
VALUATION ON A FILTERED MODULE

M.H. ANJOM SHOA*, M.H. HOSSEINI†

*Mohammad Hassan Anjom SHoa, University of Biirjand, anjomshoamh@birjand.ac.ir,
anjomshoamh@yahoo.com

†Mohammad Hossein Hosseini, University of Birjand
mhosseini@birjand.ac.ir

ABSTRACT. In this paper we show if R is a filtered ring and M a filtered R module then we can define a valuation on a module for M . Then we show that we can find an skeleton of valuation on M , and we prove some properties such that derived from it for a filtered module.

Key Words: Filtered module, Filtered ring, Valuation on module, skeleton of valuation.

1 Introduction

In algebra valuation module and filtered R -module are two most important structures. We know that filtered R -module is the most important structure since filtered module is a base for graded module especially associated graded module and completion and some similar results([1],[2],[3],[7],[8]). So, as these important structures, the relation between these structure is useful for finding some new structures, and if M is a valuation module then M has many properties that have many usage for example, Rees valuations and asymptotic primes of rational powers in Noetherian rings and lattices([4],[10]).

In this article we investigate the relation between filtered R -module and valuation module. We prove that if we have filtered R -module then we can find a valuation R -module on it. For this we define $\nu : M \rightarrow \mathbb{Z}$ such that for every $t \in M$, and by lemma(3.1), lemma(3.2), lemma(3.3), lemma(3.4) and theorem(3.1) we show ν has all properties of valuation on R -module M . Also we show if M is a filtered R -module then it has a skeleton of valuation, continuously we prove some properties for M that derived from skeleton of valuation([6],[9]).

2 Preliminaries

Definition 2.1. A filtered ring R is a ring together with a family $\{R_n\}_{n \geq 0}$ of additive subgroups of R satisfying in the following conditions:

- i) $R_0 = R$;
- ii) $R_{n+1} \subseteq R_n$ for all $n \geq 0$;
- iii) $R_n R_m \subseteq R_{n+m}$ for all $n, m \geq 0$.

Definition 2.2. Let R be a ring together with a family $\{R_n\}_{n \geq 0}$ of additive subgroups of R satisfying the following conditions:

- i) $R_0 = R$;
- ii) $R_{n+1} \subseteq R_n$ for all $n \geq 0$;
- iii) $R_n R_m = R_{n+m}$ for all $n, m \geq 0$,

Then we say R has a strong filtration.

Definition 2.3. Let R be a filtered ring with filtration $\{R_n\}_{n \geq 0}$ and M be a R -module with family $\{M_n\}_{n \geq 0}$ of subgroups of M satisfying the following conditions:

- i) $M_0 = M$;
- ii) $M_{n+1} \subseteq M_n$ for all $n \geq 0$;
- iii) $R_n M_m \subseteq M_{n+m}$ for all $n, m \geq 0$,

Then M is called filtered R -module.

Definition 2.4. Let R be a filtered ring with filtration $\{R_n\}_{n \geq 0}$ and M be a R -module together with a family $\{M_n\}_{n \geq 0}$ of subgroups of M satisfying the following conditions:

- i) $M_0 = R$;
- ii) $M_{n+1} \subseteq M_n$ for all $n \geq 0$;
- iii) $R_n M_m = M_{n+m}$ for all $n, m \geq 0$,

Then we say M has a strong filtration.

Definition 2.5. Let M be an R -module where R is a ring, and Δ an ordered set with maximum element ∞ and $\Delta \neq \{\infty\}$. A mapping v of M onto Δ is called a valuation on M , if the following conditions are satisfied:

- i) For any $x, y \in M$, $v(x + y) \geq \min\{v(x), v(y)\}$;

- ii) If $v(x) \leq v(y)$, $x, y \in M$, then $v(ax) \leq v(ay)$ for all $a \in R$;
- iii) Put $v^{-1}(\infty) := \{x \in M | v(x) = \infty\}$. If $v(az) \leq v(bz)$, where $a, b \in R$, and $z \in M \setminus v^{-1}(\infty)$, then $v(ax) \leq v(bx)$ for all $x \in M$
- iv) For every $a \in R \setminus (v^{-1}(\infty) : M)$, there is an $a' \in R$ such that $v((a'a)x) = v(x)$ for all $x \in M$

Definition 2.6. Let M be an R -module where R is a ring, and let ν be a valuation on M . A representation system of the equivalence relation \sim_ν is called a skeleton of ν .

Definition 2.7. A subset S of M is said to be ν -independent if $S \cap \nu^{-1}(\infty) = \emptyset$, and $\nu(x) \notin \nu(Ry)$ for any pair of distinct elements $x, y \in S$. Here, we adopt the convention that the empty subset \emptyset is ν -independent.

Proposition 2.1. Let M be an R -module where R is a ring, and let $v : M \rightarrow \Delta$ be a valuation on M . Then the following statements are true:

- i) If $\nu(x) = \nu(y)$ for $x, y \in M$, then $\nu(ax) = \nu(ay)$ for all $a \in R$;
- ii) $\nu(-x) = \nu(x)$ for all $x \in M$;
- iii) If $\nu(x) \neq \nu(y)$, then $\nu(x + y) = \min\{\nu(x), \nu(y)\}$;
- iv) If $\nu(az) = \nu(bz)$ for some $a, b \in R$ and $z \in M \setminus \nu^{-1}(\infty)$, then $\nu(ax) = \nu(bx)$ for all $x \in M$;
- v) If $\nu(az) < \nu(bz)$ for some $a, b \in R$ and $z \in M$, then $\nu(ax) < \nu(bx)$ for all $x \in M \setminus \nu^{-1}(\infty)$;
- vi) The core ν^{-1} of ν is prime submodule of M ;
- vii) The following subsets constitute a valuation pair of R with core $(M : \nu^{-1}(\infty))$:

$$A_\nu = \{a \in A | \nu(ax) \geq \nu(x) \text{ for all } x \in M\},$$

$$P_\nu = \{a \in A | \nu(ax) \geq \nu(x) \text{ for all } x \in M \setminus \nu^{-1}(\infty)\}$$

Proof. see proposition 1.1 [6] □

Definition 2.8. The pair (A_ν, P_ν) as in Proposition (2.1) is called the valuation pair of R induced by ν or the induced valuation pair of ν .

3 Valuation derived from filtered module

In this section we use the four following lemmas for showing the existence of valuation on filtered module. Let R be a ring with unit and R a filtered ring with filtration $\{R_n\}_{n>0}$ and M be filtered R -module with filtration $\{M_n\}_{n>0}$.

Lemma 3.1. *Let M be filtered R -module with filtration $\{M_n\}_{n>0}$. Now we define $\nu : M \rightarrow \mathbb{Z}$ such that for every $t \in M$ and $\nu(t) = \min \{i \mid t \in M_i \setminus M_{i+1}\}$. Then for all $x, y \in M$ we have $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$.*

Proof. For any $x, y \in M$ such that $\nu(x) = i$ also $\nu(y) = j$, and $\nu(x + y) = k$, so we have $x + y \in M_k \setminus M_{k+1}$. Without losing the generality, let $i < j$ so $M_j \subset M_i$ hence $y \in R_i$. Now if $k < i$, then $k + 1 \leq i$ and $M_i \subset M_{k+1}$ so $x + y \in M_i \subset M_{k+1}$ it is contradiction. Hence $k \geq i$ and so we have $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$. □

Lemma 3.2. *Let M be filtered R -module with filtration $\{M_n\}_{n>0}$. Now we define ν as lemma(3.1). If $\nu(y) \leq \nu(x)$, $x, y \in M$, then $\nu(ax) \leq \nu(ay)$ for all $a \in R$;*

Proof. Let $\nu(x) = i$ and $\nu(y) = j$, since $\nu(x) \geq \nu(y)$ then $M_j \supseteq M_i$. Since R is filtered ring, there exists $k \in \mathbb{Z}$ such that $a \in R_k$ so

$$ax \in R_k M_i \subseteq M_{k+i}$$

$$ay \in R_k M_j \subseteq M_{k+j}$$

we have $i + k \geq j + k$ by $i \geq j$, then $\nu(ax) \geq \nu(ay)$ for all $a \in R$. □

Lemma 3.3. *Let M be filtered R -module with filtration $\{M_n\}_{n>0}$. Now we define ν as lemma(3.1). Put $v^{-1} := \{x \in M \mid \nu(x) = \infty\}$. If $\nu(az) \leq \nu(bz)$, where $a, b \in R$, and $z \in M \setminus v^{-1}(\infty)$, then $\nu(ax) \leq \nu(ay)$ for all $x \in M$.*

Proof. Since $a, b \in R$ and $z \in M$ then there exist $i, j, k \in \mathbb{Z}$ such that $a \in R_i$, $b \in R_j$ and $z \in M_k$ hence

$$az \in R_i M_k \subseteq M_{i+k}$$

$$bz \in R_j M_k \subseteq M_{j+k}$$

Now if $\nu(az) \leq \nu(bz)$ then

$$k + i \leq k + j \implies i \leq j \implies R_j \subseteq R_i$$

So we have $\nu(ax) \leq \nu(bx)$ for all $x \in M$ □

Lemma 3.4. *Let M be filtered R -module with filtration $\{M_n\}_{n>0}$. Now we define ν as lemma(3.1). For every $a \in R \setminus (\nu^{-1}(\infty) : M)$, there is an $a' \in R$ such that $\nu((a'a)x) = \nu(x)$ for all $x \in M$.*

Proof. Let $x \in \nu^{-1}(\infty)$ then for all $a', a \in R$ $\nu((a'a)x) = \nu(x) = \infty$.
Now let $x \notin \nu^{-1}(\infty)$ and for all $a' \in R$ we have $\nu((a'a)x) \neq \nu(x)$. So if $a' \in R \setminus (\nu^{-1}(\infty) : M)$, then $a'a \in R \setminus (\nu^{-1}(\infty) : M)$ and hence $\nu((a'a)x) \neq \infty$.
Let $a \in R_k$, $a' \in R_{k'}$ and $x \in M_i$, then $a'a \in R_{k+k'}$ so $(a'a)x \in M_{i+k+k'}$.
We may have one of following conditions:

- 1) $\nu((a'a)x) < \nu(x)$.
- 2) $\nu(x) < \nu((a'a)x)$

Now if we have (1) then $i + k + k' < i$, it is contradiction .

Consequently $a' \in R_{k'}$ and $a \in R_k$ for $k \in \mathbb{Z}$ then

$$a'a \in R_{k'+k} \implies (a'a)x \in R_{k+k'}M_i \subseteq M_{i+k'+k}.$$

Since $M_{k'+k+i} \subseteq M_i$ hence $(a'a)x \in M_i$. So we have $\nu((a'a)x) < i$ therefore $\nu(x) > \nu((a'a)x)$, it is contradiction with (2). By now we have $\nu(x) = \nu((a'a)x)$. □

Theorem 3.1. *Let R be a filtered ring with filtration $\{R_n\}_{n>0}$, and M be a filtered R -module with filtration $\{M_n\}_{n>0}$. Now we define $\nu : M \rightarrow \mathbb{Z}$ such that for every $t \in M$ and $\nu(t) = \min \{i | t \in M_i \setminus M_{h+1}\}$. Then ν is a valuation on M .*

Proof. i) By lemma (3.1) we have For any $x, y \in M$, $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$;

ii) We have If $\nu(x) \leq \nu(y)$, $x, y \in M$, then $\nu(ax) \leq \nu(ay)$ for all $a \in R$ by lemma(3.2);

iii) Put $\nu^{-1} := \{x \in M | \nu(x) = \infty\}$. If $\nu(ax) \leq \nu(bz)$, where $a, b \in R$, and $z \in M \setminus \nu^{-1}(\infty)$, then then by lemma (3.3) $\nu(ax) \leq \nu(ay)$ for all $x \in M$;

iv) For every $a \in R \setminus (\nu^{-1}(\infty) : M)$, then by lemma(3.4) there is an $a' \in R$ such that $\nu((a'a)x) = \nu(x)$ for all $x \in M$.

So by definition(2.5) ν is a valuation on M if has those conditions. □

Corollary 3.1. *If M be a filtered R -module, then $\nu : M \rightarrow \mathbb{Z}$ has all of properties that explained in Proposition(2.1).*

Proposition 3.1. *If R is a strongly filtered ring and M is a strongly filtered R -module and there exist valuation $\nu : M \rightarrow \mathbb{Z}$ on M , then R should be a trivial filtered R -module.*

Proof. By definition(2.5)(iv) and theorem(3.1) we have for every $a \in R \setminus (v^{-1}(\infty) : M)$, there is an $a' \in R$ such that $v((a'a)x) = v(x)$. Now if $v(a) = i$, $v(a') = j$ and $v(x) = k$ then $i + j + k = k$ so $i + j = 0$, consequently $R_i = R$ for every $i > 0$. \square

Proposition 3.2. *Let M be an R -module, where R is a ring. Then there is a valuation on M , if and only if there exists a prime ideal P of R such that $PM_P \neq M_P$, where M_P is the localization of M at P .*

Proof. see (Proposition 1.3 [6]) \square

Corollary 3.2. *Let M be an filtered R -module, where R is a filtered ring. Then there exists a prime ideal P of R such that $PM_P \neq M_P$, where M_P is the localization of M at P .*

Proof. By theorem(3.1) there is an valuation on M , then by proposition(3.2) there exists a prime ideal P of R such that $PM_P \neq M_P$, where M_P is the localization of M at P . \square

Corollary 3.3. *Let M be an filtered R -module, where R is a filtered ring. Then there is a skeleton on M .*

Proof. By theorem(3.1) there is a valuation on M , then by definition(2.6) we have there is a skeleton on M . \square

Proposition 3.3. *Let M be an filtered R -module where R is a filtered ring, and ν a valuation on M . If Λ is a skeleton of ν , then the following conditions are satisfied:*

- i) Λ is a ν -independent subset of M ;
- ii) For every $x \in M\nu_{-1}(\infty)$, there exists a unique $\lambda \in \Lambda$ such that $\nu(x) = \nu(R\lambda)$.

Proof. By corollary(3.3) Λ is a skeleton of ν and by proposition(1.4, [6]) we have the above conditions. \square

Proposition 3.4. *Let M be an filtered R -module where R is a filtered ring, and ν a valuation on M . If Λ is a skeleton of ν . If $a_1\lambda_1 + \dots + a_n\lambda_n = 0$ where $a_1, \dots, a_n \in R$ and $\lambda_1 \dots \lambda_n \in \Lambda$ are mutually distinct, then $a_i \in (\nu_{-1}(\infty) : M), i = 1, \dots, n$.*

Proof. By corollary(3.3) Λ is a skeleton of ν and by proposition(1.5, [6]) we have If $a_1\lambda_1 + \dots + a_n\lambda_n = 0$ where $a_1, \dots, a_n \in R$ and $\lambda_1 \dots \lambda_n \in \Lambda$ are mutually distinct, then $a_i \in (\nu_{-1}(\infty) : M), i = 1, \dots, n$. \square

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