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Average Error Based Adaptive Regularization Control For The Gradient Constancy Variational Stereo Model

Izhar Ali Amur[1*], Khuda Bux Amur[2], Muzaffar Bashir Arain[2], Mohsin Ali Amur[1], K.N Memon[2]

[1*]Department of General Faculty, Shaheed Benazir Bhutto University, Shaheed Benazir Abad, Nawabshah ; [2]Department Mathematics and Statistics, QUEST Nawabshah, SBA.

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Abstract The study of the stereo vision problems is most crucial and challenging task in image processing and computer vision. The Stereo Vision problems address the investigation of the correspondence between the two images of same scene (stereo pairs) captured from two different views. Generally, these problems are inverse and ill posed. To deal with such problems the energy-based regularization techniques are considered as an efficient and most successful approaches. The adaptive finite element method is used here as discretization method for the partial differential equation obtained from the optimization of the designed energy functional. Such type of the regularization generally depends on the smoothness parameters, and their suitable choice. The choice of the smoothing parameter in an adaptive way and specifically their choice as a scalar function over the whole computational domain is an interesting idea. In this work, a variational model based on the gradient constancy assumption is proposed, moreover a post optimization method (mesh refinement strategy) is designed which is based on a priori estimate called average absolute disparity error estimate. The post optimization is based on an adaptive intelligent algorithm which is efficient in identifying the less regular regions of the computed disparity image and reduces the value of the parameter to refine the grid. Consequently, the smoothness appears in the solution which is the main goal.

***Correspondence Author Email Address:**
amur.izhar@gmail.com

1 Introduction

Stereo vision is an important and challenging part of computer vision research [10, 17, 20, 23]. Stereo Vision problems address the investigation of correspondence between the two images of same scene (stereo pairs) captured from two different views [23]. These views are denoted as right and left views respectively. The shift in the coordinates between the corresponding pixels is called disparity. For each pixel in the left image, one has to determine the corresponding disparity (change of its position w.r.t right image). The corresponding problem is therefore to compute the missing depth information of the scene. The main application of these corresponding problems is to distinguish the stationary objects from the moving objects in the scene, thus, avoiding the obstacles.

The ideas of the image motion are very useful in the edge enhancement in 3D reconstruction, robot navigation (autonomous behavior), medical imaging, video processing, moreover it is also possible to apply in the fluid mechanics techniques using image scenes [6, 11, 12, 21, 23].

Different approaches have been proposed in the literature since 1976, but due to the uncertainty in the diffusion models and reliability of computation for the disparity at whole domain of the computation, moreover the lack of efficiency and the quality of the results the problem have become challenging for the image community [1, 3, 8, 11, 13, 22–24, 27, 29].

Generally, these techniques are divided into four categories, feature based methods where the corresponding corners in the images are matched, area based methods [23] where the pixels of the similar like patches are matched, Fourier based methods where the phase information is used in the Fourier domain, and energy based techniques generally known as optimizations techniques which are used to find the disparity map using some optimization techniques by minimizing the suitable energy functional [1–3, 6–8, 22–24, 29].

To achieve the goal some explicit assumptions are made to compute the dense flow field. Usually, these approaches contain a data term and a regularization term. The data term assumes the brightness doesn't change with respect to image shift [13] and the regularization term reduces the fluctuations in the displacement field [2, 5, 6, 24, 29].

The energy based approaches are further divided in to probabilistic [17, 23] that minimize the discrete energy functional and variational techniques [1–3, 6, 8, 24, 29]that are used to minimize the continuous energy functionals. The probabilistic models are used to compute the displacement field and disparity in terms of optimizing a discrete energy function, which may be done by dynamic programming [18] or belief propagation [16] and graph cuts [17] approaches.

The author [19] has observed some severe drawbacks for the above approaches due to their strict smoothness assumptions like violation of the piece wise constant disparity specifically in the slanted surfaces. However, these restrictions do not apply on the energy based methods, variational approaches. It is observed from the literature that many methods have been proposed and implemented to estimate the displacement field that describes the correspondence between the pixels in two images [2, 13, 14, 18, 22, 24, 27].

To solve these problems, the famous methods like the partial differential equations (PDEs) and more precisely the variational methods have been considered as most successful techniques that minimize a certain cost function [8, 13, 15, 27, 28]. The variational approach in the imaging techniques consists in the looking for an optimal solution of the problem as minimizing an energy functional, for example, the Horn and Schunck functional as the first variational models (1981) for the computation of the optical flow, which consists of solving in the least squares sense an equation which expresses the constancy of the grey level for each pixel of an image in a video sequence [13]. These variational models consist of the data term and a regularization term. The regularization popularized by the pioneering work of Tikhonov to solve the ill-posed problems [1, 26]. These problems are generally inverse and ill posed as they depend on the data (image pairs, camera fields etc..) may be incomplete, moreover they are unstable for even small variations in the data [1–3, 9, 18]. The main classical idea for the solution of such inverse and ill-posed problems is the optimization and regularization where the Thikonov regularization is a famous method to handle such problems [4, 25]. Such type of regularization strategies was extended to total variation TV [9, 18] and Perona-Malik [15]. These strategies generally depend upon the regularization parameters and their optimal choice to control the diffusion process and obtained the smoothed and stable version of computed solution.

The main interest of this work is to study the role of the selection of parameters for imaging problems and the suitable construction of a mathematical model using the gradient constancy assumption for disparity map computation. The regularization of such smoothing parameters in adaptive way is the main task and interesting aspect of the solution of the stereo vision problems. The regularization of the ill-posed problems generally depends on the parameters, and their suitable choice which plays a significant role in the theory of regularization. In most of the smoothing methods the selection of the regularization parameters is proposed as uniform over the whole domain of the computation. The choice of parameter in adaptive way and specifically as a scalar function over the computational domain is an interesting idea. Such adaptive choice was initially introduced by [5] and was extended to the nonlinear problems by [19].

In this work, we propose a model based on the gradient constancy assumption and a post optimization method that allows this choice of parameters to be made in adaptive way and locally, such adaptation will be performed by a priori estimate called average absolute disparity error estimate.

2 PROBLEM DESCRIPTION

2.1 General structure

The stereo image pair is a real valued function defined as

$$f(\mathbf{x}) = \Omega \times \mathbf{R} \to \mathbf{R} \tag{1}$$

where $\mathbf{x} = (x, y)^T, \mathbf{x} \in \Omega$ describes the position within in the domain $\Omega \in \mathbf{R}^2$. The $f(\mathbf{x})$ denotes the image intensities (grey values). The problem is to estimate the one directional dense disparity map u from the minimizing the following variational energy functional

$$F(u) = \int_\Omega (M(f(\mathbf{x}), \alpha(\mathbf{x}) |\nabla u|^2) d\mathbf{x} \tag{2}$$

here $\mathbf{x} = (x, y)^T$ and $\nabla := (\partial x, \partial y)^T$ denotes the spatial gradient and $\alpha(\mathbf{x}) > 0$ (i.e. Strictly positive). The above energy functional Eq.2 is further divided in to two parts (i.e. The data term and the regularization term). The PDE's (Partial differential equations) based imaging techniques were derived from the idea of regularization which is generally based on the constancy assumptions called the brightness constancy assumption and the gradient constancy assumption or the combination of both assumptions.

The brightness constancy assumption: let the image particle moves along with constant brightness only in x-axis defined as

$$f(\mathbf{x}) = f(x+u, t+1) \tag{3}$$

By applying the First order Taylor's expansion on Eq.3 yields the following equation

$$f(\mathbf{x}) = f(\mathbf{x}) + f_x u + f_t \tag{4}$$

the Eq. 4 may be written as

$$f_x u + f_t = 0 \tag{5}$$

where f_x and f_t stands the derivative with respect to x and t respectively.

Gradient constancy assumption: As the brightness constancy assumption does not perform very well with respect to the brightness changes in the real world image scenes. To deal with such issues, the following gradient of the image brightness is considered as modification in the usual choice data term. Mathematically one can write

$$\nabla f = \nabla f(x+u, t+1) \tag{6}$$

By using first order Taylor's expansion on Eq. 6 we have the following equation

$$0 = \nabla f_x u + \nabla f_t \tag{7}$$

with the combination of brightness constancy assumption and the gradient constancy assumption one can consider and moreover we can design the data term by combining the Eq. 4 and Eq. 7 as the following

$$M(u) = (\lambda_1 |f(x+u) - f(x)|^2) + (\lambda_2 |\nabla f(x+u) - \nabla f(x)|^2) \tag{8}$$

The regularization term: Generally, in computer vision problems the addition of the regularization term in the energy functional is standard way to overcome the ill-posed problems. The regularization term is considered here as an essential tool to deal with the ill-posed problems and to reduce the high oscillations in the computed disparity map. The first attempts go back to the authors Tikhono and Arinstien [25]. Then this idea was applied on the optic flow problems by the Horn and Schunck [13]. Recently a stereo variational model appeared as [6], with the isotropic disparity-driven regularization approach. We have proposed here $\alpha(\mathbf{x})|\nabla u|^2$ as the regularization (smoothness) term which play the role of the smoothing in the disparity map u by the fill-in the missing information and reduce the unwanted interference between the data term and the smoothness term. Finally, by adding the designed data term with regularization term the following energy functional is obtained

$$F(u) = \int_\Omega [\{\lambda_1 |f(x+u) - f(x)|^2\} + \{\lambda_2 |\nabla f(x+u) - \nabla f(x)|^2\} + \alpha(\mathbf{x})(|\nabla u|^2)] d\mathbf{x} \tag{9}$$

3 Minimization

3.1 Euler Lagrange Equation

From the methods of calculus of variation the optimization of the energy functional yields the Euler Lagrange's equation. For the simplicity one can re-write the functional 9 in the following way

$$F(u) = \int_\Omega [\{\lambda_1 |f_x u + t_t|^2\} + \{\lambda_2 |\nabla f_x u + \nabla f_t|^2\} + \alpha(\mathbf{x})(|\nabla u|^2)] d\mathbf{x} \tag{10}$$

For better readability and simplicity we write the above eq.10 in the form of **H** as:

$$\mathbf{H} = \lambda H_1 + \alpha(x)H_2 \tag{11}$$

where $H_1 = |f_x u + t_t|^2 + |\nabla f_x u + \nabla f_t|^2$ and $H_2 = |\nabla u|^2$ After the simplifying the terms involving the gradients one can write

$$\frac{\partial H_1}{\partial u} = 2(f_x^2 u + f_t f_x) + 2(f_{xx}^2 u + 2 f_{xx} f_{xy} u + f_{xy}^2 u + f_t f_{xx} + f_t f_{xy}) \tag{12}$$

here

$$\frac{\partial}{\partial x}\left(\frac{\partial H_1}{\partial u_x}\right) = 0, \frac{\partial}{\partial y}\left(\frac{\partial H_1}{\partial u_y}\right) = 0$$

Similarly

$$\frac{\partial H_2}{\partial u} = 0, \frac{\partial}{\partial x}\left(\frac{\partial H_2}{\partial u_x}\right) = 2(u_{xx}), \frac{\partial}{\partial y}\left(\frac{\partial H_2}{\partial u_y}\right) = 2(u_{yy})$$

Finally, by substituting the above computed values in Eq.10 the following Euler-Lagrange equation

$$\lambda_1(f_x^2 u + f_t f_x) + \lambda_2(f_{xx}^2 u + 2 f_{xx} f_{xy} u + f_{xy}^2 u + f_t f_{xx} + f_t f_{xy}) - \alpha(\mathbf{x})(u_{xx} + u_{yy}) = 0 \tag{13}$$

the above problem Eq. 13 is considered with black boundaries, i.e.

3.2 WEAK FORMULATION

It is observed that generally the finite difference methods have been extensively used to find the solution of partial differential equations (PDE's) in imaging techniques [6, 7, 13, 24, 29], instead this work have been conducted with the adaptive finite element method with post optimization strategy. Therefore our next step to is to write the weak formulation for the derived model(i.e., Eq. 13). The main role of the weak formulation is to reduce the order of the differentiation. For weak formulation, we have defined here X as a natural space on which we wish to find the solution of **u** and Set $g = f_t f_x + f_t f_{xx} + f_t f_{xy}$ therefore, one can derive the following weak formulation for the proposed problem Eq. 13

$$\int_\Omega \alpha(x)\nabla u.\nabla w dx - \int_\Omega (f_x^2 u.w + f_{xx}^2 u.w + 2 f_{xx} f_{xy} uw + f_{xy}^2 uw) d\mathbf{x} = \int_\Omega g.w d\mathbf{x} \tag{14}$$

here $H^1(\Omega) = \{u \in L^2(\Omega), \nabla u \in L^2(\Omega)^2\}$ and, $L^2(\Omega)$ is the space of square integrable functions. For our simplicity we create the following notations for the weak formulation

$$a(u, w) = l(g, w), \qquad \forall \quad w \in H^1 \tag{15}$$

here

$$\begin{cases} \alpha(u, w) = \int_\Omega \alpha(\mathbf{x})\nabla u.\nabla w d\mathbf{x} + \int_\Omega (f_x^2 u.w + f_{xx}^2 u.w + 2 f_{xx} f_{xy} uw + f_{xy}^2 uw) d\mathbf{x} \\ l(g, w) = \int_\Omega g.w d\mathbf{x}, \ \forall \ w \in H^1 \subseteq X \end{cases} \tag{16}$$

3.3 DISCRETIZATION

The 2-D finite element method is used here as discretization method. Suppose that the domain Ω_h is considered as a triangular domain with the mesh T_h made up of triangular elements T. The mesh size is considered as $h > 0$. In the usual triangulation procedure the intersection of two different elements is always empty, a vertex, or a whole edge. We introduce the following discrete space as $X_h \subset X$, which denotes the space of continuous piece-wise finite elements over T_h that is

$$X_h := \{w_h \in C^0(\overline{\Omega})| \quad \forall\, T \in T_h, w_{h|T} \in P_1(T)\} \tag{17}$$

Where $P_1(T)$ denotes space of all polynomials in R^2 having degree equal to one. The discrete problem reads the Eq. 15. For our simplicity we develop the following notation for the discretization

$$a(u_h, w_h) = l(g_h, w_h), \forall\, w_h \in X_h \tag{18}$$

here

$$\begin{cases} a(u_h, w_h) = \int_\Omega \alpha(\mathbf{x})\nabla u_h.\nabla w_h d\mathbf{x} + \int_\Omega (f_x^2 u_h.w_h + f_{xx}^2 u_h.w_h + 2f_{xx}f_{xy}u_h w_h + f_{xy}^2 u_h w_h)d\mathbf{x} \\ l(g_h, w_h) = \int_\Omega g_h.w_h d\mathbf{x} \;\; \forall \;\; w_h \in X_h \end{cases} \tag{19}$$

3.4 Computational Algorithm and Mesh Adaption

The inverse and ill-posed problems generally depend on the scaling parameters for regularization. In usual methods the practitioners extensively choose these parameters as uniform over the whole domain. In this work idea of the selection of the parameters is given as the non-uniform distribution in the form of a function of the space over the domain. Such choice is different moreover the choice is given as a post optimization process which is again different from the usual methods where they optimize then discretize where as in this case we discretize then optimize. The work is conducted in the spirit of the authors [5, 18] with slight modification as the application of a priori error estimate instead of a posteriori error estimate. The work is based on local adaptive process as the control for the selection of the value of the parameter $\alpha(\mathbf{x})$. The designed strategy it reduces the value of α in those image regions where the error may occur very large (refines the mesh) and remain stable in the other regions. Algorithm contains the following steps:

1. we start with the Cartesian grid T_h^0 w.r.to. the corresponding image and then we have.
2. compute $u_{\alpha_{0,h}}$ on T_h^0 (mesh) with $\alpha_h = \alpha_0$ a large constant value.
3. the new choice of and prepare the new mesh T_h^1

$$\alpha_K^{n+1} = \max(\frac{\alpha_K^n}{(1 + k * |AADE| - 0.1)}, \alpha_{trh}) \tag{20}$$

The philosophy behind the above formula is that if the AADE in the damaged region is large for example occurring more than 10% then the proposed adaptive strategy will decrease the value of α consequently the mesh becomes refined and error decreases over there. Here the value of k is chosen experimentally.

4 Results and discussions

In this paper the main results obtained from the average error based regularization strategy have been presented and discussed in detail. The proposed mathematical model is designed using the gradient constancy assumption. The discretization of the continuous problem is performed with adaptive finite element method, moreover the given post regularization process is performed by the designed adaptive algorithm. The adaptive algorithm is based on the error-based mesh adaptive process in the spirit of using the a priori error estimate instead of a posteriori error estimate which is easy and suitable for the synthetic images. The simulations have been performed with the software FreeFem ++ (online available at http://www.freefem.org), which is a programming language for the solution of PDEs (partial differential equations) using FEM (finite element method). The selected domain of the computation is a triangular grid where the finite elements are triangles. The experiment has been conducted on the synthetic stereo image pair called Teddy stereo image pair as shown in figure. 1. The reason for the choice the synthetic images is the availability of the ground truth which allows the computation of the average adaptive error and its applicability in the designed adaptive procedure. The inverse and ill-posed problems generally depend on the regularization parameters. The general choice of such parameters has been observed in the literature as uniform (constant) on the whole domain of the computed image.

Figure 1. *(a)* Left image of Teddy stereo pair *(b)* Ground truth *(c)* Right image of Teddy stereo pair

This computational work is divided in two categories, the first part of the work is dedicated to the selection of the parameters as uniform on the whole domain, where one is interested to analyze the performance of the discrete model Eq. 19 and the second part of computation is based on mesh adaption preformed by the proposed algorthim. Furthermore to check the performance of the proposed model we compute average absolute disparity error as the following

$$\mathrm{AADE}(u, t^{gt}) = \frac{1}{|\Omega|}\sum_{i=1}^{n} |u_{\alpha,h} - u^{gt}|^{2} \tag{21}$$

4.1 Example 01

In our first example the performances of proposed algorithm have been analyzed with the uniform selection of the parameters over the whole domain of the computation. The simulations have been produced with the various values of α whereas the constant values for all other scaling parameters have been fixed as one. The main interest in the parameter α is to establish the connection between constant choice of the parameters and the automatic choice of the α with the main adaptive regularization algorithm. The six readings of the experiment have been conducted and the average absolute disparity error (AADE) has been computed for each reading. The values of the parameter and computed results at each reading are given in the Table. 1, Figure. 2 and the Figure. 3(a-f).

Table 1. Results for Average absolute disparity error with uniform selection of values of α .

Reading No	1	2	3	4	5	6
α	100	50	10	1	0.1	0.01
AADE	0.5123	0.4543	0.4226	0.4216	0.4213	0.4345

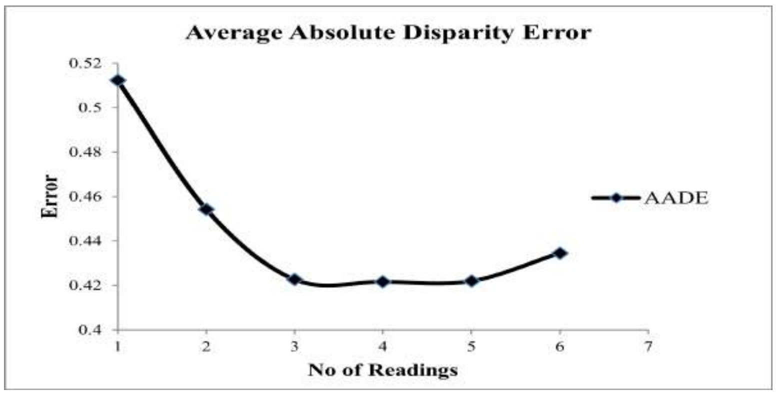
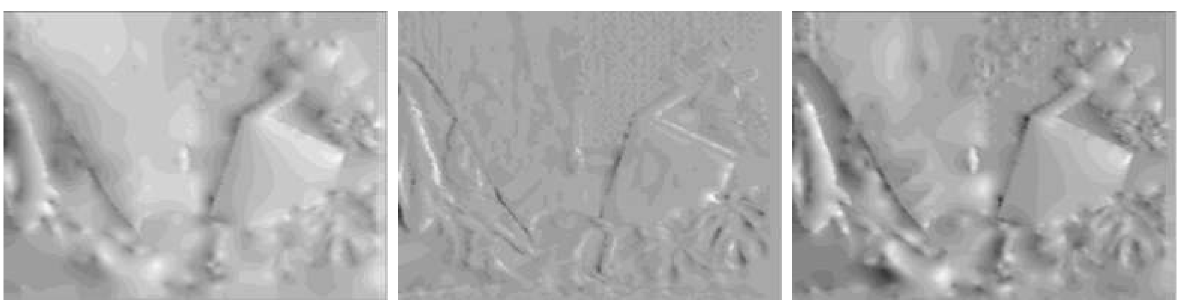
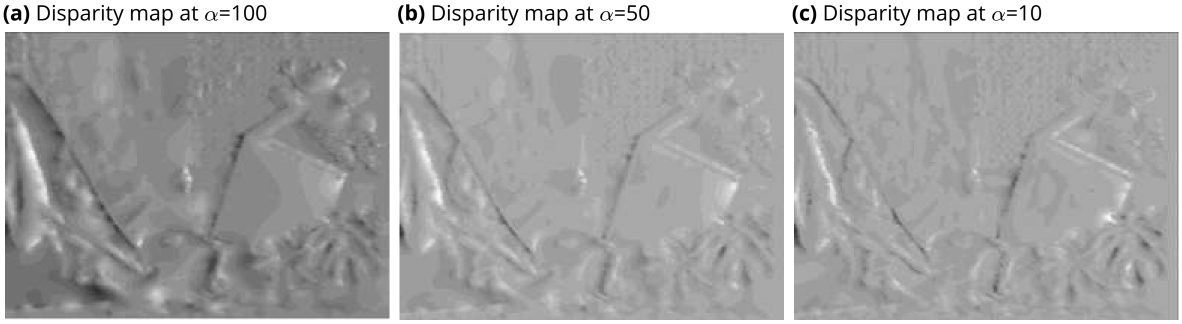


Figure 2. AADE on various iterations for Teddy stereo pair with uniform selection of scaling parameter



(a) Disparity map at α=100 **(b)** Disparity map at α=50 **(c)** Disparity map at α=10



(d) Disparity map at α=1.0 **(e)** Disparity map at α=0.1 **(f)** Disparity map at α=0.01

Figure 3. Results for the disparity map using with uniform selection of scaling parameter

From the above results(i.e.,**Figure 3** (a-f)), one can see that AADE decreases with the smaller values of the α on the computational domain. Such a interesting behavior of the solution assures the significant role of diffusion scaling parameter in inverse problems. It has been also observed that with the very smaller values of the pixel brightness also decreases, which effects the useful information in the computed image. Simply one can conclude that our proposed algorthm with uniform selection of scaling parameter give better results when 0.01 $\leq \alpha \leq$ 1.

4.2 Example 02

In our second example the proposed model based on the gradient constancy assumption and a post optimization method that allows this choice of parameters to be made in locally and in adaptive way has been computed successfully. The algorithm with its intelligent choice of regularization parameters showed a dramatic improvement in the solution at every adaptive step and revealed the precise effects of smoothness with the isotropic mesh adaptation process. Moreover the six adaptive iterations have been performed to check the clear performance of the designed algorithm. The algorithm is initialized with the large regularization parameter α = 1000 and all other parameters are fixed. The designed algorithm is average error based, therefore the results for the average absolute disparity error and maximum average absolute disparity error are given at each adaptive step of the computation one can see the Table (2 and 3) and the Figures. 4 and Figure. 5. The disparity map results at each step of the adaptive process are given in the Figure.6(a-f) and Figure. 7(a-f). Each image is assigned a caption showing the purpose of the image at various adaptive steps.

Table 2. Results for Average absolute disparity error at different iterations

Iteration No	1	2	3	4	5
AADE	0.4359	0.4252	0.4193	0.4149	0.4203

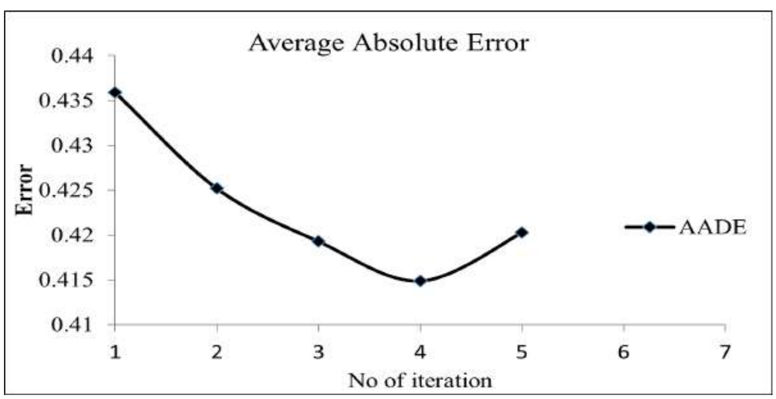


Figure 4. Result for average absolute disparity error

Table 3. Results for Maximum Average absolute disparity error at different iterations

Iteration No	1	2	3	4	5
AADE	0.8244	0.8212	0.8199	0.8198	0.8207

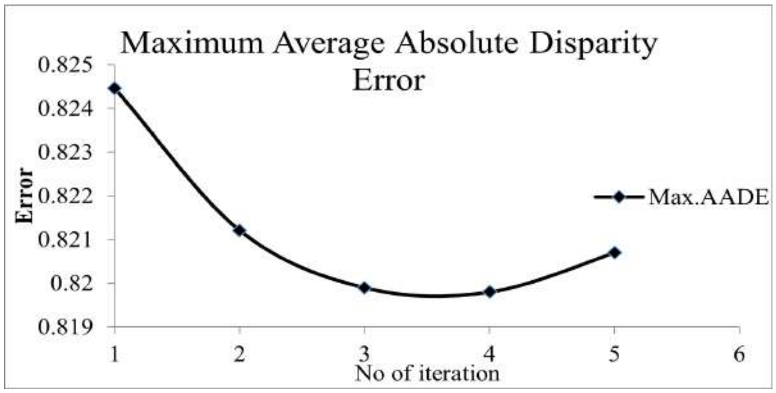
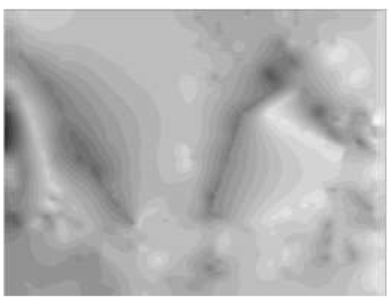
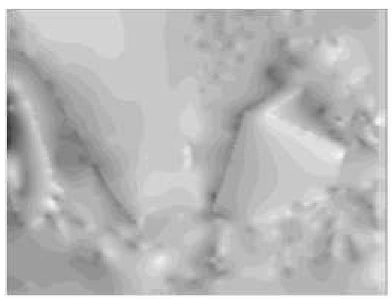


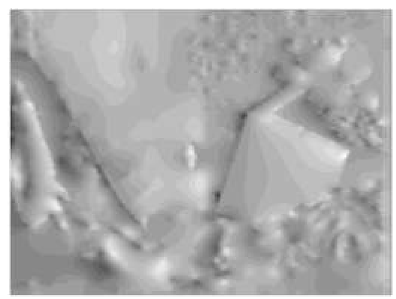
Figure 5. Results for Maximum Average absolute disparity error



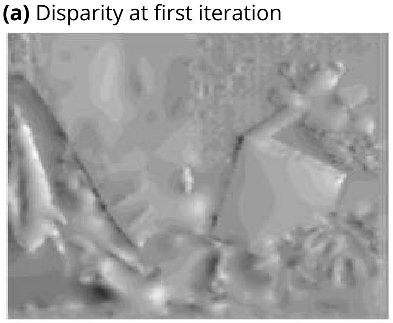
(a) Disparity at first iteration



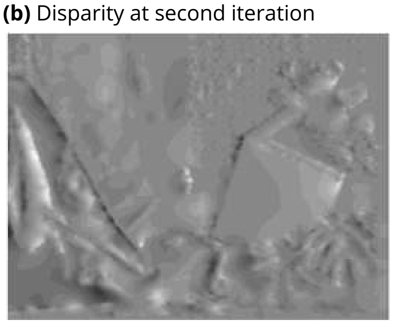
(b) Disparity at second iteration



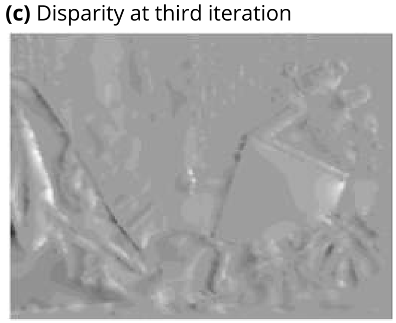
(c) Disparity at third iteration



(d) Disparity at fourth iteration

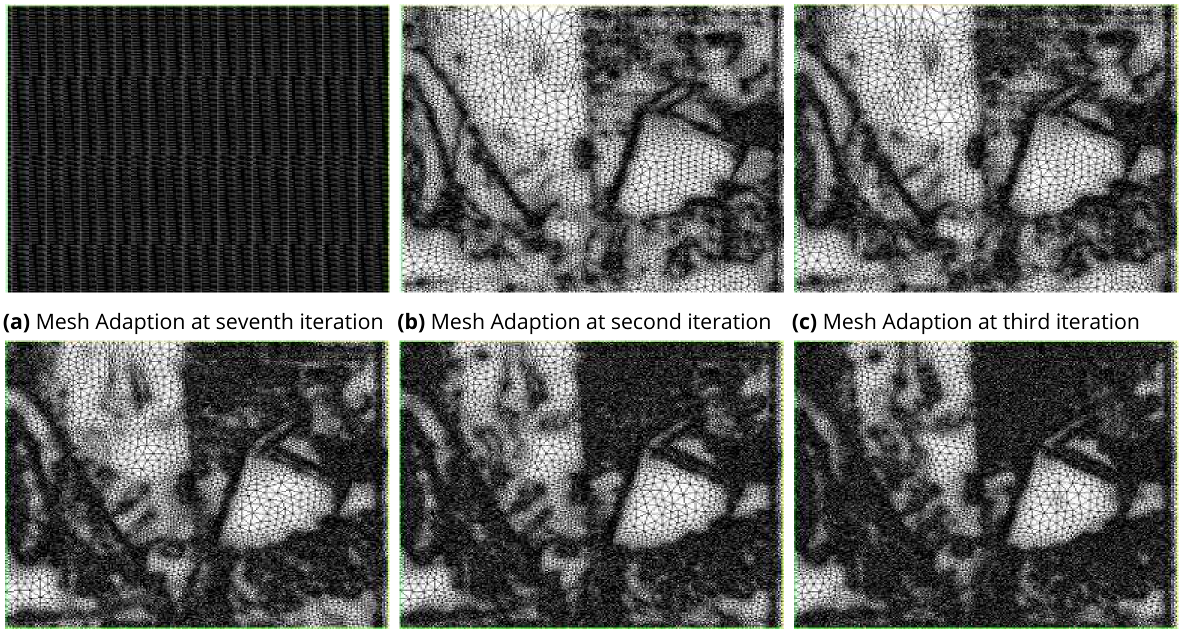


(e) Disparity at fifth iteration



(f) Disparity at sixth iteration

Figure 6. Results for the disparity map using with uniform selection of scaling parameter



(a) Mesh Adaption at seventh iteration **(b)** Mesh Adaption at second iteration **(c)** Mesh Adaption at third iteration

(d) Mesh Adaption at fourth iteration **(e)** Mesh Adaption at fifth iteration **(f)** Mesh Adaption at Sixth iteration

Figure 7. Mesh results for the disparity map at different iterations with local selection of scaling parameter

From the overall performance of the designed post optimization process for gradient constancy stereo model, it has been observed that the proposed method yields the satisfactory results and such regularization plays a significant role as good error based mesh refinement method. The regularization process is prepared to control the values of in the damaged (less regular) regions of the computed image. This algorithm makes the intelligent choice for the values of the smoothing parameter in less regular regions of the disparity map image and decrease over there. Consequently the refinement in the triangular mesh appears and keeps the value constant in the regions where motion does not appear and the regions where the solution is already refined with small error. The overall average disparity error map decreases at each adaptive step. As it is clearly observed that after fourth adaptive iteration the method starts to diverge as the adaptive error starts to increase and the solution starts to damage quickly. From the performance of the method one can say that the main goal of the work is achieved up to a good satisfactory level. The work is interesting and it is possible to apply the idea on various other applications like image denoising, edge detection and optic flow.

5 Conclusion

In this work a novel adaptive method based on gradient constancy assumption has been applied for the computation of disparity depth from the stereo image pairs. The method is based on an adaptive Finite Element Method based discretization along with the post optimization process by an intelligent adaptive algorithm was successfully tested to determine the scalar valued disparity map (stereo depth) from the pair of images of the same scene. Initially the proposed algorithm was successfully tested with the constant (uniform) choice of the smoothing parameters. Thereafter the proposed model based on the gradient constancy assumption and a post optimization method that allows this choice of parameters to be made in locally and in adaptive way has been computed successfully. The algorithm with its intelligent choice of regularization parameters showed a dramatic improvement in the solution at every adaptive step and revealed the precise effects of smoothness with the isotropic mesh adaptation process. From the overall performance of the designed post optimization process for the gradient constancy stereo model, it has been observed that the proposed method yields the satisfactory results, and such regularization plays a significant role as good error based mesh refinement method. The regularization process has been prepared to control the values of α in the damaged (less regular) regions of the computed image. The results were compared with its related existing models were observed satisfactory. The smoothness term requires the crucial attention, that may be modified in the light of the famous authors and mathematical imaging community. The work is interesting, and it is possible to apply the idea to various other applications like image denoising, edge detection, optic flow and so on. As the modifications in mathematical model and settings of regularization parameters, especially in local and adaptive way yields the betterment in visual quality of result but the improvements are still required, particularly in modeling aspect and selection of this novel algorithm with sharp error estimates and threshold settings, which is under consideration as future work.

Author Contributions

Izhar Ali as the first author wrote the first draft of the manuscript, reviewed the literature, performed simulations, and analyse the results. **Khuda bux Amur** designed the methodology and supervised the whole work. **Muzaffar Bashir Arain** reviewing and writing. **Mohsin Ali Amur** editing and final revision of the manuscript. **K. N Memon** final revision of the manuscript.

Compliance with Ethical Standards

It is declared that all authors don't have any conflict of interest.

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References

[1] Amur, K. [2012], 'Some regularization strategies for an ill-posed denoising problem', *International Journal of Tomography and Statistics* **19**(1), 46–59.

[2] Amur, K. [2013], 'A posteriori control of regularization for complementary image motion problem', *Sindh University Research Journal-SURJ (Science Series)* **45**(3).

[3] AMUR, K., CHANDIO, M. and ALI, E. [2013], 'A novel mesh adaptation technique for the computation of stereo depth using texture less simple stereo pairs', *Sindh University Research Journal-SURJ (Science Series)* **45**(4).

[4] Aubert, G., Kornprobst, P. and Aubert, G. [2006], *Mathematical problems in image processing: partial differential equations and the calculus of variations*, Vol. 147, Springer.

[5] Belhachmi, Z. and Hecht, F. [2011], 'Control of the effects of regularization on variational optic flow computations', *Journal of Mathematical Imaging and Vision* **40**, 1–19.

[6] Ben-Ari, R. and Sochen, N. [2007], Variational stereo vision with sharp discontinuities and occlusion handling, *in* '2007 IEEE 11th International Conference on Computer Vision', IEEE, pp. 1–7.

[7] Brox, T., Bruhn, A., Papenberg, N. and Weickert, J. [2004], High accuracy optical flow estimation based on a theory for warping, *in* 'Computer Vision-ECCV 2004: 8th European Conference on Computer Vision, Prague, Czech Republic, May 11-14, 2004. Proceedings, Part IV 8', Springer, pp. 25–36.

[8] Bruhn, A., Weickert, J. and Schnörr, C. [2005], 'Lucas/kanade meets horn/schunck: Combining local and global optic flow methods', *International journal of computer vision* **61**, 211–231.

[9] Chambolle, A. [2004], 'An algorithm for total variation minimization and applications', *Journal of Mathematical imaging and vision* **20**, 89–97.

[10] Grimson, W. E. L. [1985], 'Computational experiments with a feature based stereo algorithm', *IEEE Transactions on pattern analysis and machine intelligence* (1), 17–34.

[11] Hamid, M. S., Abd Manap, N., Hamzah, R. A. and Kadmin, A. F. [2022], 'Stereo matching algorithm based on deep learning: A survey', *Journal of King Saud University-Computer and Information Sciences* **34**(5), 1663–1673.

[12] Hamzah, R. A., Ibrahim, H. and Hassan, A. H. A. [2017], 'Stereo matching algorithm based on per pixel difference adjustment, iterative guided filter and graph segmentation', *Journal of Visual Communication and Image Representation* **42**, 145–160.

[13] Horn, B. K. and Schunck, B. G. [1981], 'Determining optical flow', *Artificial intelligence* **17**(1-3), 185–203.

[14] Ji, P., Li, J., Li, H. and Liu, X. [2021], 'Superpixel alpha-expansion and normal adjustment for stereo matching', *Journal of Visual Communication and Image Representation* **79**, 103238.

[15] Kichenassamy, S. [1997], 'The perona–malik paradox', *SIAM Journal on Applied Mathematics* **57**(5), 1328–1342.

[16] Klaus, A., Sormann, M. and Karner, K. [2006], Segment-based stereo matching using belief propagation and a self-adapting dissimilarity measure, *in* '18th International Conference on Pattern Recognition (ICPR'06)', Vol. 3, IEEE, pp. 15–18.

[17] Kolmogorov, V. and Zabih, R. [2002], Multi-camera scene reconstruction via graph cuts, *in* 'Computer Vision—ECCV 2002: 7th European Conference on Computer Vision Copenhagen, Denmark, May 28–31, 2002 Proceedings, Part III 7', Springer, pp. 82–96.

[18] Lei, C., Selzer, J. and Yang, Y.-H. [2006], Region-tree based stereo using dynamic programming optimization, *in* '2006 IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR'06)', Vol. 2, IEEE, pp. 2378–2385.

[19] Li, G. and Zucker, S. W. [2006], Differential geometric consistency extends stereo to curved surfaces, *in* 'Computer Vision–ECCV 2006: 9th European Conference on Computer Vision, Graz, Austria, May 7-13, 2006, Proceedings, Part III 9', Springer, pp. 44–57.

[20] Lin, X., Wang, J. and Lin, C. [2020], Research on 3d reconstruction in binocular stereo vision based on feature point matching method, *in* '2020 IEEE 3rd International Conference on Information Systems and Computer Aided Education (ICISCAE)', IEEE, pp. 551–556.

[21] Marr, D. and Poggio, T. [1976], 'Cooperative computation of stereo disparity: A cooperative algorithm is derived for extracting disparity information from stereo image pairs.', *Science* **194**(4262), 283–287.

[22] Mozerov, M. G. and Van De Weijer, J. [2015], 'Accurate stereo matching by two-step energy minimization', *IEEE Transactions on Image Processing* **24**(3), 1153–1163.

[23] Scharstein, D. and Szeliski, R. [2002], 'A taxonomy and evaluation of dense two-frame stereo correspondence algorithms', *International journal of computer vision* **47**, 7–42.

[24] Slesareva, N., Bruhn, A. and Weickert, J. [2005], Optic flow goes stereo: A variational method for estimating discontinuity-preserving dense disparity maps, *in* 'Pattern Recognition: 27th DAGM Symposium, Vienna, Austria, August 31-September 2, 2005. Proceedings 27', Springer, pp. 33–40.

[25] Tihonov, A. N. [1963], 'Solution of incorrectly formulated problems and the regularization method', *Soviet Math.* **4**, 1035–1038.

[26] Tikhonov, A. N., Goncharsky, A., Stepanov, V. V. and Yagola, A. G. [1995], *Numerical methods for the solution of ill-posed problems*, Vol. 328, Springer Science & Business Media.

[27] Xiang, X., Zhang, R., Zhai, M., Xiao, D. and Bai, E. [2018], 'Scene flow estimation based on adaptive anisotropic total variation flow-driven method', *Mathematical Problems in Engineering* **2018**.

[28] Yan, T., Gan, Y., Xia, Z. and Zhao, Q. [2019], 'Segment-based disparity refinement with occlusion handling for stereo matching', *IEEE Transactions on Image Processing* **28**(8), 3885–3897.

[29] Zimmer, H., Bruhn, A., Valgaerts, L., Breuß, M., Weickert, J., Rosenhahn, B. and Seidel, H.-P. [2008], Pde-based anisotropic disparity-driven stereo vision., *in* 'VMV', pp. 263–272.