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A Modified Hybrid Method For Solving Non-Linear Equations With Computational Efficiency

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Abstract This paper proposes a modified hybrid method for solving non-linear equations that improves computational efficiency while maintaining accuracy. The proposed method combines the advantages of the traditional Halley's and mean-based methods, resulting in a more efficient algorithm. The modified hybrid method starts with Halley's method and then switches to the mean-based method for rapid convergence. To further improve the efficiency of the algorithm, the proposed method incorporates a dynamic selection criterion to choose the appropriate method at each iteration. Numerical experiments are performed to evaluate the performance of the proposed method in comparison to other existing methods. The results show that the modified hybrid method is computationally efficient and can achieve high accuracy in a shorter time than other commonly used methods having similar features. The proposed method is applicable to a wide range of non-linear equations and can be used in various fields of science and engineering where non-linear equations arise. The modified hybrid method provides an effective tool for solving non-linear equations, offering significant improvements in computational efficiency over existing methods.

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1 Introduction

Numerical methods for finding roots, also referred to as root-finding algorithms, are a category of methods for solving nonlinear equations. Finding the roots of a nonlinear equation.The values of the variables

that give the equation its real identity can be done using these techniques. Numerical methods for root finding have a huge range of applications. They are frequently used in disciplines like computer science, engineering, economics, physics, and mathematics. For example, they can be used to find the roots of a polynomial equation, calculate the intersection of two curves, or solve an optimization problem. They can also be used to solve equations with multiple solutions and in the presence of uncertainty. Root-finding numerical methods are also used in various engineering and scientific problems. In mechanical engineering, for example, they are used to design and analyze structures and components. In electrical engineering, root-finding algorithms can be used to design electrical circuits and analyze the behavior of power systems. In physics, root-finding algorithms are used to calculate the trajectories of particles and the behavior of fluids and gases. In addition, root-finding numerical methods can be helpful to solve optimization problems. These problems involve finding the optimal solution, or the one that maximizes or minimizes a certain function, given certain constraints. In economics, for example, root-finding algorithms can be used to optimize the production of goods and services.
Applied mathematics, physical sciences, biological sciences, and so many other disciplines of engineering have mathematical models in terms of nonlinear equations of the following type:

$$f(x) = 0. \tag{1}$$

Here we are faced with the difficulty of obtaining an optimal solution with a higher order of convergence at a favorable CPU time. Many researchers provided us with various numerical algorithms for that purpose. In other words, we say that a numerical method is reasonably good enough if it achieves an optimal solution in the least amount of time as compared to other existing methods with similar characteristics. Everyone nowadays wants to do more work in less time. The most commonly used classical techniques for solving nonlinear equations of type f(x) = 0 include the Newton-Raphson Method, the Regula-False Method, Halley's Method, and many others. As we see, Newton's Raphson method (NRM) is one of the best techniques for solving nonlinear equations with quadratic convergence, having two function evaluations per iteration [14], as shown in Eq^n : (1).

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 1, 2, 3 \ldots \tag{2}$$

In 2021, Qureshi et al.[19] used the Halley technique (HA), one of the old techniques with third-order convergence and three functions of evaluation.

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}, n = 1, 2, 3 \ldots \tag{3}$$

The Newton method is well known for being superior to multiple iterative techniques that are reliable for increasing the order of convergence. Therefore, Weerakoon and Fernando [22] modified a new two-step technique for nonlinear equations.

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{\frac{1}{2}[f'(x_n)+f'(y_n)]}. \end{aligned}\right\}, n = 1, 2, 3 \ldots \tag{4}$$

In Eq^n : (3), first the NRM and then the Newton-like method are used with the arithmetic mean of $f^{'}(x)$ and $f^{'}(y)$ in the denominator. This method leads to a third order of convergence. When the arithmetic mean is changed by the generalized mean [24] heronian mean [20] geometric mean [10], harmonic mean [15], and contraharmonic mean [1], the simple root protects the third order of convergence. In 2020, Naseem et al. [12] adopted the multi-variational iteration approach, a recent three-step technique to resolve the nonlinear scalar equations. Their nominated technique has six function evaluations with a ninth order of convergence, and 1.4422 is the efficiency index of this technique. The method is summarized as $N9_2$, which is shown below:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n) f''(x_n)}{2f'^3(x_n)} \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(y_n) - \beta f(y_n)}, \beta = 1. \end{aligned} \tag{5}$$

In 2021, Qureshi et al. [19] developed a three-step technique for solving nonlinear equations and systems of nonlinear equations. Their method shows performance in analyzing dynamical behavior and real-life problems related to the engineering and science fields. The scheme was shorted as $N9_1$.

$$\left.\begin{aligned} y_n &= x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)} \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n) + f(z_n)}{f'(y_n)} \end{aligned}\right\}, n = 1, 2, 3 \ldots \tag{6}$$

In [7], J.P. Jaiswal and S. Panday, in 2013, proposed a new three-step iterative scheme for solving nonlinear equations. With five function evaluations, the scheme approaches eighth-order convergence. This method is listed as N8, as given below:

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \left(\frac{f(y_n)}{f'(x_n)}\right) \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}. \end{aligned}\right\} n = 0, 1, 2 \ldots \tag{7}$$

Now, in Section 2, we'll look at the proposed method's construction and order of convergence. Section 3 provides a numerical comparison of the proposed method to other existing methods. Sec. 4 will show a comparison of real-life problems, and Sec. 5 will conclude the entire work.

2 Material and Methods

Under the observation of the above literature, we were inspired and developed a new three-step hybrid iterative technique initiated by the the classical Halley method Eq^n : (2), which merged with the two-step technique Eq^n : (3). Every type of merging is not applicable; we have to merge distinct techniques very carefully as we get higher order of convergence with fewer functions of evaluations; that is our main priority.

Halley method Eq^n : (2) :

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}, n = 1, 2, 3 \ldots \tag{8}$$

To be combined with the third-order mean-based two-step technique Eq^n : (3), as follows:

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{\frac{1}{2}[f'(x_n)+f'(y_n)]}. \end{aligned}\right\}, n = 1, 2, 3 \ldots \tag{9}$$

The purpose of presently used method tries to decrease function evaluation costs as much as feasible while enhancing convergence order. Such types of estimations are taken by so many researchers [2, 3, 6, 8, 9, 11, 16, 18, 21, 23]. Hence, after merging Eq^n : (8) and Eq^n : (9), We acquire the proposed method shown below:

$$\begin{aligned} y_n &= x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)}{\frac{1}{2}[f'(y_n) + f'(z_n)]}, \end{aligned} \tag{10}$$

where $n = 1, 2, 3 \ldots$ with an efficiency index of $9^{\frac{1}{6}} \approx 1.4422$ having six function evaluations (three simple evaluations, two first-order evaluations, and one second-order derivative evaluation) and order of convergence to be 9 (proved later). The proposed hybrid method is abbreviated as NP9.

Order of convergence

Theorem 1:

Assume that γ is the root of a differentiable function $f : \Re \subset R \to R$ for an open interval. As a result, the three-step iterative technique $NP9$ such that Eq^n : (10) achieves ninth-order convergence, and the resulting error term is,

$$\varepsilon_{n+1} = \frac{\varepsilon_n^9 f''[\gamma]^8}{128 f'[\gamma]^8} + O\left(\varepsilon_n^{10}\right)$$

where $\varepsilon_n = x_n - \gamma$

Proof: Suppose that γ is the root of $f(x_n)$, x_n be n^{th} nearly to the root by NP9 and $\varepsilon_n = x_n - \gamma$ be the error term after n^{th} iteration. Utilizing the Taylor's Series for $f(x_n)$ about γ, we have

$$f(x_n) = f'[\gamma]\varepsilon_n + \frac{1}{2}f''[\gamma]\varepsilon_n^2 + O[\varepsilon_n]^3 \tag{11}$$

By Taylor's Series for $f\prime(x_n)$ about γ, We obtained

$$f'(x_n) = f'[\gamma] + f''[\gamma]\varepsilon_n + O[\varepsilon_n]^2 \tag{12}$$

By Taylor's Series for $f\prime\prime(x_n)$ about γ, We obtained

$$f''(x_n) = f''[\gamma]\varepsilon_n + \frac{1}{2}f'''[\gamma]\varepsilon_n^2 + O[\varepsilon_n]^3 \tag{13}$$

By using Eq^n : (11), Eq^n : (12) and Eq^n : (13) in first step of Eq^n : (10)

$$\sigma_n = \frac{\varepsilon_n^3 f''[\gamma]^2}{4f'[\gamma]^2} - \frac{3\varepsilon_n^4 f''[\gamma]^3}{8f'[\gamma]^3} + O[\varepsilon_n]^5 \tag{14}$$

By Taylor's series for $f(y_n)$ about γ, we obtained

$$f(y_n) = f'[\gamma]\sigma_n + \frac{1}{2}f''[\gamma]\sigma_n^2 + O[\sigma_n]^3 \tag{15}$$

By Taylor's series for $\frac{1}{f(\gamma_n)}$ about γ, we obtained

$$\frac{1}{f'(y_n)} = \frac{1}{f'[\gamma]} - \frac{f''[\gamma]\sigma_n}{f''[\gamma]^2} + O[\sigma_n]^2 \tag{16}$$

Multiplying Eq^n : (15) and Eq^n : (16) we get

$$\frac{f(y_n)}{f'(y_n)} = \sigma_n - \frac{\sigma_n^2 f''[\gamma]}{2f'[\gamma]} + \frac{\sigma_n^3 f''[\gamma]^2}{2f'[\gamma]^2} + \frac{\sigma_n^4 f''[\gamma]^3}{2f'[\gamma]^3} + O[\sigma_n]^5 \tag{17}$$

Using Eq^n : (17) in Second step of Eq^n : (10)

$$\varepsilon_n = \frac{\sigma_n^2 f''[\gamma]}{2f'[\gamma]} - \frac{\sigma_n^3 f''[\gamma]^2}{2f'[\gamma]^2} + \frac{\sigma_n^4 f''[\gamma]^3}{2f'[\gamma]^3} + O[\sigma_n]^5 \tag{18}$$

By Taylor's series for $f(z_n)$ about γ, we obtained

$$f(z_n) = f'[\gamma]\varepsilon_n + \frac{1}{2}f''[\gamma]\varepsilon_n^2 + O[\varepsilon_n]^3 \tag{19}$$

Differentiate Eq^n : (19), we have

$$f'(z_n) = f'[\gamma] + f''[\gamma]\varepsilon_n + O[\varepsilon_n]^2 \tag{20}$$

Differentiate Eq^n : (15), we have

$$f'(y_n) = f'[\gamma] + f''[\gamma]\sigma_n + O[\sigma_n]^2 \tag{21}$$

Using Eq^n : (19), Eq^n : (20) and Eq^n : (21) in $3^r d$ step of Eq^n : (10)

$$\varepsilon_{n+1} = \frac{\varepsilon_n^9 f''[\gamma]^8}{128f'[\gamma]^8} + \mathrm{O}\left(\varepsilon_n^{10}\right) \tag{22}$$

The above Eq^n : (22) confirmed the 9^{th} convergence.

3 Numerical Comparisons

In table 1, we compare the convergence order, evaluation function, and efficiency index with those which implemented in other existing schemes. Table 2 contains non-linear functions $f(x)$ with an initial guess x_0 and approximate roots x^*, and finally, from tables 3–7, the solutions of the functions given in Table 2. Here, error analysis (ε), an approximate root (x^*), function evaluations (FV), computational cost (COC), the number of iterations (I), and CPU time are compared in tables 3–7. One of the most important factors in numerical analysis is the efficiency index, denoted as $E = \rho^{1/(2n+3n^2+n^3)}$ where E shows efficiency index, ρ shows order of convergence and $2n + 3n^2 + n^3$ is cost of function evaluations, i.e., n shows the number of simple functions, the coefficient of n^2 shows the number of first-order derivatives, and the coefficient of n^3 shows the number of second-order derivatives used per iteration, respectively. The proposed technique outperforms the compared existing method in terms of time, but the N8 method fails at $f_1(x) - f_4(x)$ where $f_5(x)$ Converge to other roots and perform more evaluations than other methods.

Table 1. Under consideration the comparison of efficiency index for the techniques:

Method	Order	FV	E	Recent Function Evaluations per iteration for $n \geq 1$
NP9	9	6	1.4422	$2\,n + 3n^2 + n^3$
NRM	2	2	1.4142	$n + n^2$
HA	3	3	1.4422	$n + n^2 + n^3$
$N9_1$	9	6	1.4422	$3\,n + 2n^2 + n^3$
$N9_2$	9	6	1.4422	$3\,n + 2n^2 + n^3$
N8	8	5	1.5157	$3\,n + 2n^2$

Table 2. Testing functions with initial guesses x_0 and approximate roots x^*

Functions $f(x)$	x_0	x^*
$f_1(x) = x^2 - e^x - 3x + 2$	4	0.2575302854398
$f_2(x) = x^3 - 10$	5	2.154434690031884
$f_3(x) = e^x \sin x - 2x - 5$	0.3	-2.523245230732555
$f_4(x) = (x - 1)^3 - 1$	7	2
$f_5(x) = x^5 + x - 10000$	0	6.308777129972689

Table 3. Numerical comparison of $f_1(x) = x^2 - e^x - 3x + 2$ at initial guess $x_0 = 4$

Method	I	x^*	ϵ	COC	CPU Time
NP9	4	0.2575302854398	0	24	2.12e-3
HRM	8	0.2575302854398	0	16	3.22e-1
HA	6	0.2575302854398	0	18	1.11e-2
N8	–	Fails	–	–	–
$N9_1$	4	0.2575302854398	0	24	0.34e-2
$N9_2$	7	0.2575302854398	0	42	2.12e-3

In this numerical compression proposed method (NP9) gives approximate Solution in four iterations also $N9_1$ approximate solution in four iterations but proposed method takes much smaller CPU time as above discuss that the proposed method is time efficient method.

Table 4. Numerical comparison of $f_2(x) = x^3 - 10$ at initial guess $x_0 = 5$

Method	I	x^*	ϵ	COC	CPU Time
NP9	4	2.154434690031884	0	24	6.16e-4
HRM	8	2.154434690031884	0	16	4.22e-2
HA	7	2.154434690031884	0	21	2.01e-2
$N8$	–	Fails	–	–	–
$N9_1$	4	2.154434690031884	0	24	7.34e-4
$N9_2$	6	2.154434690031884	0	36	3.32e-3

Table No: 4 also shows time efficiency of proposed method as here $N9_1$ take $7.34e^{-}4$ seconds but proposed method required $6.16e^{-}4$ seconds for giving optimal solution.

Table 5. Numerical comparison of $f_3(x) = e^x sinx - 2x - 5$ at initial guess $x_0 = 0.3$

Method	I	x^*	ϵ	COC	CPU Time
NP9	4	-2.523245230732555	0	24	0.10e-2
HRM	4	-2.523245230732555	0	32	7.12e-3
HA	11	-2.523245230732555	0	33	4.06e-4
$N8$	–	Fails	–	–	–
$N9_1$	5	-2.523245230732555	0	30	1.33e-2
$N9_2$	5	-2.523245230732555	0	30	2.22e-2

Table 6. Numerical comparison of $f_4(x) = (x - 1)^3 - 1$ at initial guess $x_0 = 7$

Method	I	x^*	ϵ	COC	CPU Time
NP9	1	2	0	24	0.10e-3
HRM	4	2	0	20	3.12e-2
HA	7	2	0	21	1.11e-1
$N8$	–	Fails	–	–	–
$N9_1$	4	2	0	24	1.33e-3
$N9_2$	6	2	0	36	5.32e-3

Table 7. Numerical comparison of $f_5(x) = x^5 + x - 10000$ at initial guess $x_0 = 5$

Method	I	x^*	ϵ	COC	t
NP9	14	6.308777129972	0	78	2.12e-3
HRM	40	6.308777129972	8.8817841970e-16	80	3.22e-1
HA	23	6.308777129972	8.8817841970e-16	69	1.11e-2
$N8$	50	6.308777129972	1.1490350395e+01	250	8.88e-1
$N9_1$	4	6.308777129972	0	24	0.34e-2
$N9_2$	7	6.308777129972	0	42	2.12e-3

4 Applications of Non-linear Problems in Daily life

The real-life problems related to the engineering and science fields are also resolve with the proposed method, and a comparison with the existing methods will be shown in tables 8–10. For example, the applications of open channel flow, the van der Waals equation, and the force acting between particles kinetic equation problems and beam deigning are used by many authors [4][15][7][13].

4.1 Open-Channel Flow

It is still difficult to design common and ecological way to connect the water stream with factors influencing the flow inside open channels such as trenches, seepage ditches, drains, and sewers. A stream rate is defined as the volume of stream passing a specific point through a space during a specified time period. But another risky circumstance arises when the viable channel becomes clogged. Succeeding that, Manning's condition becomes an important factor for the water stream in an unlocked channel flowing under steady stream positions:

$$Q = \frac{\sqrt{m}}{n} AR^{\frac{2}{3}} \tag{23}$$

where m is the channel's slant, A is its cross-sectional space, R is the channel's pressure-driven sweep, and n is Manning's unpleasantness co-efficient. It is discovered that $A = Wh$ for a rectangular channel with width W and channel depth h, and that:

$$R = \frac{Wh}{(W + 2h)} \tag{24}$$

With these values Eq^n : (24) become:

$$Q = \frac{\sqrt{m}}{n} \, Wh \, [\frac{Wh}{(W + 2h)}]^{\frac{2}{3}} \tag{25}$$

According to the necessity where as one requirement is to decide the profundity of water in the channel for a given amount of water, the above condition can be revised.

$$f_1(h) = \frac{\sqrt{m}}{n} \, Wh \, [\frac{Wh}{(W + 2h)}]^{\frac{2}{3}} - Q \tag{26}$$

The depth of water h in the channel has been assessed while expecting the remainder of the boundaries as $Q = 14.15m^3/s$, $W = 4.572m$, $n = 0.017$ and $m = 0.0015$. The underlying supposition is taken to be $h_0 = 8.5m$. The mathematical outcomes under various plans are displayed in Table:8

Table 8. Mathematical outputs for the example 1 with the initial guess $h_0 = 8.5m$

Method	I	x^*	ϵ	COC	CPU Time
NP9	3	1.465091220295	0	78	2.12e-3
HRM	6	1.465091220295	2.22044604925e-16	80	3.22e-1
HA	5	1.465091220295	2.22044604925e-16	69	1.11e-2
$N8$	3	1.465091220295	2.22044604925e-16	250	8.88e-1
$N9_1$	3	1.465091220295	2.22044604925e-16	24	0.34e-2
$N9_2$	3	1.465091220295	2.22044604925e-16	42	2.12e-3

4.2 Well Known Kinetic Equation Problem

The equation of kinetic problem has the following form:

$$e^{\frac{2100}{T}} = 1.111011T \tag{27}$$

Where T denotes the Temperature of system (26) has been derived from the stirred reactor with the cooling coils [17]. By Putting $T = x$, The (26) could be expressed as the following non-linear function:

Table 9: Mathematical outputs for the example 1 with the initial guess $h_0 = 8.5m$.

$$f_2(h) = h^2 e^{\frac{21000}{h}} - 1.11 * 10^{11} \tag{28}$$

Through which the system's temperature may be calculated. Starting with the initial guess $x_0 = 230$, we iterate the procedure and Table 9 illustrates the associated results from various iteration approaches.

Table 9. Mathematical outputs for the example 4.2 with the initial guess $h_0 = 430$

Method	I	x^*	ϵ	COC	CPU Time
NP9	6	5.517738249303266e+02	0	78	4.004e-3
HRM	19	5.517738249303266e+02	0	80	1.26e-3
HA	50	5.517738249303266e+02	4.984398600310442	69	2.92e-3
$N8$	4	5.517738249303266e+02	0	250	6.63e-3
$N9_1$	50	fails	–	–	–
$N9_2$	50	4.936395122601401e+02	1.444658554135941e+01	42	5.22e-3

4.3 Beam Designing Model

We take into consideration the beam positioning problems from [5], which provides a nonlinear function as

$$f_3(h) = h^4 + 4h^3 - 24h^2 + 16h + 16 \tag{29}$$

According to the fundamental theorem of algebra, the provided function $f_4(h)$ must have exactly four roots (zeroes), as it is a four-degree polynomial. Using the proposed algorithms, we approximate the necessary root using the initial guess $x_0 = 0.50$, and the numerical results are displayed in Table 10.

Table 10. Mathematical outputs for the example 4.3 with the initial guess $h_0 = -0.75$

Method	I	x^*	ϵ	COC	CPU Time
NP9	6	5.517738249303266e+02	0	78	4.004e-3
HRM	19	5.517738249303266e+02	0	80	1.26e-3
HA	50	5.517738249303266e+02	4.984398600310442	69	2.92e-3
$N8$	4	5.517738249303266e+02	0	250	6.63e-3
$N9_1$	50	fails	–	–	–
$N9_2$	50	4.936395122601401e+02	1.444658554135941e+01	42	5.22e-3

5 Concluding Remarks

By combining one of the oldest methods, the Halley's method, with the two-step method, a new ninth order method is developed. In this case, the two-step approach and Halley's method are both third order convergent. Combining these two approaches yields our proposed methodology (NP9), a new ninth order technique with six function evaluation required as per iteration.The asymptotic error and order of convergence of the newly proposed hybrid approach have been theoretically driven using Taylor's expansion and confirmed by computational order of convergence.The estimated efficiency index is 1.4422.A thorough comparison of the suggested technique with other methods currently in use for various non-linear methods taken from the literature has also been done. . In comparison to other ninth order methods, the proposed ninth order method offers competitive results and, in most circumstances, outperforms them. Numerous scholars have designed higher performing versions of the classical Newton method that were inspired by the existing inaccuracy. The primary objective we have in the current research is to make Newton's approach perform better by effectively merging it with the existing two-step method.

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Shehzad Ali: Writing- Original draft preparation, Software, Methodology, Visualization.
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Asif Ali Shaikh: Supervision.
Bahadur Ali: Validation.

Compliance with Ethical Standards

It is declare that all authors don't have any conflict of interest. It is also declare that this article does not contain any studies with human participants or animals performed by any of the authors. Furthermore, informed consent was obtained from all individual participants included in the study.

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