot(2, 1, 1);
plot(x, y1);
xlim([0, 100]);
ylim([80, 4.5]);
xticks([25, 50, 75, 100]);
yticks([80, 85, 87.5, 100]);
title('Carrying␣Capacity␣A(x)');
xlabel('x');
ylabel('A(x)');

subplot(2, 1, 2);
plot(x, y2);
xlim([0, 100]);
ylim([80, 4.5]);
xticks([25, 50, 75, 100]);
yticks([80, 85, 87.5, 100]);
title('Initial␣Population␣u0(x)');
xlabel('x');
ylabel('u0(x)');
set(gcf,'Position', [100, 100, 800, 300]);
```

We might anticipate that at first, the population will fluctuate as it approaches its maximum. We utilize the ’Manipulate’ function to demonstrate how the population distribution varies over time to see if this occurs. If you want to witness the changes as time passes, simply press "Reset and Play."

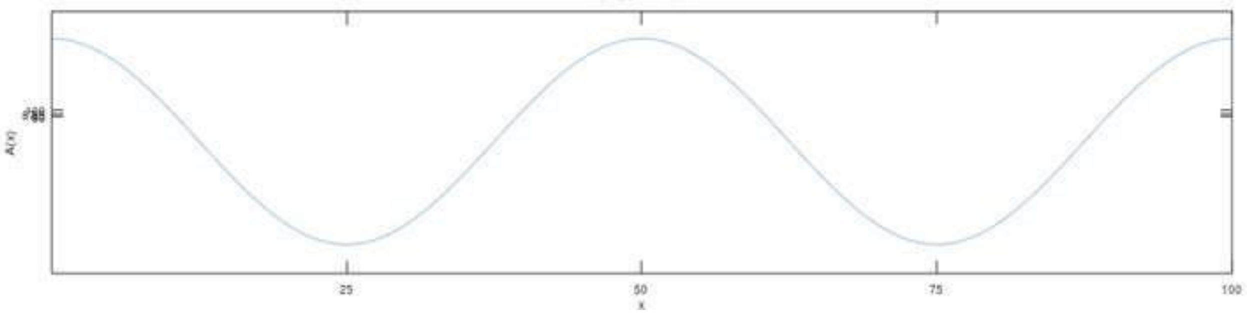


Figure 3. (a) Carrying capacity of growth

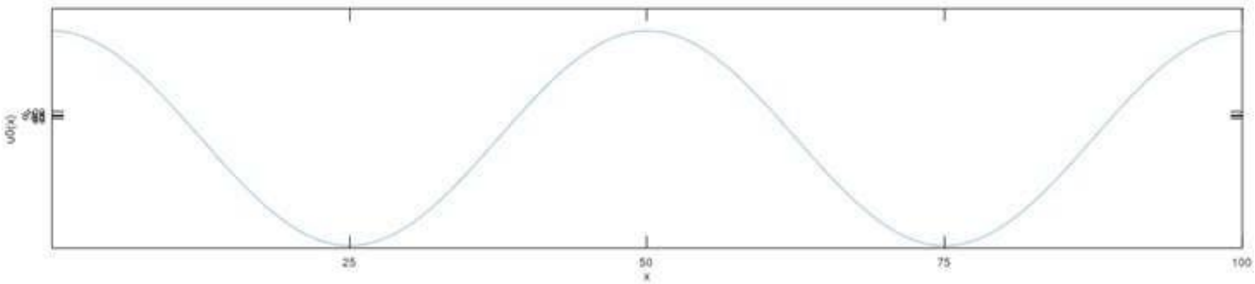


Figure 3. (b) Initial population

Let's think about a fictitious situation: Would we expect the new initial profile curve to eventually resemble A(x) if we modified the initial population distribution while maintaining the carrying capacity at each place (x)? Can we explain why this expectation exists?

Consider the following: What happens to the stage that grosses of beginning population near achieve its balanced condition if we increase k (the growth constant)? The inquiry can be summed up once more with a "phase diagram."

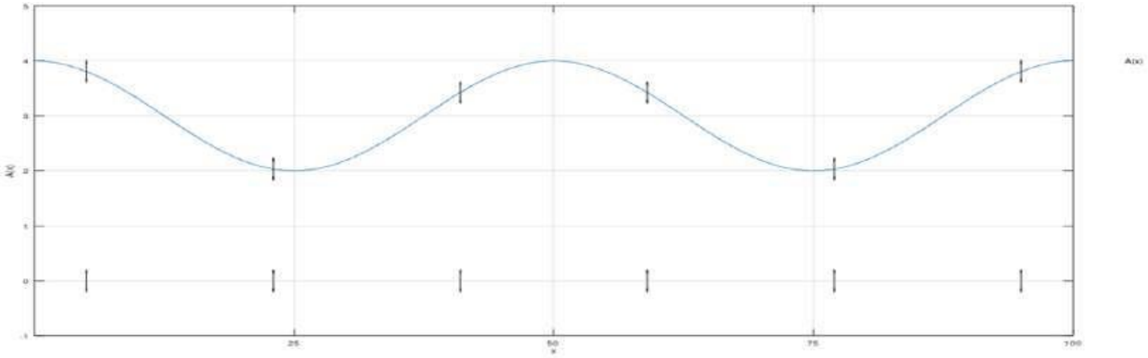


Figure 4. Stability of equilibrium conditions

The following MATLAB code generates a plot showing the stability of equilibria for the differential equation $\frac{dU}{dt} = k \cdot (A(x) - U)$:

```
%Parameters
k = 0.04;
%DefinethefunctionA(x)
A = @(x) 3 +cos (2 *pi * x / 50);
%Definethexvalues
x =linspace (0, 100, 100);
%CalculateA(x)values
A_values = A(x);
%Createtheplot
plt =plot (x, A_values);
xlabel(’x’ );
ylabel(’A(x)’ );
%Settitlefontsizeto18
title(’Stability␣of␣Equilibria␣for␣dU/dt␣=␣k␣*␣(A(x)␣-␣U)’      ,’FontSize’  , 18);
%Addarrows
offset  = 0.2;
holdon;
forx_val  = 5:18:95
    arrow_x  = x_val;
    arrow_y  = A(x_val) + 4 * offset;
    quiver(arrow_x, A(arrow_x), 0, offset,’k’  )
    arrow_x  = x_val;
    arrow_y  = A(x_val) - 4 * offset;
    quiver(arrow_x, A(arrow_x), 0, -offset,’k’  );
    arrow_x  = x_val;
    arrow_y  = 0 + 4 * offset;
    quiver(arrow_x, 0, 0, offset,’k’  );
    arrow_x  = x_val;
    arrow_y  = 0 - 4 * offset;
    quiver(arrow_x, 0, 0, -offset,’k’  );
end
%Addlabels
text(105, 4,’A(x)’  ,’HorizontalAlignment’  ,’left’  );
%Setplotproperties
axis([0, 100, -1, 5]);
set(gca,’XTick’  , [25, 50, 75, 100]);
set(gca,’YTick’  , [-1, 0, 1, 2, 3, 4, 5]);
gridon;
%Addplotlabel
plot_label  = [’Stability␣of␣Equilibria␣for␣dU/dt␣=␣k␣*␣(A(x)␣-␣U)’      ];
title(plot_label,’FontSize’  , 18);
```

Like before, the graph shows the locations where things balance out, denoted by areas where the population has changed over time. $\frac{\partial u}{\partial t}$ is zero everywhere. Two functions are represented by these points:

one in which the population is constant, $u(t, x) = 0$, and another where it reaches a maximum, $u(t, x) = A(x)$. The arrows show that if there are originally people at each place $u_0(x) > 0$ (if $u_0(x)$ is larger than zero), the population progressively increases until it reaches its maximum for each site over time:

$$\lim_{t \to \infty} u(t, x) = A(x).$$

This shows that the equilibrium point for this equation, $A(x)$, is stable.

One further point to consider is that there is a different model of population expansion where the maximum $\frac{\partial u}{\partial t} = k(x)u(A - u)$ remains constant. However, depending on the location, the population's rate of expansion, or the growth constant k, can change.

4 MIGRATING POPULATIONS

In the previous illustration, the profile curve's evolution simply depended on the values of the function $u(t, x)$. Let's now have a look at another partial differential equation (PDE) where the steepness of the profile curve at a given position x affects the change over time at that location. The fundamental version of this equation is as follows, and it links the steepness of u to how quickly it varies over time:

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}, \quad \text{with} \quad u(0, x) = u_0(x) \tag{6}$$

Thinking of this partial differential equation (PDE) in terms of a population that is migrating–where the parameter c controls how quickly the movement occurs–will help us to comprehend its solution. This is demonstrated in the output from the 'Manipulate' tool that is shown below (change c's speed using the 'speed' buttons). Consider it as simulating the migration of gray whales along the Californian coast to make a connection to the real world. It is a good idea to use the "Change" button before adjusting the speed when experimenting through the graphic.

When using DSolve to address the PDE

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x},$$

we obtain a formula known as the "migration equation" that expresses the solution. As usual, we begin with a preliminary population profile, $u(0, x) = u_0(x)$.

The solution to the PDE $\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$ is given by:

$$u(t, x) = u_0(-ct + x)$$

You may have noticed in earlier classes how changing $f(x)$ to $f(x - a)$ results in differences in the graph. Let's consider the y-value $y_0 = f(0)$ on the $f(x)$ graph; the identical y-value may be seen on the $f(x - a)$ graph, but now it shows up at $x = a$, since $y_0 = f(0) = f(a - a)$. Similarly, we can see that $y_1 = f(x_1) = f((x_1 + a) - a)$, so the y-value y_1, which was at x_1 on the $f(x)$ graph, will be at $x_1 + a$ on the $f(x - a)$ graph. The $f(x - a)$ graph thus appears as a horizontal shift of the $f(x)$ graph to the right by a units, and the result applies for all x_1 values. In our context, $u_0(x - ct)$ results in a graph that's a rightward shift of $u_0(x)$ by ct units. Consider that displacement = speed × time.

Using the chain rule, we can demonstrate that $u_0(x - ct)$ for any starting population arrangement fulfills the migration equation $u(t, x)$. This model, sometimes known as the "traveling wave equation," is useful for explaining phenomena like the well-known stadium wave. Jean le Rond d'Alembert was the first to provide partial differential equations (PDEs) explaining wave behavior and techniques to deal with those equations during his research on moving strings.

Furthermore, the equation in the field of fluid dynamics $\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$ symbolizes a condensed form of the transport equation. This formula describes how dissolved materials flow with a current.

5 SEASONAL MIGRATION

Gray whales move in seasonal cycles from the southern Baja Peninsula, close to the Tropic of Cancer, to the Chukchi Sea, north of the Arctic Circle. The periodicity of this migration can be represented by shifting the original graph using a periodic function, resulting in:

$$u(t, x) = u_0(x - A\sin(ct))$$

Now, this $u(t, x)$ complies with the periodic migration equation's conditions:

$$\frac{\partial u}{\partial t} = -Ac\cos(ct)\frac{\partial u}{\partial x}, \quad u(0, x) = u_0(x) \geq 0 \tag{7}$$

With $u_0(x)$, you may note that the term is the same as the original profile curve. The term $-Ac\cos(ct)$ grew out of the chain rule. You might be curious as to what component of the graph is affected by the parameter A, given that the parameter c still determines the migration's speed.

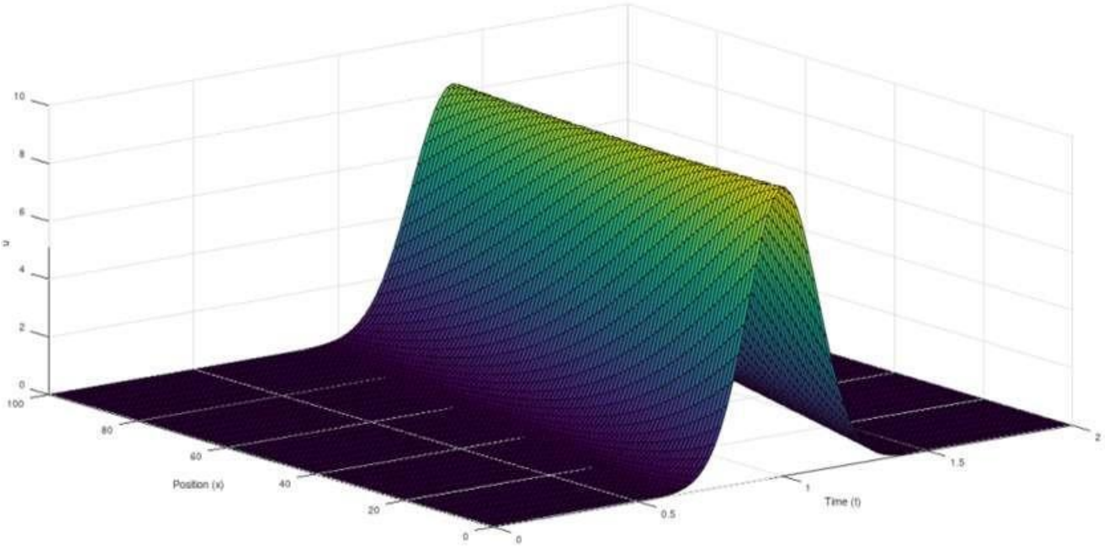


Figure 5. Seasonal human migration

The following MATLAB code simulates the solution to an advection-diffusion equation, representing the seasonal migration of a population. The parameter A represents the amplitude, c is the speed of migration, L is the length of the domain, and T is the total time of simulation. The initial condition is given by the probability density function (PDF) of a normal distribution.

```
%Parameters
A = 10;%Amplitude
c = 10;%Speed
L = 100;%Lengthofthedomain
T = 2;%Totaltime
Nx = 100;%Numberofspatialpoints
Nt = 100;%Numberoftimepoints
%Spatialandtimegrids
x =linspace (0, L, Nx);
t =linspace (0, T, Nt);
%Initializethesolutionmatrix
U =zeros (Nx, Nt);
%Initialcondition(PDFofnormaldistribution)
mu = 50;
sigma = 6;
U(:, 1) = A *exp (-(x - mu).^2 / (2 * sigma^2));
%Solvetheadvection-diffusionequationusingfinitedifferencemethod
dx = x(2) - x(1);
dt = t(2) - t(1);
alpha = c * dt / dx;

forn =  2:Nt
    fori = 2:Nx-1
        U(i, n) = U(i, n-1) - alpha * (U(i, n-1) - U(i-1, n-1));
    end
end
%Plotthesolution
surf(t, x, U’);
xlabel(’Time␣(t)’ );
ylabel(’Position␣(x)’ );
zlabel(’u’);
title(’Seasonal␣Migration␣Solution’ );
```

6 DISPERSION:

Populations tend to begin dispersing throughout a wider area when there is overcrowding in one particular location. By making the population graph’s change over time proportional to the curve of the graph, we may construct a model. Notice that if the graph curves downhill and the tangent line is horizontal, a local maximum can be identified through the second derivative test (having a negative second derivative). This helps explain why this method works. Therefore, the population will decline in the future around a local maximum, according to the model. The dispersion equation provides an expression for this idea:

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2}, \quad d > 0 \tag{8}$$

The dispersion coefficient can be represented by the proportionality constant "d." The graph of this equation's so-called fundamental solution resembles a bell curve.

$$u(t, x) = \frac{1}{\sqrt{4\pi dt}} \exp\left(\frac{-x^2}{4dt}\right) \tag{9}$$

Starting from a certain time point, its width, indicated by the standard deviation, grows over time $t_0 > 0$. This proves that this function is a dispersion equation solution.

```
function u = fundamentalSolution(d, t, x)
    u = (1 / (4 * pi * d * t)) * exp(-x^2 / (4 * d * t));
end
d = 0.1; % Choose your value for \(d\)
t = 0.1; % Choose your value for \(t\)
x = 0.1; % Choose your value for \(x\)
```

Quicker dispersion is caused by a larger dispersion coefficient. The dispersion coefficient buttons and the plus and minus buttons for time can be used to show the difference between the graph with a dispersion coefficient of 5 at time 20 and the graph with a distribution value of 20 at time 5.

The following MATLAB code calculates and plots the dispersion solution with given parameters. The parameter *d* represents the dispersion coefficient, *t* is the time, *L* is the position offset, and *x* is the spatial variable. The resulting plot shows the dispersion solution.

```
%Defineparameters
d = 5;
t = 1;
L = 50;
x =linspace(0, 100, 100);
%Calculatethedispersionsolution
u = (20 / (4 *pi* d * t)) *exp(-((x - L).^2) / (4 * d * t));
%Plotthesolution
plot(x, u);
xlabel('x');
ylabel('u');
title('Dispersion␣Solution');
axis([0, 100, 0, 2.75]);
```

In these diagrams, the value of the overall population is represented by the region underneath the curve, and over time, the dispersion process produces a population that is constantly divided. Our distribution equation, which is sometimes referred to as the formula for the chemical diffusion equation, shows how the dye concentration in a liquid gradually levels out over time.

The analysis of heat loss from an element is where this equation's origins can be found. The heat equation was created in 1807 when Joseph Fourier presented his research on the mathematics of heat conduction. It was formally published in a treatise in 1822. [6]

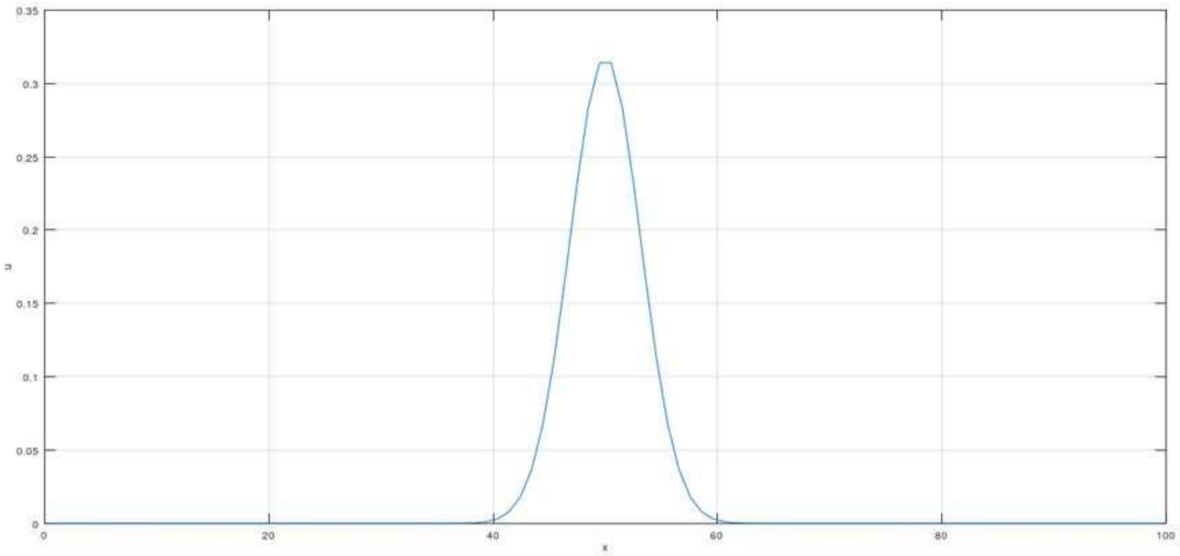


Figure 6. Solutions and dispersions are mixtures of two or more compounds

7 COMBINED MODELS:

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} - c\frac{\partial u}{\partial x}, \quad d > 0 \tag{10}$$

As you might guess, a fundamental solution means replacing x with $x - ct$:

$$u(x, t) = \frac{1}{\sqrt{4\pi dt}} \exp\left(\frac{-(x - ct)^2}{4dt}\right) \tag{11}$$

```
% Define parameters
c = 5;
d = 10;
t = 1;
x = linspace(0, 100, 100);

% Calculate the advection -dispersion solution
u = (20 / (4 * pi * d * t)) * exp(-((x - 20 - c * t).^2) / (4 * d * t));

% Plot the solution
plot(x, u);
xlabel('x');
ylabel('u');
title('Advection -Dispersion ␣Solution ');
axis([0, 100, 0, 2]);
```

To observe the initial data's evolution over time, just click the "Reset and Play" button. Studying invasive species makes use of a more exciting integrated model.

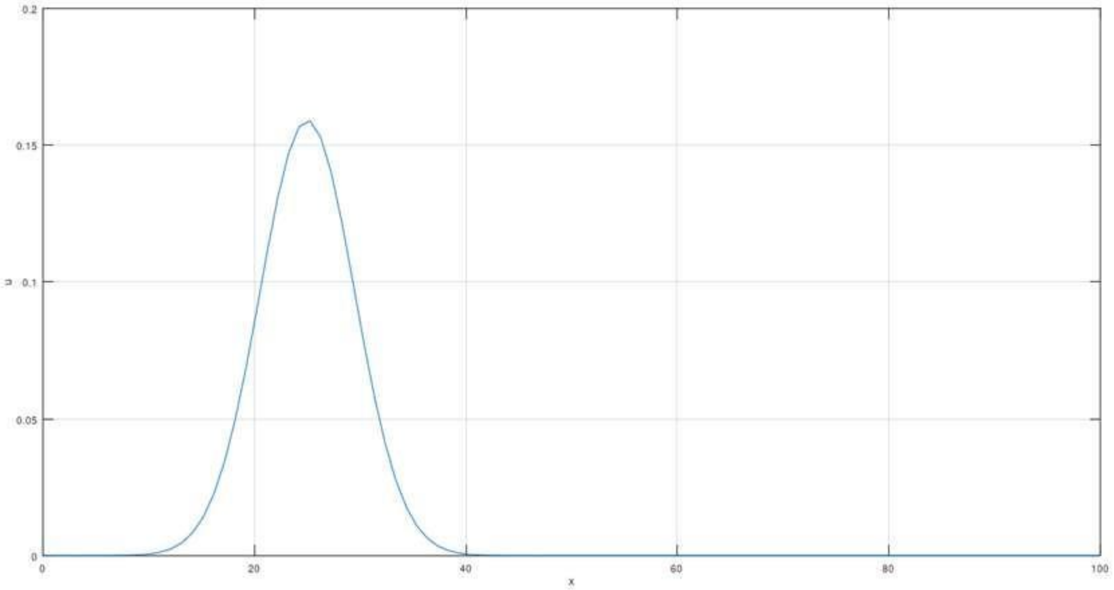


Figure 7. The solutions of advection-dispersion equation

Data on the increasing number of gypsy moths in Wisconsin over a period of ten years (1996-2006) is shown in the graphic below. [4]

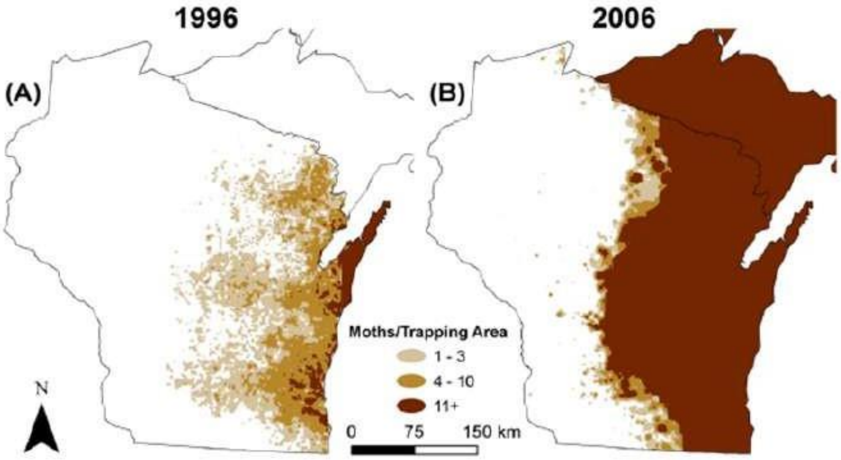


Figure 8. Explain the years and showing moth/trapping areas.

The darkening spots on the maps represent the maximum population a region can support statewide, which is at least eleven moths per trapping area.

We used the letter "x" to represent the distance from the Mississippi River, which runs along the state's western boundary and is approximately 180 miles away from Lake Michigan. To represent this, we mix dispersion and logistic growth in a partial differential equation (PDE).

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} + ku(A - u), \quad d, k, A > 0. \tag{12}$$

The graph below shows how the initial profile curve changes over time when we solve this equation numerically (imagine using Euler's method on the full profile curve). The margin of the gypsy moth population on the two maps does not match closely, which serves as an indication that the model and the data do not match up properly.

The potential effects of randomness heighten the complexity of biological systems. However, the idea of a Fisher-Kolmogorov equation variant that incorporates dispersion and logistic growth appears to capture the majority of the crucial aspects of the gypsy moth data.

The general gypsy moth population is not stable, as it would be if simply dispersion were at work, which is an important last point to make. The growth component $ku(A - u)$ on the right side of the equation in this model balances out dispersion. As the population grows over time, it covers the entire state and is beginning to reach each location's maximum capacity A.

8 SUMMARY:

The lesson focuses on the dynamic evolution of a profile curve and uses DSolve and Manipulate to show three basic partial differential equations. Complex population dynamics situations, such as location-dependent carrying capacity and migratory patterns, are simulated using these equations [7,21].

This method aids in a deeper comprehension of processes that occur in the real world and is helpful for students learning multivariate calculus is a basic textbook on partial differential equation contains other applications to population dynamics [13].

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Author Contributions

Faraz Ahmed Soomro: Conceptualization, Software implementation, Writing- Original draft preparation.
Israr Ahmed: Investigation, Visualization, Software validation, Supervision.
Waheed Ali Panhwar : Writing-Reviewing, Editing, Supervision
Shah Rukh Soomro: Methodology, Data curation, Editing.

Compliance with Ethical Standards

It is declare that all authors don't have any conflict of interest. It is also declare that this article does not contain any studies with human participants or animals performed by any of the authors. Furthermore, informed consent was obtained from all individual participants included in the study.

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