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Modified Bracketing Iterative Method for Solving Nonlinear Equations

Muhammad Imran Soomro[1], **Zubair Ahmed Kalhoro**[1], **Abdul Wasim Shaikh**[1], **Sanaullah Jamali**[2*], **Owais Ali**[1]

[1]Institute of Mathematics and Computer Science, University of Sindh, Allama I.I. Kazi Campus, Jamshoro-76080, Sindh, Pakistan; [2]University of Sindh, Laar Campus Badin

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Abstract Non-linear equations, depicted as curves in numerous everyday situations, have long piqued the curiosity of researchers and engineers due to their frequent occurrence in practical problems. Despite attempts to tackle these equations both analytically and numerically, analytical methods often fall short when the equation's degree exceeds five, prompting the adoption of numerical approaches to yield approximate solutions. Consequently, this study places emphasis on segmenting intervals into smaller sub-intervals, with a particular focus on employing the Regula-Falsi method to integrate these segmented intervals, thereby enhancing its convergence rate. Furthermore, by utilizing the Regula-Falsi formula for interval segmentation, the number of iterations and computational time required are minimized. Additionally, the effectiveness of the proposed method is verified through numerical experiments involving various equation types, including algebraic, trigonometric, exponential, logarithmic, and transcendental equations, comparing the outcomes with established methods. The findings demonstrate that the proposed algorithm not only efficiently segments intervals but also enhances accuracy and reduces errors when these segmented intervals are utilized in conventional bracketing methods.

***Correspondence Author Email Address:**
sanaullah.jamali@usindh.edu.pk
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1 Introduction

The fundamental goal of this research is to present a derivative-free method for approximating solutions to nonlinear problems. We investigate the Regula-Falsi approach, particularly in cases when the root of a nonlinear equation f(x) = 0 must be found inside a bracketing interval determined by initial estimations. These bracketing methods are well-known for their consistent convergence features, which are obtained by iteratively decreasing the interval between guesses to find the equation's root. The popularity of bracketing approaches arises from their persistent convergence to the root, as demonstrated by several academics [7, 9, 11, 31] Furthermore, a unique technique based on Muller's algorithm has been developed to meet the issues of nonlinear equation resolution. While classic Muller's approach uses the final three points of an iterative sequence to generate an interpolating polynomial, it has a drawback: there is no global convergence. In order to overcome the aforementioned limitation and achieve worldwide convergence, a bracketing approach is implemented, as described by [32]. Furthermore, [10] proposes an innovative approach based on interpolation methods. The suggested approach, based on Newton's backward interpolation, exhibits quadratic convergence and has been proved to be effective in addressing a wide range of problem sets in the literature. Additionally, [6] proposes a numerical solution for nonlinear equations, emphasizing the Ridders method in particular. In order to increase the Ridders approach's accuracy and efficiency, the research looks at ways to combine it with the Bisection Method and Newton-Raphson techniques. Root-finding techniques are essential in many practical applications, as demonstrated by a comparative study of iterative approaches for solving non-linear equations [34]. The convergence rate, or the speed at which a given technique converges, determines the efficiency of numerical methods. This research compares and contrasts the limits of the Bisection technique, the Newton-Raphson method, and the Regula-Falsi method in order to determine which is better. The limitations of these methods have allowed the most recent research in the area of iterative processes for solving non-linear equations to be presented. The recent development of the area of iterative approaches is examined, as well as its potential future applications.

And [33] describes a unique computing method, a modified iteration of the bisection method, that is specifically designed for solving nonlinear equations. [16] introduced the modified bisection approach for handling nonlinear equations. A visual graph was supplied for the number of sub-intervals and different error boundaries, along with an estimate of the number of iterations. The new technique is independent of the function, unlike Newton's method or the fixed point method. That is, no unique circumstances are needed. It is sufficient that the answer even exists. Thus, it may be applied to a wide range of tasks. [27] introduced an effective technique for identifying all real roots of a single-variable function within a specified bounded domain. The method suggested employs adaptive mesh refinement to determine bracketing intervals, relying on the bisection criterion for root discovery. Each bracketing interval encompasses a single root. [18] has devised an enhanced parabolic technique, validated through computational experiments, ensuring accuracy. This novel method merges the rapid quadratic progression of parabolic methodologies with the capacity for steady convergence, particularly suited for functions with gradual variations akin to the bisection method. [5] suggested two new iterative methods to accelerate convergence after using the conventional regula-falsi approaches. These techniques guarantee that the resulting series of iterations $\{(x_n - x^*)\}_{n=1}^{\infty}$ and the series of diameters $(b_n - a_n)_{n=1}^{\infty}$ both show quadratic convergence towards zero. [35]

A new method, similar to the regula-falsi method, is presented in this study that aims to locate the single root x^* of a nonlinear equation $f(x) = 0$ in the interval $[a, b]$ with global convergence. It is shown that this novel approach exhibits quadratic convergence. In particular, the iterative sequence $(x_n - x^*)$ and the interval diameter sequences $(b_n - a_n)$ both exhibit quadratic convergence towards zero. To find the root of a continuous real function inside a certain interval, two potential approaches are presented [4]. Combinations of bisection, rational interpolation, and linear interpolation are used in these techniques. These algorithms perform extremely well in terms of their asymptotic behavior. Finding the root of a function requires only four or five times the number of function evaluations necessary by the bisection approach, and usually significantly fewer. In [2], two novel hybrid algorithms were presented. The advantages of bracketing and open approaches are combined in these algorithms. Particularly, the first hybrid algorithm combines the Regula-Falsi Method approach with a modified secant technique (FP-MSe), whereas the second combined algorithm combines the Regula-Falsi Method approach with a trigonometric secant method (FP-TMSe). These suggested algorithms perform better than both Bisection Method (BM) and Regula-Falsi Method (RFM) techniques, according to numerical results. Moreover, the provided algorithms outperform Newton-Raphson (NR), secant (Se), and trisection (Tri) methods in terms of average runtime and number of iterations. The superiority of the suggested algorithms over other pertinent techniques is confirmed by the implementation results.

Let x_n represent the root-approximations found for a certain issue; the following formula [8] computes the COC.

$$COC = \frac{\ln\left(\frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}\right)}{\ln\left(\frac{x_{n-1} - x_{n-2}}{x_{n-2} - x_{n-3}}\right)} \tag{1}$$

2 Derivation of Proposed Algorithm

The proposed research work begins with suggesting an improvement in the RFM, i.e. decomposing the interval into sub-intervals to improve the number of iterations, CPU time, accuracy and COC.
Considering a continuous real mapping $f(x) = 0$, we have a nonlinear equation. Assume that $f(a).(b) < 0$ and that the roots of the aforementioned equation are located in the interval $[a, b]$. The following iterative formulae is used to approximate the root:

Case 1: If $f(a) < 0$ and $f(b) > 0$, then a root lies between $[a, b]$.

$$\text{Step 1: } S_n = a + 2(b - a) \cdot (a + b + 2)^{-1} \tag{2}$$

(a): If $f(S_n) < 0$, then $S_n = a$, and the interval $[a, b]$ shortens to $[S_n, b]$.

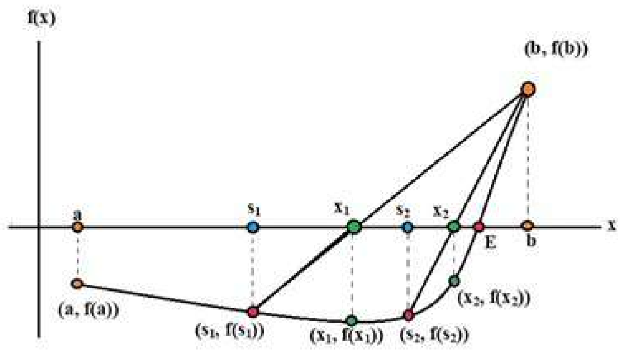

Figure 1. Geometrical Representation of PM **Case 1(a).**

Figure 1 demonstrates how to find roots inside closed intervals using the Proposed Method (PM) shown geometrically. First, select two initial guesses, a and b, and an exact root, E, lies in between them. The intermediate value property is met if $f(a)$ is negative and $f(b)$ is positive. To proceed, the interval $[a, b]$ is shortened to $[S_1, b]$, where $f(S_1)$ is positive, signifying that the root is located between S_1 and b. The first approximation of the root is then obtained by drawing a chord between $(S_1, f(S_1))$ and $(a, f(a))$, crossing the x-axis at x_1. The approximation is improved by iterations of this procedure until the required precision is attained.

$$\text{Step 2: } x_n = (S_n \cdot f(b) - b \cdot f(S_n)) \cdot (f(b) - f(S_n))^{-1} \tag{3}$$

(b): If $f(S_n) > 0$, then $S_n = b$, and the interval $[a, b]$ shortens to $[a, S_n]$.

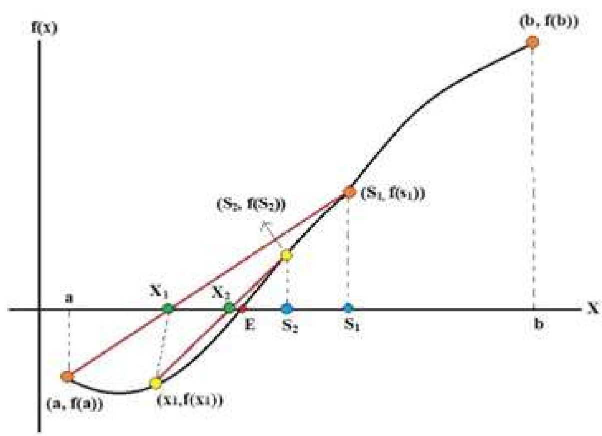

Figure 2. Geometrical Representation of PM **Case 1(b).**

Figure 2 demonstrates how to find roots inside closed intervals using the Proposed Method (PM) shown geometrically. First, select two initial guesses, a and b, and an exact root, E, lies in between them. The intermediate value property is met if $f(a)$ is negative and $f(b)$ is positive. To proceed, the interval $[a, b]$ is shortened to $[S_1, b]$, where $f(S_1)$ is positive, signifying that the root is located between S_1 and b. The first approximation of the root is then obtained by drawing a chord between $(S_1, f(S_1))$ and $(a, f(a))$, crossing the x-axis at x_1. The approximation is improved by iterations of this procedure until the required precision is attained.

$$\text{Step 2: } x_n = (a \cdot f(S_n) - S_n \cdot f(a)) \cdot (f(S_n) - f(a))^{-1} \tag{4}$$

Case 2: If $f(a) > 0$ and $f(b) < 0$, then a root lies between $[a, b]$.

$$\text{Step 1: } S_n = b + 2(a - b) \cdot (a + b + 2)^{-1} \tag{5}$$

(a): If $f(S_n) < 0$, then $S_n = b$, and the interval $[a, b]$ shortens to $[a, S_n]$.

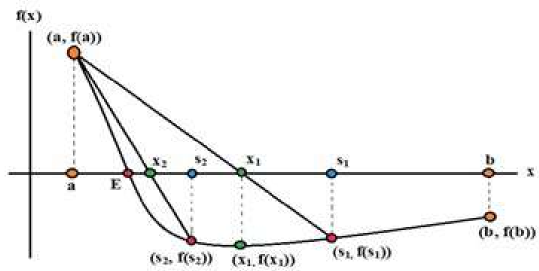

Figure 3. Geometrical Representation of PM **Case 2(a).**

Figure 3 shows the geometric representation of PM. E is the exact root between the closed interval $[a, b]$, and beginning points a and b are established. The property of intermediate value is met when $f(a) > 0$ and $f(b) < 0$. The root falls between a and S_1, hence using subinterval S_1, the interval $[a, b]$ is reduced to $[a, S_1]$. This is because $f(S_1) < 0$. When the x-axis is intersected at x_1, a chord drawn across $(S_1, f(S_1))$ and $(a, f(a))$ yields the exact root, x_1, if $f(x_1) = 0$, otherwise an approximate root (first approximation). The root is between x_1 and b if $f(x_1) < 0$. Subinterval S_2 should be repeated till the required precision is obtained.

$$\text{Step 2: } x_n = (a \cdot f(S_n) - S_n \cdot f(a)) \cdot (f(S_n) - f(a))^{-1} \tag{6}$$

(b): If $f(S_n) > 0$, then $S_n = a$, and the interval $[a, b]$ shortens to $[S_n, b]$.

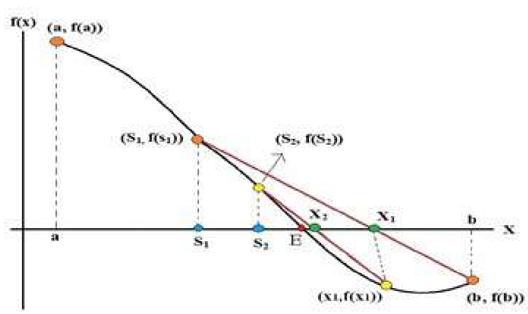

Figure 4. Geometrical Representation of PM **Case 2(b).**

The PM approach is shown in Figure 4, with the starting points a and b and their exact root E shown. The property of intermediate value is met when $f(a) > 0$ and $f(b) < 0$. Since $f(S_1) > 0$, the subinterval S_1 is utilized to shorten $[a, b]$ to $[a, S_1]$, moving the interval closer to the root. $(S_1, f(S_1))$ and $(b, f(b))$ are represented as chords that meet the x-axis at x_1. x_1 is the root if $f(x_1) = 0$; otherwise, it approximates the root. The root is between x_1 and S_1 if $f(x_1) < 0$. S_2 is used in this manner again until the required precision is obtained.

$$\text{Step 2: } x_n = (S_n \cdot f(b) - b \cdot f(S_n)) \cdot (f(b) - f(S_n))^{-1} \tag{7}$$

3 Numerical Experiments

Below examples were taken from literature [1–35].

Additionally, the numerical computations shown in the tables and graphs were done using Origin 2021 and Maple 2022 software, respectively. An Intel(R) Core(TM) i3-4010U laptop with 4.00GB of RAM and a 1.70GHz processor was used to record the results.

Example 1: $x^3 - 2x - 5 = 0$ with $[2, 3]$ being the initial bracket.

Table 1. Results form **Example 1** With a Fixed Error Threshold.

Method	No. of Iter.	Solution	Time (sec.)	COC
Bisection Method (BM) [33]	42	2.094551481542339	0.125829	1
Regula-Falsi Method (RFM) [35]	28	2.094551481542256	0.159014	1.008384
Modified Algorithm (MA) [1]	08	2.094551481542327	0.075506	1.282041
Hybrid Closed Algorithm (HCA) [29]	08	2.094551481542325	0.074995	1.312882
Proposed Method (PM)	06	2.094551481542324	0.050222	1.329093

Example 2: $e^x - 5 = 0$ with $[1, 2]$ being the initial bracket.

Table 2. Results of **Example 2** With a Fixed Error Threshold.

Method	No. of Iter.	Solution	Time (sec.)	COC
BM	40	1.609437912434260	0.067076	1
RFM	17	1.609437912433912	0.079787	1.009941
MA	08	1.609437912434100	0.074765	1.298226
HCA	08	1.609437912434100	0.061495	1.302945
PM	07	1.609437912434100	0.047396	1.364252

Table 3. Error Comparison of **Example 1** With Fixed Number of Iterations.

Method	No. of Iter.	Approx. Solution	Error
BM	7	2.101562500000000	10^{-2}
RFM	7	2.094460845766487	10^{-3}
MA	7	2.094551481541535	10^{-9}
HCA	7	2.094551481542271	10^{-10}
PM	7	2.094551481542327	10^{-13}

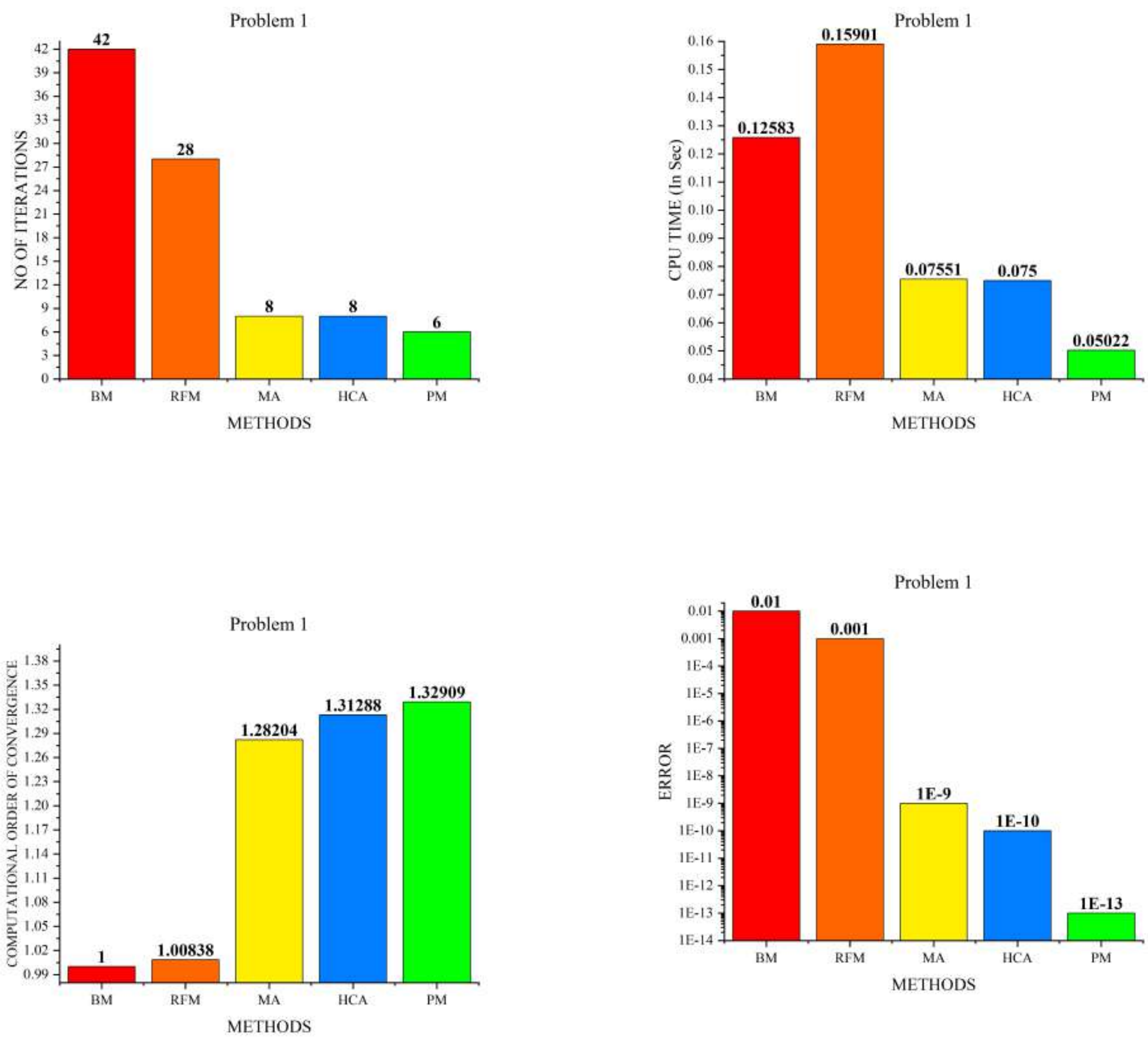

Figure 5. Graphical Representation of tables 1 & 3 of **Example 1**.

Table 4. Error Comparison of **Example 2** With Fixed Number of Iterations.

Method	No. of Iter.	Approx. Solution	Error
BM	7	1.617187500000000	10^{-1}
RFM	7	1.609433332813999	10^{-4}
MA	7	1.609437912433751	10^{-9}
HCA	7	1.609437912433983	10^{-9}
PM	7	1.609437912434100	10^{-11}

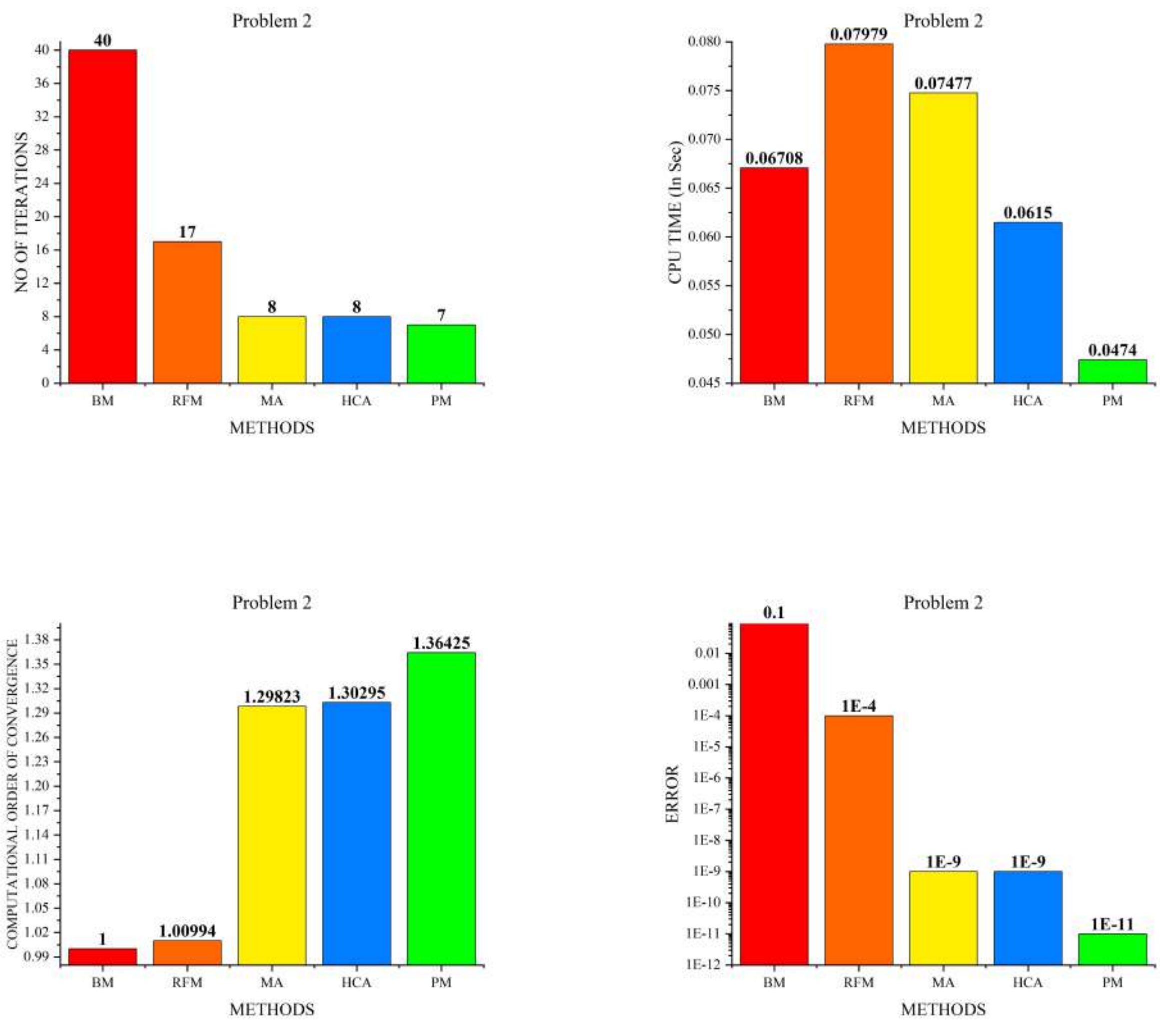

Figure 6. Graphical Representation of tables 2 & 4 of **Example 2**.

Example 3: $\log(x) \cdot e^{x} - 2 = 0$ with $[1, 2]$ being the initial bracket.

Table 5. Results of **Example 3** With a Fixed Error Threshold.

Method	No. of Iter.	Solution	Time (sec.)	COC
BM	39	1.537201702578386	0.088633	1
RFM	20	1.537201702577560	0.084905	1.015290
MA	08	1.537201702578354	0.078329	1.357066
HCA	08	1.537201702578351	0.074051	1.365064
PM	07	1.537201702578346	0.060547	1.425511

Table 6. Error Comparison of **Example 3** With Fixed Number of Iterations.

Method	No. of Iter.	Approx. Solution	Error
BM	7	1.539062500000000	10^{-1}
RFM	7	1.537160229657979	10^{-3}
MA	7	1.537201702577478	10^{-9}
HCA	7	1.537201702576625	10^{-9}
PM	7	1.537201702578346	10^{-11}

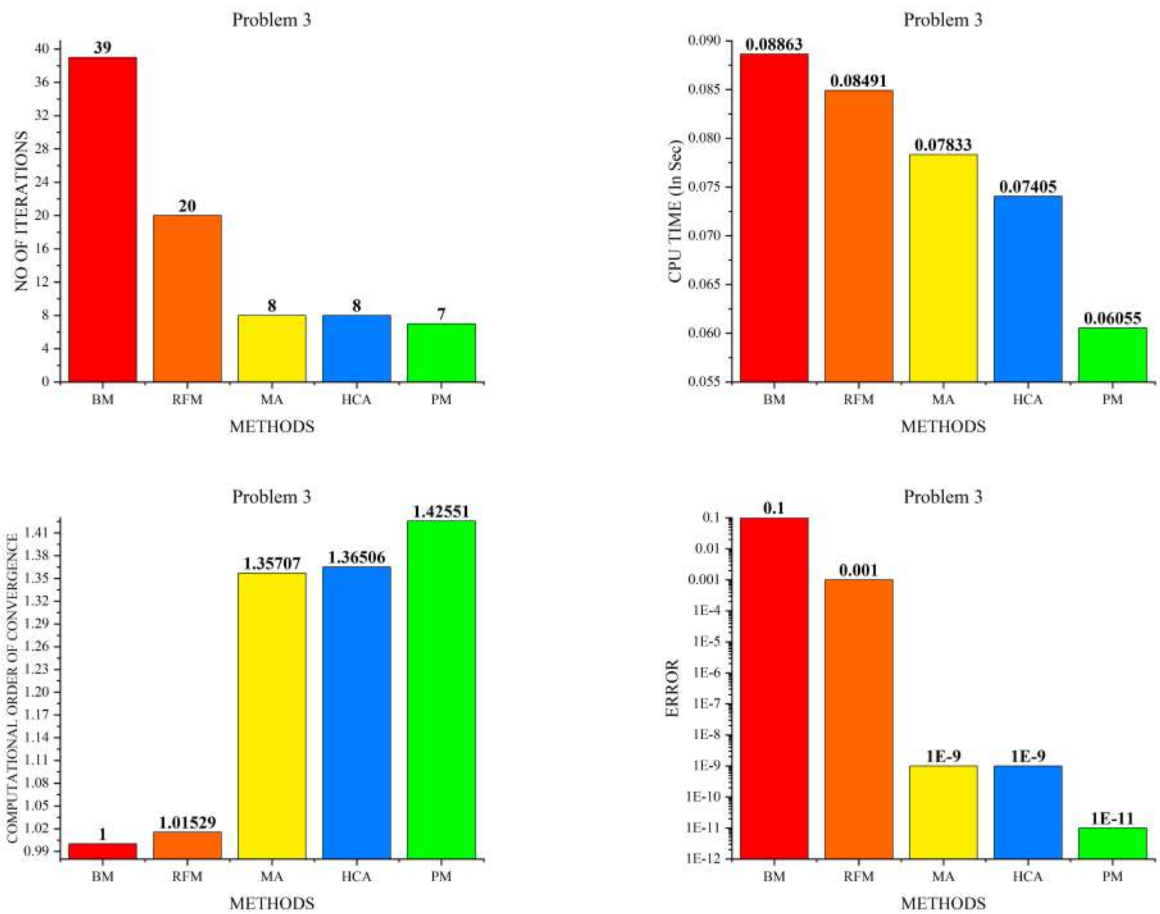

Figure 7. Graphical Representation of tables 5 & 6 of **Example 2**.

Example 4. Open-channel flow [20, 28]. The relationship between water movement and different elements affecting the flow in open channels, such canals, drainage ditches, gutters, and sewers, is still a challenge in the fields of civil and environmental engineering. The amount of water that flows through a certain location in a predetermined amount of time is referred to as the flow rate. However, things get complicated when the concerned channel tilts. In these situations, water flow in an open channel with uniform flow conditions is analyzed using Manning's equation.

$$Q = \sqrt{\frac{m}{n}} \cdot AR^{\frac{2}{3}} \tag{8}$$

Important parameters in a channel include its hydraulic radius (R), cross-sectional area (A), slope (m), and Manning's roughness coefficient (n). A channel is depicted as a rectangle with width (W) and water depth (h). The formula $A = Wx$ defines the connection between these factors.

$$R = \frac{Wx}{W + 2x}, \text{and} \quad A = Wx \tag{9}$$

Using these values of (9) in (8), then (8) becomes

$$\text{Water flow} = \sqrt{\frac{m}{n}} \cdot Wx \left[\frac{Wx}{W + 2x}\right]^{\frac{2}{3}} \tag{10}$$

The following equation may be represented in the structure of a nonlinear function to get the depth of water in the channel that corresponds to a given volume of water:

$$f(x) = \sqrt{\frac{m}{n}} \cdot Wx \left[\frac{Wx}{W + 2x}\right]^{\frac{2}{3}} - Q \tag{11}$$

The water depth h in the channel was approximated under the assumption of fixed parameters: $Q = 14.15\,\text{m}^3/\text{s}$, $W = 4.572\,\text{m}$, $n = 0.017$, and $m = 0.0015$. An initial estimate of $h_0 = 8.5\,\text{m}$ was utilized. The initial bracket $[1, 2]$ was chosen, leading to an approach towards $x^* = 1.465091220295825\ldots$.

Table 7. No of iterations and average CPU time at fixed error of **problem 4**.

Methods	No. of Iterations	Avg CPU time	Error Threshold
BM	36	9.474×10^{-3}	1×10^{-10}
RFM	9	8.277×10^{-3}	1×10^{-10}
MA	8	9.688×10^{-3}	1×10^{-10}
HCA	7	6.473×10^{-3}	1×10^{-10}
PM	5	1.804×10^{-3}	1×10^{-10}

Table 8. Absolute functional value of 4th iteration of **problems 4**.

Methods	Absolute Functional Value (4th Iteration)	Exact Root (up to 15 dp)
BM	$3.736998173987303 \times 10^{-1}$	1.465091220295825
RFM	$2.907928981343844 \times 10^{-5}$	1.465091220295825
MA	$3.334905628331057 \times 10^{-1}$	1.465091220295825
HCA	$2.794850108560176 \times 10^{-2}$	1.465091220295825
PM	$1.577587482870513 \times 10^{-8}$	1.465091220295825

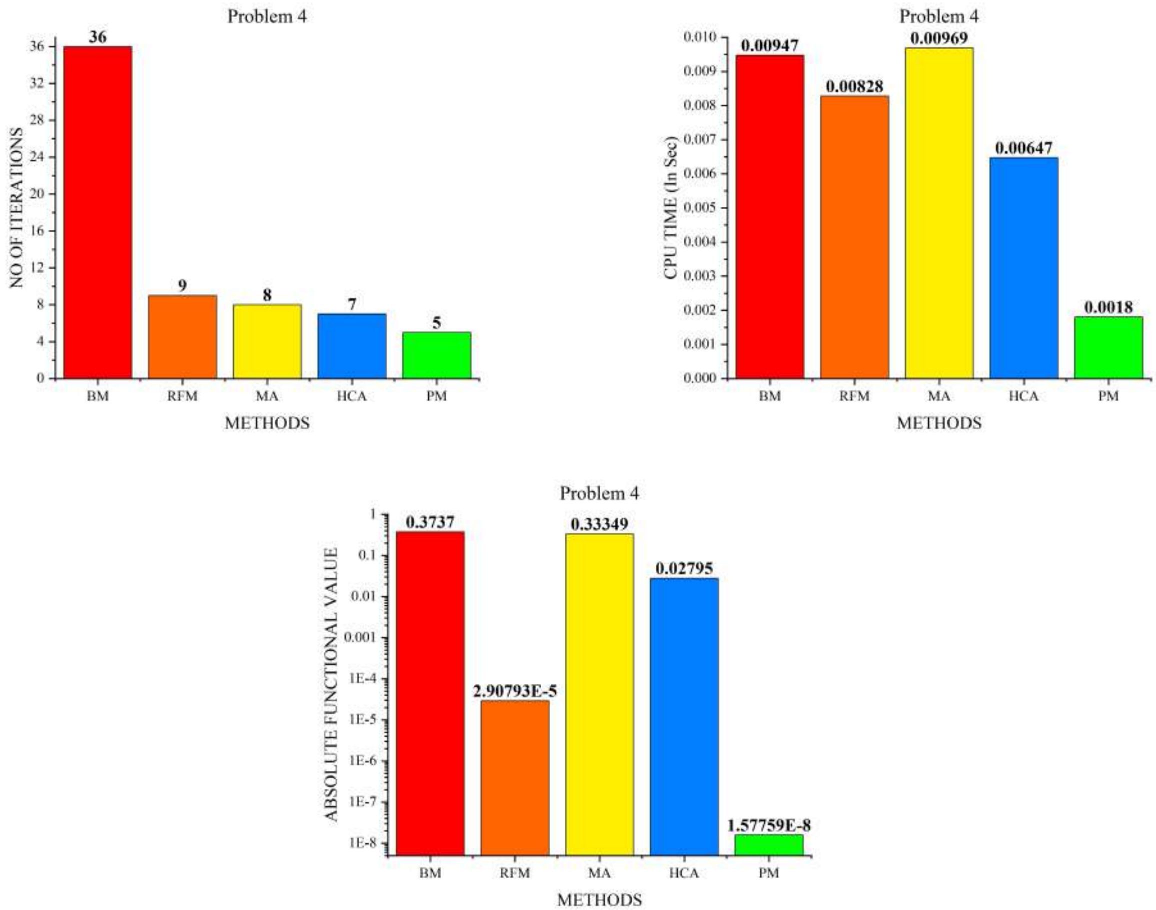

Figure 8. Graphical Representation of table 7 & 8 of **Example 4**.

Example 5: Van der Waals's Equation for Volume Determination. Van der Waals' equation is a useful tool for comprehending the behavior of ideal and actual gases in the field of chemical engineering [20, 30]. It requires the arrangement seen below:

$$(P + \frac{A_1 n^2}{x^2}) \cdot (x - nA_2) = nRT \tag{12}$$

We can easily convert the above equation into the following non-linear function by giving specific values to the parameters:

$$40x^3 - 95.26535116x^2 + 35.28x - 5.6998368 = 0 \tag{13}$$

Here, the volume, which can be found by solving problem 5, is represented by x. The polynomial has three roots by nature because of its degree of three. Only one of these roots, roughly 1.9298462428, has a positive real value, which is consistent with the idea that a gas's volume is never negative. With an initial estimate of $x_0 = [1, 2]$, we start the iteration process.

Table 9. : No of iterations and average CPU time at fixed error of **problem 5**.

Methods	No. of Iterations	Avg CPU time	Error Threshold
BM	38	6.445×10^{-3}	1×10^{-10}
RFM	9	6.776×10^{-3}	1×10^{-10}
MA	13	8.882×10^{-3}	1×10^{-10}
HCA	6	7.834×10^{-3}	1×10^{-10}
PM	6	6.61×10^{-4}	1×10^{-10}

Table 10. Absolute functional value of 4th iteration of **problem 5**.

Methods	Absolute Functional Value (4th Iteration)	Exact Root (up to 15 dp)
BM	$4.034299552968719 \times 10^{0}$	1.970784219407506
RFM	$4.479038238827116 \times 10^{-4}$	1.970784219407506
MA	$4.526453528056322 \times 10^{-1}$	1.970784219407506
HCA	$4.258268284618794 \times 10^{-2}$	1.970784219407506
PM	$1.790006265878219 \times 10^{-6}$	1.970784219407506

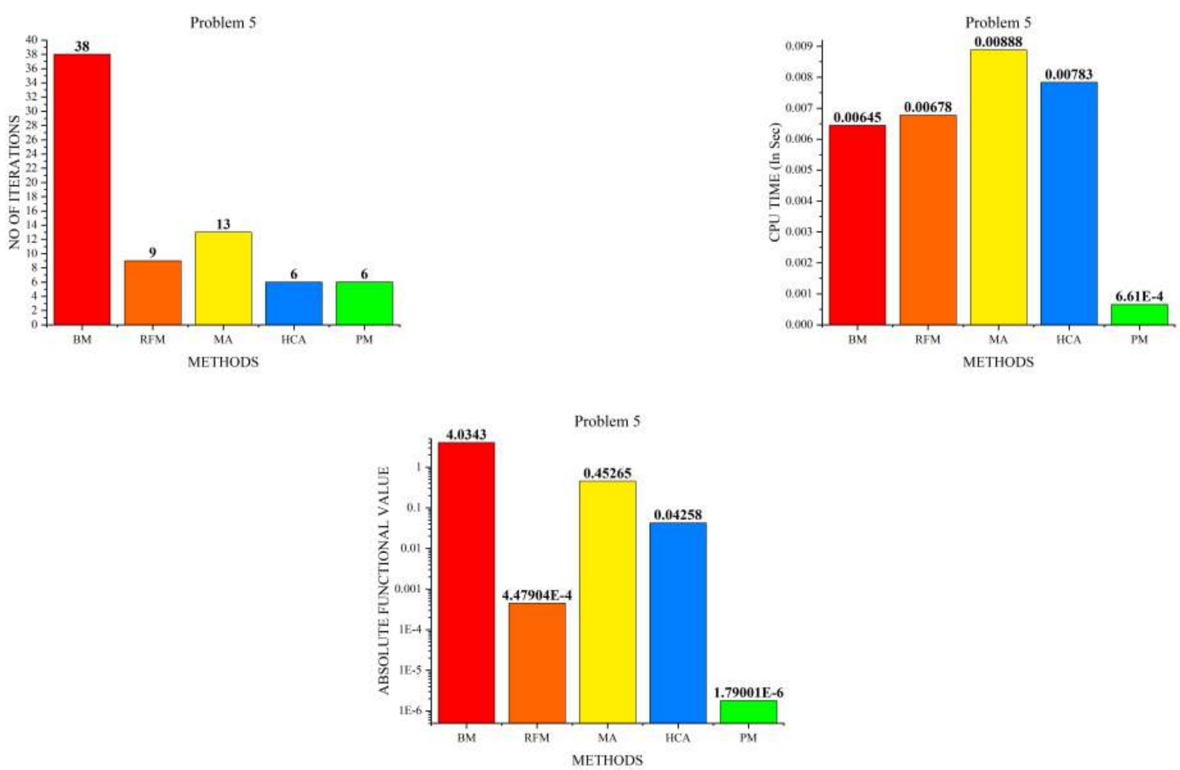

Figure 9. Graphical Representation of table 9 & 10 of **Example 5**.

4 Conclusions

Since nonlinear equations are so common in real-world applications, they are very important. Nevertheless, approaches to solving these kinds of equations frequently encounter difficulties, especially with regard to speed and durability. To overcome these obstacles, the Regula-Falsi Method (RFM), which improves performance by utilising interval decomposition techniques, has become a viable method for solving nonlinear equations. It is clear from comparison study that the Proposed Method (PM) outperforms other methods in terms of iterations, CPU time, Computational Order Convergence (COC), and numerical accuracy. When the PM is directly compared to its primary rivals, BM, RFM, MA, and HCA, it regularly performs better. Tables 1, 3, 2, 4, 5, 6, 7, 8, 9, and 10, as well as graphs 5, 6, 8, and 9 provide thorough comparisons that showcase this superiority. Throughout these experiments, the Proposed Method consistently outperforms competing techniques, demonstrating its usefulness and dependability in solving nonlinear equations. With its robustness, speed, and improved performance across numerous tests, the Proposed Method stands out as an appealing option for dealing with nonlinear equations in a variety of practical scenarios.

Author Contributions

Muhammad Imran Soomro: Conceptualization and Methodology **Zubair Ahmed Kalhoro**: Supervision, **Abdul Wasim Shaikh**: Supervision, **Sanaullah Jamali**: use latex Software, Writing- Reviewing and Editing **Owais Ali**: Validation.

Compliance with Ethical Standards

It is declare that all authors don't have any conflict of interest. It is also declare that this article does not contain any studies with human participants or animals performed by any of the authors. Furthermore, informed consent was obtained from all individual participants included in the study.

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Author Information

ORCID:
Zubair Ahmed Kalhoro: 0000-0002-8609-6298
Sanaullah Jamali: 0000-0002-1162-5909
Owais Ali: 0000-0002-9620-4600

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