

The effect of uniform and exponential streams on Magnetohydrodynamic flows of viscous fluids

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Abstract This research investigates the equations of motion that govern the behavior of an incompressible fluid with variable viscosity, heat transfer, and the presence of a transverse magnetic field. The study derives exact solutions that describe flows in which the vorticity distribution varies indirectly with the stream function perturbed by uniform and exponential streams. To obtain these solutions, the research introduces a transformation variable that converts a nonlinear system into a linear form. The study's primary objective is to contrast the velocity profile components $u(x, y)$ and $v(x, y)$ in the case of unvarying two-dimensional movements of an incompressible fluid on a plane, with varying viscosity and heat transfer. The study provides insights into the fluid's behavior under different conditions and identifies the impact of various parameters on the velocity profile components.

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1 Introduction

The study of fluid dynamics and magnetohydrodynamics (MHD) has been of great importance in various fields of engineering and science. In recent years, there has been an increasing interest in understanding the effects of different types of streams on MHD flows of viscous fluids. [1–4]. Magnetohydrodynamics (MHD) is a field of research that studies the interaction between electrically conducting fluids and magnetic fields. MHD has significant applications in various fields, such as plasma physics, astrophysics, and material processing industries. MHD flows of viscous fluids under the influence of external forces, such as uniform and exponential streams, have been of great interest to both fundamental research and practical applications [5–7]. This area of research has been studied extensively in the literature, with many researchers focusing on the study of MHD flows of viscous fluids with perturbed vorticity distribution. The study of MHD flows of viscous fluids under different external forces has led to the development of exact solutions, numerical simulations, and experimental investigations [8, 9]. In particular, the study of MHD flows of viscous fluids under the influence of external forces, such as uniform and exponential streams, is of great interest to both fundamental research and practical applications. MHD flows of viscous fluids have been studied extensively in the literature, particularly in the presence of external forcing. The study of MHD flows of viscous fluids with perturbed vorticity distribution has been the focus of many researchers. Taylor [10] was the first to propose the idea of choosing vorticity function to be related to the stream function, resulting in exact solutions that represent the disintegration of double arrays of vortices. Kampe [11] expanded on Taylor's concept by considering the vorticity in the form of $\nabla^2\psi = K\psi$. Meanwhile, Kovasznay [12] developed Taylor's idea further by linking the vorticity to the stream function, which is perturbed by a uniform stream. This approach led to an exact solution that closely resembles the downstream flow behind a two-dimensional grid. Wang [13] presented an exceptional analysis of the documented solutions for the Navier-Stokes equation. He categorized the solutions into three distinct groups: (a) movements where the non-linear inertial factors in the linear momentum equations become zero, (b) resemblance properties of the movements in such a way the motion equations reduce to a sequence of ordinary differential equations, and (c) movements where vorticity is selected to simplify the governing equations in terms of the stream function to a linear equation.

The analytical and numerical study of the Navier-Stokes equations is an important area of research that has numerous practical applications [14, 15] discussed the behaviour of flow. Lin and Tobak [16], Hui [17] and Naeem and Jamil [18] obtained more results by studying similar flows, taking $\nabla^2\psi = K(\psi - Rz)$, $\nabla^2\psi = K(\psi - Ry)$ and $\nabla^2\psi = K(\psi - Ry)$. Obtaining solutions for Newtonian and non-Newtonian fluids in [19, 20] involves assuming a particular form of vorticity distribution or stream function. The effect of uniform streams on MHD flows has been investigated by Balasubramaniam et al. [21] for a viscoelastic fluid, further he showed that the presence of a uniform stream can significantly affect the flow behaviour of viscoelastic fluids in a channel, particularly with respect to the velocity profiles and pressure drop. Similarly Thakur et al. [22] demonstrated that the uniform stream has a significant impact on the velocity profile, skin friction, and heat transfer characteristics of MHD flow in a channel. Bhattacharyya et al. [23] studied the MHD flow of a viscous fluid between parallel plates in the presence of an exponential stream, his study showed that the presence of an exponential stream significantly alters the flow behaviour, particularly with respect to the velocity profiles, shear stress, and pressure drop. Kumar et al. [24] investigated the effect of an exponential stream on the heat transfer characteristics of MHD flow of a viscoelastic fluid. They found that the exponential stream significantly enhances the heat transfer coefficient and Nusselt

number, particularly for higher values of the magnetic field strength. The study of MHD flows of viscous fluids under the influence of external forces, such as uniform and exponential streams, has been of great interest to both fundamental research and practical applications. The analytical and numerical study of the Navier-Stokes equations is an important area of research that has numerous practical applications. Many researchers have contributed to the development of exact solutions, numerical simulations, and experimental investigations of MHD flows of viscous fluids under different external forces. The effect of uniform and exponential streams on MHD flows has been investigated extensively, and the presence of such streams can significantly alter the flow behavior of viscoelastic fluids in a channel, particularly with respect to the velocity profiles, shear stress, pressure drop, and heat transfer characteristics. Overall, the study of MHD flows of viscous fluids under different external forces is an active area of research with significant practical applications, and further research in this area is needed to advance our understanding of these complex flows[25–27].

The effects of uniform and exponential streams on MHD flows of viscous fluids can have significant implications for their stability, mixing, and heat transfer properties [28–30]. In this research, we provide precise solutions to the equation that regulates the stable plane movements of a fluid with variable viscosity and heat transfer, in which the vorticity distribution is proportional to the stream function that is perturbed by a uniform and exponential stream represented as $\nabla^2\psi=K[\psi-U_1(ax+by)-U_1e^{(ax+by)}]$. The aim of this study is to investigate the motion of an incompressible fluid with variable viscosity, taking into account the presence of an attractive field and heat transfer over an infinite plane. The main objective is to determine the general solutions for flows where the vorticity distribution is directly proportional to the stream function, perturbed by uniform and exponential streams. We present a theoretical analysis of the MHD equations under the influence of external forcing and derive a set of coupled differential equations that describe the flow behavior. Also, point out that the solutions for Newtonian and non-Newtonian fluids in [19, 20] are obtained by assuming a specific form of vorticity distribution or stream function. It should be noted that to the best of our knowledge, the exact solutions resulting from this form of vorticity have not yet been considered for either Newtonian or non-Newtonian flows.

2 Basic Magnetohydrodynamic (MHD) equations and problem formulation

Here are the fundamental dimensionless equations that govern a fluid with a variable viscosity is in motion in an incompressible state and finite conductivity in a steady state, under the influence of a magnetic field [31]:

$$u_x+v_y+w_z= 0 \tag{1}$$

$$uu_x+vu_y+wu_z= -p_x+\frac{1}{Re} \left[(2\mu u_x)_x+(\mu(u_y+v_x))_y+(\mu(w_x+u_z))_z \right]+R_H \left[H_3(H_{1z}-H_{3x})-H_2(H_{2x}-H_{1y}) \right] \tag{2}$$

$$uv_x+vw_y+ww_z= -p_y+\frac{1}{Re} \left[(2\mu v_y)_y+(\mu(u_y+v_x))_x+(\mu(u_y+v_x))_x \right]+R_H \left[H_1(H_{2x}-H_{1y})-H_3(H_{3y}-H_{2z}) \right] \tag{3}$$

$$uw_x+vw_y+ww_z= -p_z+\frac{1}{Re} \left[(2\mu w_z)_z+(\mu(w_x+u_z))_x+(\mu(v_z+w_y))_y \right]+R_H \left[H_2(H_{3y}-H_{2z})-H_1(H_{1z}-H_{3x}) \right] \tag{4}$$

$$[uH_2-vH_1]_y-[wH_1-uH_3]_z=\frac{1}{R_\sigma} \left[H_{1xx}+H_{1yy}+H_{1zz} \right] \tag{5}$$

$$[vH_3-wH_2]_z-[uH_2-vH_1]_x=\frac{1}{R_\sigma} [H_{2xx}+H_{2yy}+H_{2zz}] \quad (6)$$

$$[wH_1-uH_3]_x-[wH_3-wH_2]_y=\frac{1}{R_\sigma} [H_{3xx}+H_{3yy}+H_{3zz}] \quad (7)$$

$$uT_x+vT_y+wT_z=\frac{1}{RePr} [T_{xx}+T_{yy}+T_{zz}] + \frac{Ec}{Re} \mu \left[2(u_x^2+v_y^2+w_z^2) + (u_y+v_x)^2 + (w_y+v_z)^2 + (u_z+w_x)^2 \right] + \frac{R_H Ec}{R_\sigma} \left[(H_{3y}-H_{2z})^2 + (H_{1z}-H_{3x})^2 + (H_{2x}-H_{1y})^2 \right] \quad (8)$$

$$H_{1x}+H_{2y}+H_{3z}=0 \quad (9)$$

where u, v, w are the velocity components H_1, H_2, H_3 the components of the magnetic field vector \vec{H} , p the pressure, T the temperature, μ the viscosity all being function of x, y and Re the Reynolds number, Pr the Prandtl number, Ec the Eckert number, R_H the magnetic pressure number and R_σ the magnetic Reynolds number. Considering the flow to be plane transverse flow. We have

$$(u, v, w) = (u, v, 0), (H_1, H_2, H_3) = (0, 0, H), (\cdot)_z = 0 \quad (10)$$

The system of equations (1)-(9), utilizing equations (10) becomes,

$$u_x+v_y=0 \quad (11)$$

$$uu_x+vu_y=-\left[p+R_H\frac{H^2}{2}\right]_x+\frac{1}{Re}\left[(2\mu u_x)_x+(\mu(u_y+v_x))_y\right] \quad (12)$$

$$uv_x+wv_y=-\left[p+R_H\frac{H^2}{2}\right]_y+\frac{1}{Re}\left[(2\mu v_y)_y+(\mu(u_y+v_x))_x\right] \quad (13)$$

$$uH_x+vH_y=\frac{1}{R_\sigma} [H_{xx}+H_{yy}] \quad (14)$$

$$uT_x+vT_y=\frac{1}{RePr} [T_{xx}+T_{yy}] + \frac{Ec}{Re} \mu \left[2(u_x^2+v_y^2) + (u_y+v_x)^2 \right] + \frac{R_H Ec}{R_\sigma} (H_x^2+H_y^2) \quad (15)$$

Equation (11) implies the existence of the stream function ψ such that.

$$u=\psi_y, v=-\psi_x \quad (16)$$

The system of equations (11)-(15) on utilizing equation (16), transform into the following system of equations.

$$J_x=-\psi_x\omega+\frac{1}{Re} [\mu(\psi_{yy}-\psi_{xx})]_y \quad (17)$$

$$J_y=-\psi_y\omega+\frac{1}{Re} [\mu(\psi_{yy}-\psi_{xx})]_x-\frac{4}{Re} [\mu\psi_{xy}]_y \quad (18)$$

$$\psi_yH_x-\psi_xH_y=\frac{1}{R_\sigma} [H_{xx}+H_{yy}] \quad (19)$$

$$\psi_yT_x-\psi_xT_y=\frac{1}{RePr} [T_{xx}+T_{yy}] + \frac{Ec}{Re} \mu \left[4(\psi_{xy})^2 + (\psi_{yy}-\psi_{xx})^2 \right] + \frac{R_H Ec}{R_\sigma} [H_x^2+H_y^2] \quad (20)$$

wherever the function vorticity ω and the general energy function J are provided as follow

$$\omega = -(\psi_{xx}+\psi_{yy}) \quad (21)$$

$$J = \rho + R_H \frac{H^2}{2} + \frac{1}{2} (\psi_x^2 + \psi_y^2) - \frac{2 \mu \psi_{xy}}{Re} \quad (22)$$

After resolving the system of equations (17)-(20), we can obtain the pressure p by using equation (22). Our main focus is to find the answer of the system of equations (17)-(20) as the distribution vorticity is relative to the flow function and disturbed by a uniform and an exponent stream. To achieve this goal, we set

$$\Psi_{xx} + \Psi_{yy} = K \left[\Psi - U_1 (ax + by) - U_2 e^{(ax+by)} \right], \quad (23)$$

where $K, a, b \neq 0$, $a \neq b$ and U are actual quantities. Utilizing agitated stream function

$$\Psi = \psi - U_1 (ax + by) - U_2 e^{(ax+by)}, \quad (24)$$

and retaining equation (23), the equation (21) becomes

$$\omega = -K\Psi. \quad (25)$$

Equation (17) and (18), employing equation (24) and (25) enhances.

$$J_x = \left[\frac{K\Psi^2}{2} \right]_x + aK\Psi \left[U_1 (ax+by) + U_2 e^{(ax+by)} \right] + M_y \quad (26)$$

$$J_y = \left[\frac{K\Psi^2}{2} \right]_y + bK\Psi \left[U_1 (ax+by) + U_2 e^{(ax+by)} \right] - \frac{4}{Re} \times \left[\mu \Psi_{xy} + ab \left(U_1 (ax+by) + U_2 e^{(ax+by)} \right) \right]_y + M_x \quad (27)$$

where

$$M = \frac{1}{Re} \left[\mu \Psi_{yy} - \Psi_{xx} + (b^2 - a^2) \left(U_1 (ax+by) + U_2 e^{(ax+by)} \right) \right]$$

Equation (26) and (27), on using the integrability condition $J_{xy} = J_{yx}$ provide.

$$M_{xx} - M_{yy} + k (b\Psi_x - a\Psi_y) \left[U_1 (ax + by) + U_2 e^{(ax+by)} \right] - \frac{4}{Re} \times \left[\mu \Psi_{xy} + ab(U_1 (ax + by) + U_2 e^{(ax+by)}) \right]_{yx} = 0 \quad (28)$$

Used for the movement of an uncompressed fluid with flexible thickness, heat transfer, and transverse magnetic field, the equation needs to be fulfilled by the function Ψ and viscosity μ is given by equation (28). This equation applies when the vorticity dissemination is relative to the flow function, disturbed by a uniform and an exponent stream. Additionally, by using equation (19), (20) and (21) we get.

$$\left[\Psi_y + b \left(U_1 (ax + by) + U_2 e^{(ax+by)} \right) \right] H_x - \left[\Psi_x + a \left(U_1 (ax+by) + U_2 e^{(ax+by)} \right) \right] H_y = \frac{1}{R\sigma} [H_{xx} + H_{yy}] \quad (29)$$

$$\begin{aligned} & \left[\Psi_y + b \left(U_1 (ax + by) + U_2 e^{(ax+by)} \right) \right] T_x - \left[\Psi_x + a \left(U_1 (ax+by) + U_2 e^{(ax+by)} \right) \right] T_y = \frac{1}{RePr} \\ & \left[T_{xx} + T_{yy} \right] \frac{Ec}{\mu} Re \left[4 \left(\Psi_{xy} + ab(U_1 (ax + by) + U_2 e^{(ax+by)}) \right) \right]^2 + \left(\Psi_{yy} - \Psi_{xx} + (b^2 - a^2) \left(U_1 (ax + by) + \right. \right. \\ & \left. \left. U_2 e^{(ax+by)} \right)^2 + \frac{Ec}{R_H} R\sigma \left[H_x^2 + H_y^2 \right] \right] \end{aligned} \quad (30)$$

equation (23) employing, equation (24) becomes.

$$\Psi_{xx} + \Psi_{yy} - K\Psi = - \left(a^2 + b^2 \right) \left[U_1 (ax+by) + U_2 e^{(ax+by)} \right] \quad (31)$$

Leading the conversion variable as

$$\xi = ax + by \quad (32)$$

converting the equations (28)-(31), into another independent variable ξ , we get

$$\Psi_{\xi\xi} - \Lambda\Psi = -U_1\xi - U_2e^\xi \tag{33}$$

Where

$$\Lambda = \frac{k}{(a^2+b^2)} \quad \text{and} \quad (\mu\Psi)_{\xi\xi} = 0 \tag{34}$$

$$H_{\xi\xi} = 0 \tag{35}$$

$$T_{\xi\xi} + PrEc\mu\Psi^2\Lambda^2(a^2+b^2) + \frac{EcR_H Re Pr}{R_\sigma} H_\xi^2 = 0 \tag{36}$$

3 Exact Solutions

We consider three specific cases in this portion to show some exact results to the equations (33)-(36):

Case I: $\Lambda = -n^2 \quad n > 0$

Case II: $\Lambda = m^2 \quad m > 0$

Case III: $\Lambda = 0$

We take into consideration these cases independently and then conclude the solution of the equation (33)-(36), our plan is that first we find Ψ from equation (33) and employ this Ψ to determine μ, H, T, ψ, u, v and ρ from system of equations (22), (24), and (34)-(36).

3.1 Case I

In this scenario, the result of equation (33) can be expressed as:

$$\Psi = A_1 \sin(n(ax + by) + A_2) - \frac{U_1(ax + by)}{n^2} - \frac{U_2 e^{ax + by}}{n^2 + 1}, \tag{37}$$

where A_1 and A_2 are real constants. Equation (34) employing (37) provides

$$\mu = \frac{A_3(ax + by) + A_4}{A_1 \sin(n(ax + by) + A_2) - \frac{U_1(ax + by)}{n^2} - \frac{U_2 e^{ax + by}}{n^2 + 1}} \tag{38}$$

where A_3 and A_4 are real constants. The solution of equation (35) is

$$H = A_5(ax + by) + A_6, \tag{39}$$

where A_5 and A_6 are real constants. Equation (36), using equations (38) and (39), becomes

$$T_{\xi\xi} + EcPr\Lambda^2(a^2 + b^2)(A_3\xi + A_4)\Psi + \frac{R_H Ec Pr Re H_\xi^2}{R_\sigma} = 0. \tag{40}$$

The solution of (40) is

$$T = \frac{Ec Pr \Lambda^2 (a^2 + b^2)}{12n^3 (n^2 + 1)} [A_3 (12A_1 (n^2 + 1) (n(ax + by) \sin(n(ax + by) + A_2) - 2\cos(n(ax + by) + A_2) + n(n^2 + 1)U_1(ax + by)^4 - 12n^2 U_2 e^{ax + by} \times (ax + by - 2) + 2nA_4 A_1 (n^2 + 1) (6\sin(n(ax + by))) + (ax + by)^3) - 6n^2 U_2 e^{ax + by}] - \frac{Ec R_H Pr Re A_5^2}{2R_\sigma} (ax + by)^2 + A_7(ax + by) + A_8, \tag{41}$$

where A_7 and A_8 are real constants. The stream function Ψ for this case is given by

$$\Psi = A_1 \sin(n(ax + by) + A_2) - \frac{n^2 + 1}{n^2} U_1(ax + by) - \frac{2 + n^2}{n^2 + 1} U_2 e^{ax + by}. \quad (42)$$

It represents a sine stream $\sin(n(ax + by) + A_2)$ in the +ve x-direction and perturbation that is not intermittent in x and y. The velocity component distribution equations (5) and (26) and pressure from equation (10), are provided by

$$u = A_1 n b \cos(n(ax + by) + A_2) - \frac{n^2 + 1}{n^2} b U_1 - \frac{2 + n^2}{n^2 + 1} b U_2 e^{ax + by}, \quad (43)$$

$$v = -A_1 a n \cos(n(ax + by) + A_2) + \frac{a(n^2 - 1)}{n^2} U_1 + \frac{a(2 + n^2)}{n^2 + 1} U_2 e^{ax + by}, \quad (44)$$

$$\begin{aligned} p = \frac{K}{2n^4(n^2 + 1)^2} & \left[A_1 \left(n(1 + n^2)^2 \right) (n^3 A_1 \sin(n(ax + by) + A_2) - 2U_1(n(ax + by) \sin((n(ax + by) + A_2) \right. \\ & + \cos(n(ax + by) + A_2) - 2n(n^4 U_2 (n \sin(n(ax + by) + A_2))) + \cos((n(ax + by) + A_2)) - U_1(1 + n^2)^3 \\ & \cos(n(ax + by) + A_2) - n^3(2 + n^2) U_2 e^{ax + by} (n \cos(n(ax + by) + A_2) - \sin((n(ax + by) + A_2)))) + U_1(1 + n^2)^2 \\ & \left. (U_1(1 + n^2)(ax + by)^2 + 2n^2 U_2(e^{ax + by})) + n^2(1 + n^2)(2 + n^2) U_2 e^{ax + by} 2U_1(ax + by) - 1) + n^2 e^{ax + by} - \right. \\ & \left. \frac{(a^2 + b^2)}{2aRe(1 + n^2)} \left[2b\mu(A_1 n^2(1 + n^2)) \sin(n(ax + by) + A_2) + (2 + n^2)(U_2 e^{ax + by}) - 2aRe n^4(1 + n^2)^3 \right. \right. \\ & \left. \left. (A_1 n^3(1 + n^2) \cos(n(ax + by) - A_2) - (n^2 + 1)^2 U_1 - (2 + n^2)n^2 U_2(e^{ax + by})^2 - R_H \frac{H^2}{2} + A_9. \right. \right. \end{aligned} \quad (45)$$

where A_9 is a real constant.

3.2 Case II

In this instance

$$\Psi = B_1 e^{m(ax + by)} + B_2 e^{-m(ax + by)} + \frac{U_1(ax + by)}{m^2} - \frac{U_2 e^{ax + by}}{1 - m^2}, \quad (46)$$

where B_1 and B_2 are real constants equation (34) employing (46) gives

$$\mu = \frac{B_3(ax + by) + B_4}{B_1 e^{m(ax + by)} + B_2 e^{-m(ax + by)} + \frac{U_1(ax + by)}{m^2} - \frac{U_2 e^{ax + by}}{1 - m^2}}, \quad (47)$$

where B_3 and B_4 are real constants. The solution of equation (35) is

$$H = B_5(ax + by) + B_6, \quad (48)$$

where B_5 and B_6 are real constants. Equation (36), using equations (47) and (48)

$$T_{\xi\xi} + EcPr\Lambda^2(a^2 + b^2)(B_3\xi + B_4)\Psi + \frac{R_H EcPrReH_5^2}{R_\sigma} = 0. \quad (49)$$

The solution of equation (49) is

$$T = \frac{EcPr\Lambda^2 (a^2 + b^2)}{12m^3(1 - m^2)} [B_3\{12 (1 - m^2) \{B_1 e^{m(ax+by)}(2 - m(ax + by)) - B_2 e^{-m(ax+by)} (m (ax + by) + 2) - m (1 - m^2) U_1 (ax + by)^4 - 12mU_2 (e^{m(ax+by)} (m (ax + by) - 1) m^2 e^{m(ax+by)}) - 2B_4m (1 - m^2) 6B_1 e^{m(ax+by)} + 6B_2 e^{-m(ax+by)} + U_1 (ax + by)^3 - 6n^2U_2 e^{m(ax+by)}\} - \frac{R_H EcPrReB_5^2}{2R_\sigma} (ax + by)^2 + B_7 (ax + by) + B_8, \quad (50)$$

where B_7 and B_8 are actual constants. For this instance stream function

Here we use Stream function ψ

$$\psi = B_1 e^{m(ax+by)} + B_2 e^{-m(ax+by)} + \frac{1 - m^2}{m^2} U_1 (ax + by) - \frac{2 - m^2}{1 - m^2} U_2 e^{(ax+by)}. \quad (51)$$

The velocity distribution and pressure components for this case are provided as follows:

$$u = mbB_1 e^{m(ax+by)} - mbB_2 e^{-m(ax+by)} + \frac{U_1 b (1 - m^2)}{m^2} - \frac{2 - m^2}{1 - m^2} U_2 b e^{(ax+by)}, \quad (52)$$

$$v = -maB_1 e^{m(ax+by)} + maB_2 e^{-m(ax+by)} - \frac{aU_1 (1 - m^2)}{m^2} + \frac{2 - m^2}{1 - m^2} U_2 a e^{(ax+by)}, \quad (53)$$

$$p = \frac{K}{2m^4(1 - m^2)^2} \left[nB_1 (1 - m^2)^2 e^{m(ax+by)} m^3 B_1 e^{m(ax+by)} + 2U_1 (m (ax + by) - 1) + 2m^5 B_1 U_2^2 \times e^{(m+1)(ax+by)} + m (1 - m^2) B_2 (1 - m^2) e^{-m(ax+by)} m^3 B_2 e^{-m(ax+by)} + 2U_1 (m (ax + by) - 1) - 2m^3 U_1 e^{(ax+by)(1-m)} + (1 - m^2) U_1 (1 - m^2) (m^2 + 1) 2 (B_1 e^{m(ax+by)} - B_2 e^{-m(ax+by)}) + (ax + by)^2 + 2m^2 U_1 U_2 e^{(ax+by)} (m^2 (ax + by) + 1) + m^5 U_2 e^{2(ax+by)} \right] + \frac{b}{a Re} \left[\mu (b^2 - a^2) (m^2 B_1 e^{m(ax+by)}) \right] + \left(m^2 B_2 e^{-m(ax+by)} - \frac{U_2}{1 - m^2} e^{(ax+by)} \right) - R_H \frac{H^2}{2} - \left[\frac{(a^2 + b^2)}{2} + 2 \frac{\mu ab}{Re} m B_1 e^{m(ax+by)} - m B_2 e^{-m(ax+by)} - \frac{U_1}{m^2} - \frac{U_2}{1 - m^2} e^{(ax+by)} \right]^2 + B_9, \quad (54)$$

where B_9 is a real constant.

3.3 Case III

In this instance

$$\Psi = -\frac{U_1 (ax + by)^3}{6} - U_2 e^{(ax+by)} + C_1 (ax + by) + C_2, \quad (55)$$

$$\mu = \frac{C_3 (ax + by) + C_4}{-\frac{U_1 (ax+by)^3}{6} - U_2 e^{(ax+by)} + C_1 (ax + by) + C_2}, \quad (56)$$

$$H = C_5 (ax + by) + C_6, \quad (57)$$

$$T = \frac{EcPr\Lambda^2 (a^2 + b^2)}{180} \left[C_3 \left(180U_2e^{(ax+by)} (2 - (ax + by)) + (ax + by)^3 \left(U_1 (ax + by)^3 \right) - 15 (C_1 (ax + by) + 2C_2 + C_3 (ax + by)^2 \left(\frac{3}{2}U_1 (ax + by)^3 - 30 (C_1 (ax + by) + 3C_2) - 180U_2e^{(ax+by)} \right) \right) \right] - \frac{R_H EcPrRe H_5^2}{2R\sigma} (ax + by)^2 + (ax + by) C_7 + C_8, \tag{58}$$

where C_1, C_2, \dots, C_8 are real constant. $\Lambda = 0$ corresponds to an irrotational flow and it is the following uniform flow

$$\psi = -\frac{U_1 (ax + by)}{6} (ax + by)^2 + 6 - 2U_2e^{(ax+by)} + C_1 (ax + by) + C_2. \tag{59}$$

The components of velocity distribution and pressure, in this case, are given by

$$u = -\frac{U_1 b}{2} (ax + by)^2 + 2 - 2bU_2e^{(ax+by)} + bC_1, \tag{60}$$

$$v = \frac{aU_1}{2} (ax + by)^2 + 2 + 2aU_2e^{(ax+by)} - aC_1, \tag{61}$$

$$p = K \left[U_1^2 (ax + by)^2 - \frac{U_1 U_2}{2} (ax + by)^2 e^{(ax+by)} - 2e^{(ax+by)} ((ax + by) + 1) - \frac{C_1 U_1}{8} (ax + by)^4 - \frac{C_2 U_1 (ax + by)^3}{6} - \frac{(U_1 + C_1 + C_2) U_1}{24} (ax + by)^4 + U_2 (U_1 + C_1 + C_2) e^{(ax+by)} + \frac{C_1 (U_1 + C_1 + C_2) (ax + by)^2}{2} + C_2 (U_1 + C_1 + C_2) (ax + by) + \left[\frac{b\mu (b^2 - a^2)}{aRe} - \frac{2\mu a b}{Re} \right] \left[\frac{U_1}{2} (ax + by)^2 + U_1 + C_1 \right]^2 - R_H \frac{H^2}{2} + \frac{U_1}{2} (a^2 - b^2) (ax + by) + C_9, \tag{62}$$

where C_9 is real constant.

4 Result and Discussion

This research has discovered precise solutions to the equation of motion that controls the unchanging two-dimensional movements of an incompressible fluid with heat transfer and varying viscosity. The equation $\nabla^2 \psi = K [\psi - U_1 (ax + by) - U_2 e^{(ax+by)}]$ defines a uniform and exponential stream that perturbs the vorticity distribution, which is proportional to the stream function. To analyze the physical aspects of our findings, we have created several graphs. Our primary objective has been to contrast the velocity profiles components $u(x, y)$ and $v(x, y)$ in the case of unvarying two-dimensional movements of an incompressible fluid on a plane, with varying viscosity and heat transfer. We have studied the variation of emerging parameters of interest to infer these results. The velocity components considered are only for Case-I ($\Lambda = 1$) and Case-II ($\Lambda = 1$). We have presented graphs of the velocity components $u(x, y)$ and $v(x, y)$ against x and y for various values of emergent parameters a, b, n, m and U_1 and U_2 . We have used Mathcad as computational software and for convenience, all graphs are sketched using $A_1 = 0.1, A_2 = 2, B_1 = 0.002, B_2 = 0.003, a = 2, b = 3, n = 2, m = 2, U_1 = 0.1, U_2 = 0.1$. All Figures (1-10) for case-I and Figures (11-20) (a) for case-II show the velocity components of $u(x, y)$ and $v(x, y)$ as a function of " x " and " y " treated as constants with " $y = 2$ ".

In all Figures (1-20) (b), the velocity components $u(x,y)$ and $v(x,y)$ are considered as a function of "y" and "x" treated as constant with "x= 1".

For Case-I: The graphs presented in Figures (1-4) depict the velocity components $u(x,y)$ and $v(x,y)$ at different values of the parameters "a" and "b". Interestingly, these graphs reveal that the velocity components $u(x,y)$ and $v(x,y)$ exhibit opposite behavior with respect to the parameters "a" and "b". Specifically, the first velocity component, $u(x,y)$, decreases as "a" increases while the second velocity component, $v(x,y)$, increases with "a". Similarly, as "b" increases, $u(x,y)$ decreases while $v(x,y)$ increases. Furthermore, it is observed that the absolute values of $u(x,y)$ are greater than those of $v(x,y)$ across all graphs. Moving on to Figures (5-6), these illustrate how the velocity components $u(x,y)$ and $v(x,y)$ are influenced by the parameter "n". The graph shows that $u(x,y)$ increases as "n" increases while $v(x,y)$ decreases. Finally, Figures (7-10) demonstrate the impact of the parameters U_1 and U_2 on the velocity components $u(x,y)$ and $v(x,y)$. Interestingly, both parameters have qualitatively similar effects on the velocity profiles. Specifically, $u(x,y)$ decreases with increasing U_1 and U_2 while $v(x,y)$ exhibits the opposite behaviour.

For Case-II: Figures (11-14) have been created to examine the impact of the parameters "a" and "b" on the velocity components $u(x,y)$ and $v(x,y)$. It is evident that these parameters have opposite effects on the two components of the velocity profile. Specifically, the first component $u(x,y)$ increases while the second component $v(x,y)$ decreases as "a" and "b" increase. Figures (15-16) have been prepared to explore the influence of the parameter "m" on the velocity profile. These figures show that the first component $u(x,y)$ decreases, while the second component $v(x,y)$ grows exponentially as "m" increases. To analyze the effect of the parameter U_1 on the two components of the velocity field, Figures (17-18) have been created. These figures demonstrate that both velocity components decrease with increasing U_1 . Finally, Figures (19-20) have been presented to compare the two velocity components $u(x,y)$ and $v(x,y)$ for various values of U_2 . These figures clearly illustrate that the component $u(x,y)$ decreases while the component $v(x,y)$ increases as U_2 increases. It is important to note that all material constants used in the figures are in SI units.

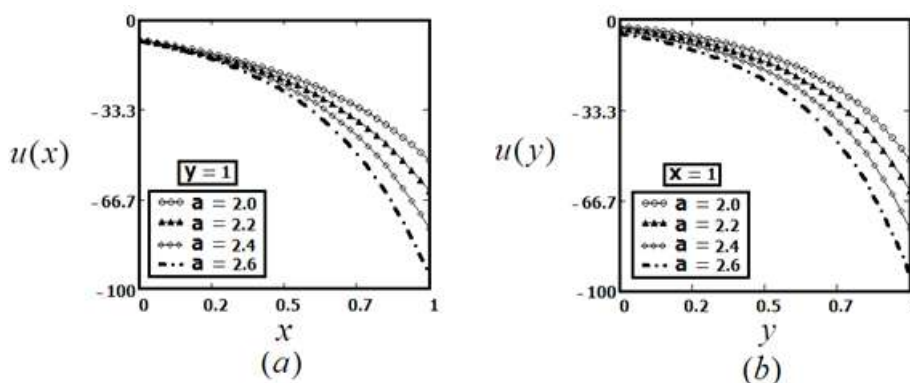


Figure 1. Velocity Profiles $u(x,y)$ for viscous fluid given by equations (43), for $A_1 = 0.1$, $A_2 = 2$, $b=3$, $n=2$, $U_1 = 0.1$, $U_2 = 0.1$ and differing values of a .

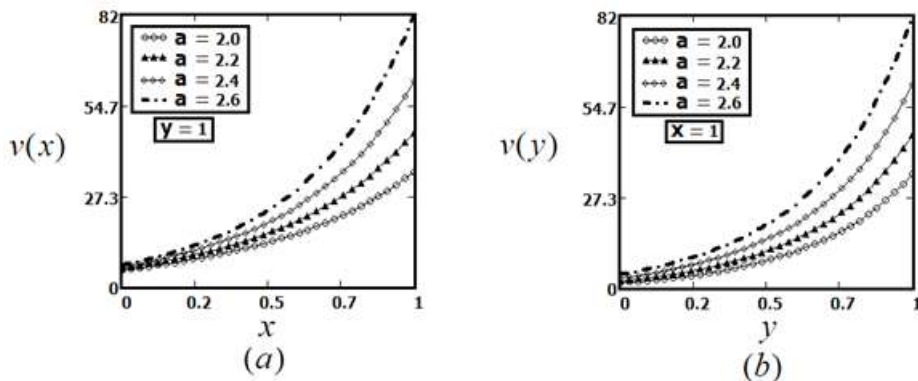


Figure 2. Velocity Profiles $v(x,y)$ for viscous fluid given by equations (44), for $A_1 = 0.1, A_2 = 2, b = 3, n = 2, U_1 = 0.1, U_2 = 0.1$ and differing values of a .

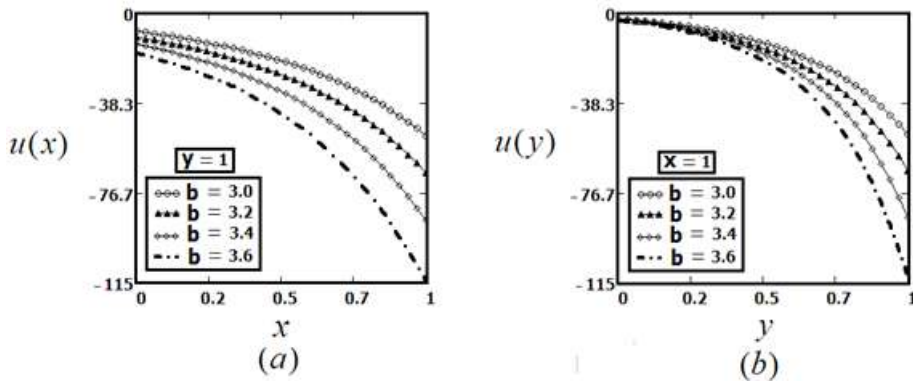


Figure 3. Velocity Profiles $u(x,y)$ for viscous fluid given by equations (43), for $A_1 = 0.1, A_2 = 2, a = 2, n = 2, U_1 = 0.1, U_2 = 0.1$ and differing values of b .

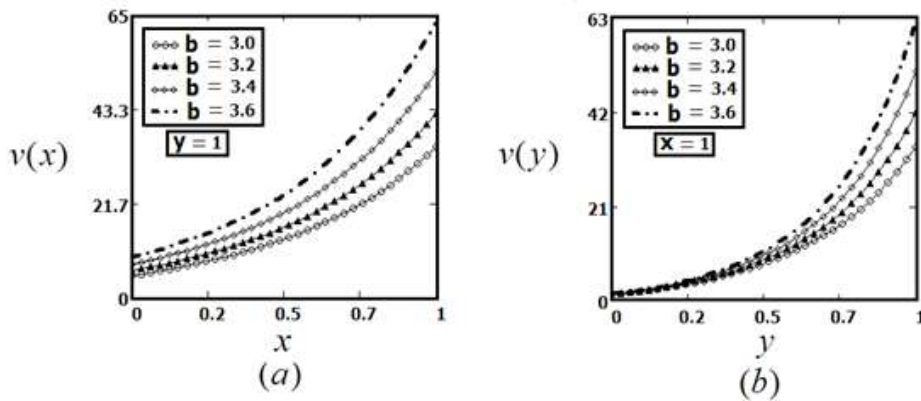


Figure 4. Velocity Profiles $v(x,y)$ for viscous fluid given by equations (44), for $A_1 = 0.1, A_2 = 2, a = 2, n = 2, U_1 = 0.1, U_2 = 0.1$ and differing values of b .

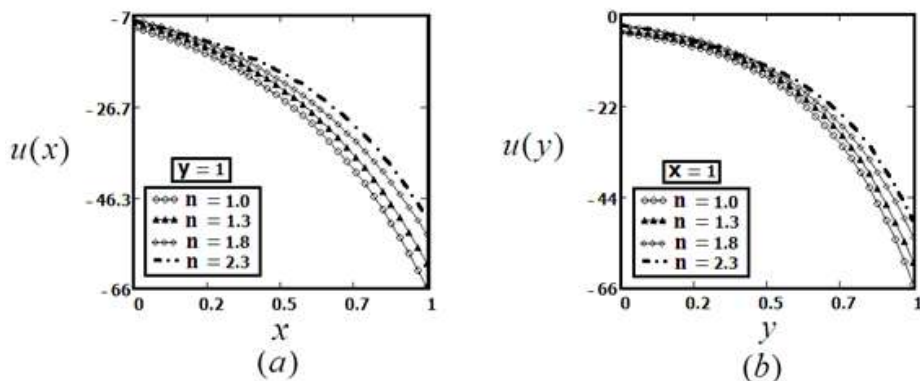


Figure 5. Velocity Profiles $u(x,y)$ for viscous fluid given by equations (43), for $A_1 = 0.1, A_2 = 2, a = 2, b = 3, U_1 = 0.1, U_2 = 0.1$ and differing values of n .

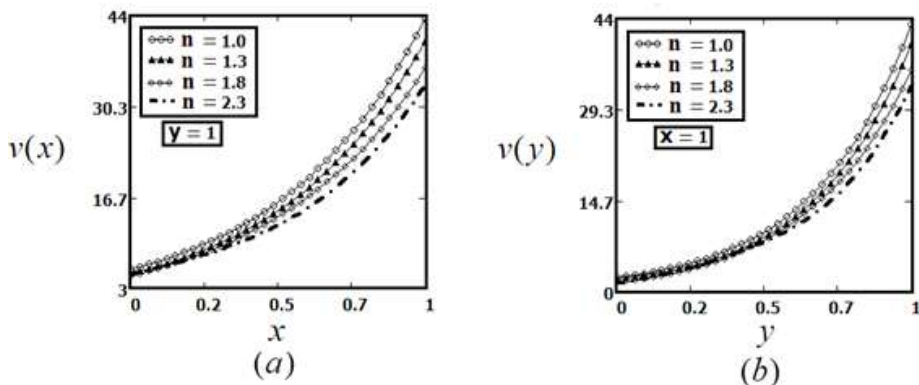


Figure 6. Velocity Profiles $v(x,y)$ for viscous fluid given by equations (44), for $A_1 = 0.1, A_2 = 2, b = 3, a = 2, U_1 = 0.1, U_2 = 0.1$ and differing values of n .

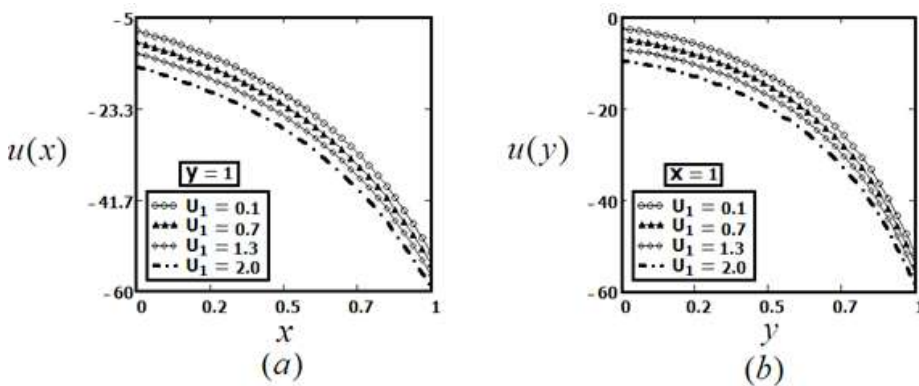


Figure 7. Velocity Profiles $u(x,y)$ for viscous fluid given by equations (43), for $A_1 = 0.1, A_2 = 2, b = 3, n = 2, a=2, U_2 = 0.1$ and differing values of U_1 .

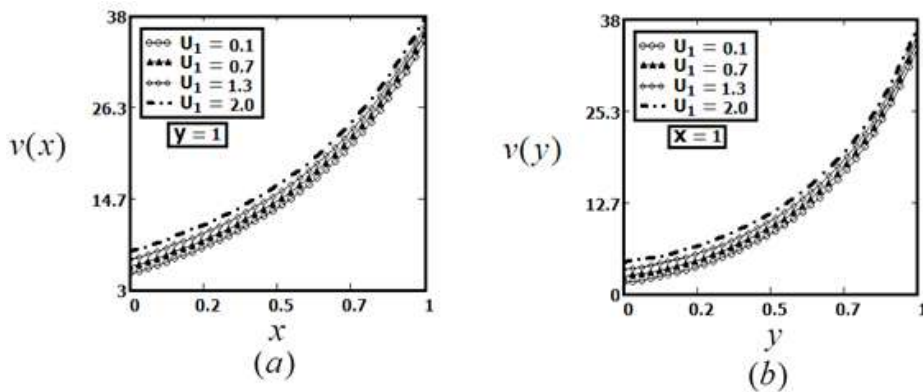


Figure 8. Velocity Profiles $v(x,y)$ for viscous fluid given by equations (44), for $A_1 = 0.1, A_2 = 2, b = 3, n = 2, a=2,$
 $U_2 = 0.1$ and differing values of U_1 .

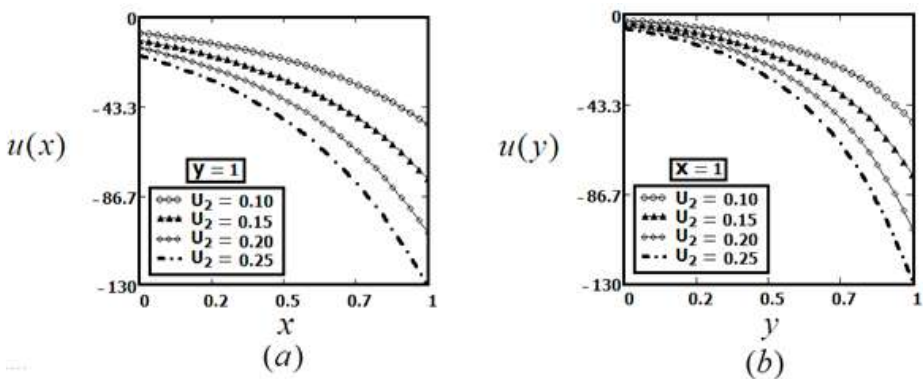


Figure 9. Velocity Profiles $u(x,y)$ for viscous fluid given by equations (43), for $A_1 = 0.1, A_2 = 2, b = 3, n = 2, a=2,$
 $U_1 = 0.1$ and differing values of U_2 .

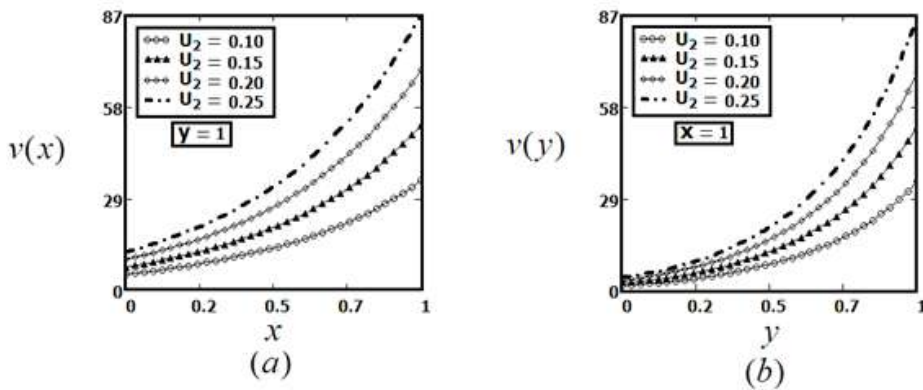


Figure 10. Velocity Profiles $v(x,y)$ for viscous fluid given by equations (44), for $A_1 = 0.1, A_2 = 2, b = 3, n = 2, a=2,$
 $U_2 = 0.1$ and differing values of U_2 .

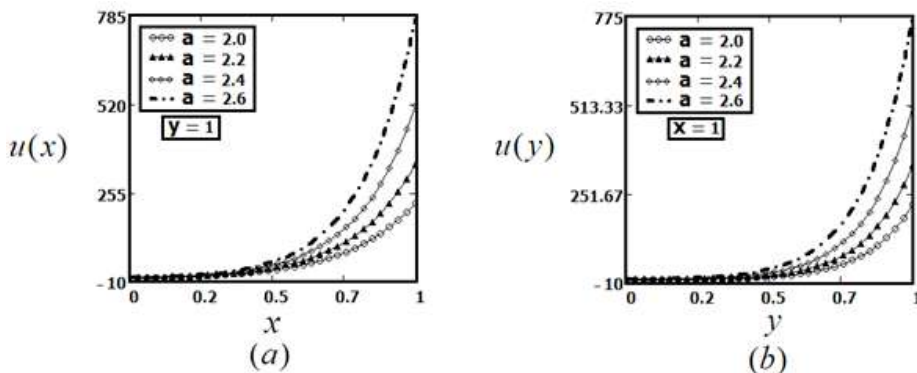


Figure 11. Velocity Profiles $u(x,y)$ for viscous fluid given by equations (52), for $B_1 = 0.002$, $B_2 = 0.003$, $b = 3$, $m = 2$, $U_1=0.1$, $U_2 = 0.1$ and differing values of a .

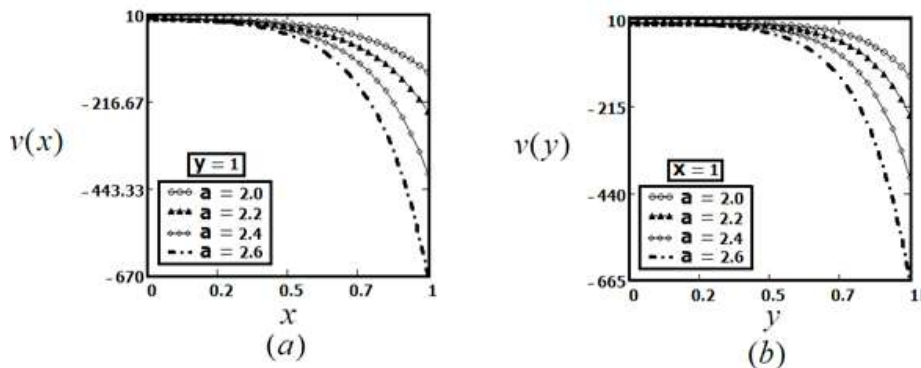


Figure 12. Velocity Profiles $v(x,y)$ for viscous fluid given by equations (53), for $B_1 = 0.002$, $B_2 = 0.003$, $b = 3$, $m = 2$, $U_1 = 0.1$, $U_2 = 0.1$ and differing values of a .

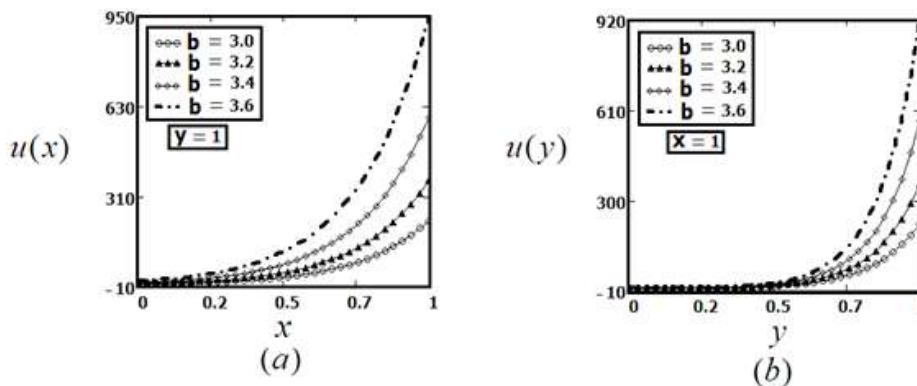


Figure 13. Velocity Profiles $u(x,y)$ for viscous fluid given by equations (52), for $B_1 = 0.002$, $B_2 = 0.003$, $a = 2$, $m = 2$, $U_1=0.1$, $U_2 = 0.1$ and differing values of b .

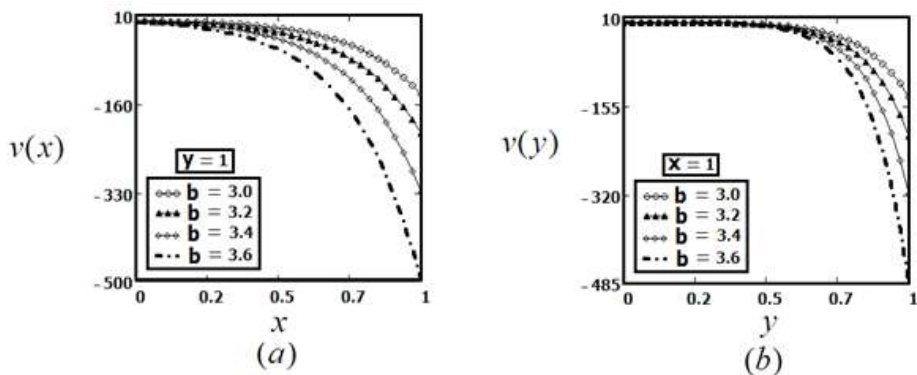


Figure 14. Velocity Profiles $v(x, y)$ for viscous fluid given by equations (53), for $B_1 = 0.002, B_2 = 0.003, a = 2, m = 2, U_1=2, U_2 = 0.1$ and differing values of b .

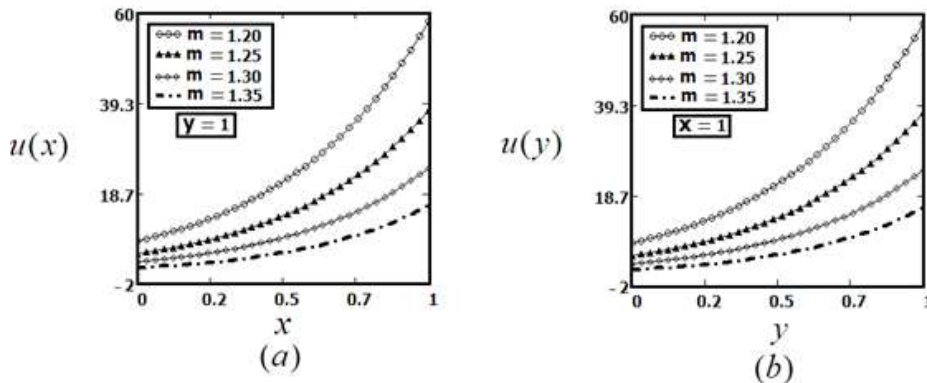


Figure 15. Velocity Profiles $u(x, y)$ for viscous fluid given by equations (52), for $B_1 = 0.002, B_2 = 0.003, a = 2, b = 3, U_1=0.1, U_2 = 0.1$ and differing values of m .

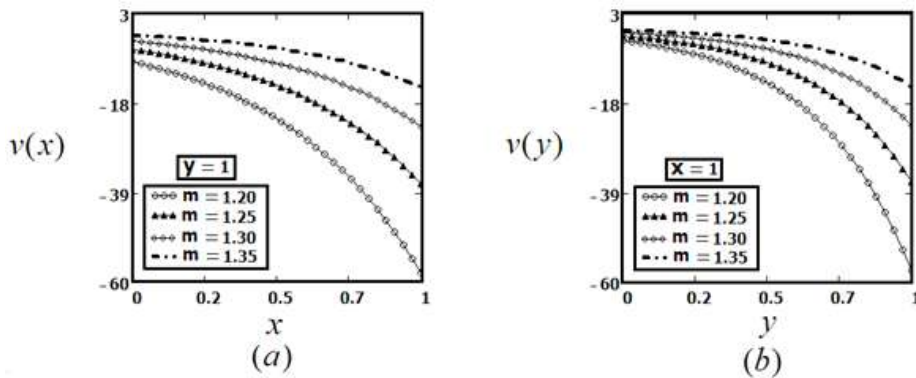


Figure 16. Velocity Profiles $v(x, y)$ for viscous fluid given by equations (52), for $B_1 = 0.002, B_2 = 0.003, a = 2, b = 3, U_1=0.1, U_2 = 0.1$ and differing values of m .

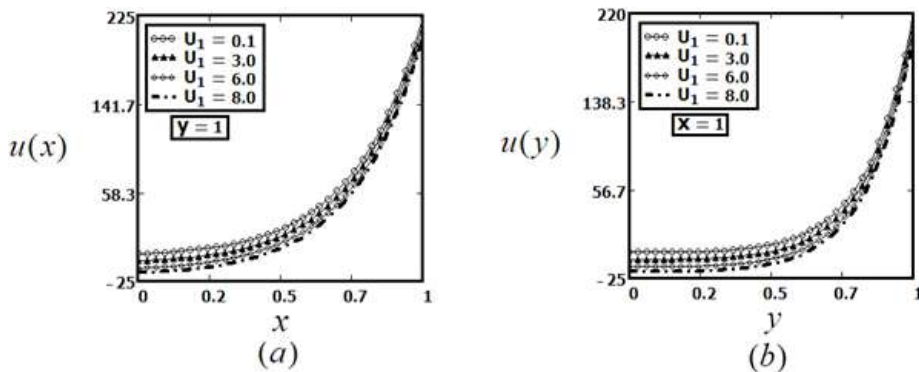


Figure 17. Velocity Profiles $u(x,y)$ for viscous fluid given by equations (52), for $B_1 = 0.002, B_2 = 0.003, a = 2, b = 3, m=2, U_2 = 0.1$ and differing values of U_1 .

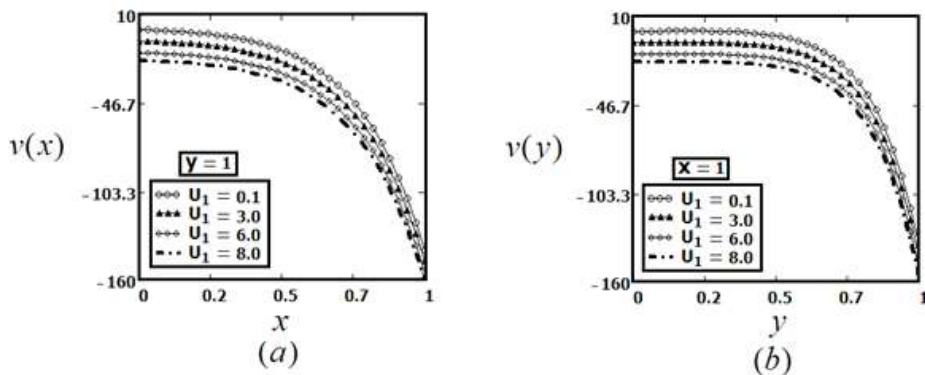


Figure 18. Velocity Profiles $v(x,y)$ for viscous fluid given by equations (53), for $B_1 = 0.002, B_2 = 0.003, a = 2, b = 3, m=2, U_2 = 0.1$ and differing values of U_1 .

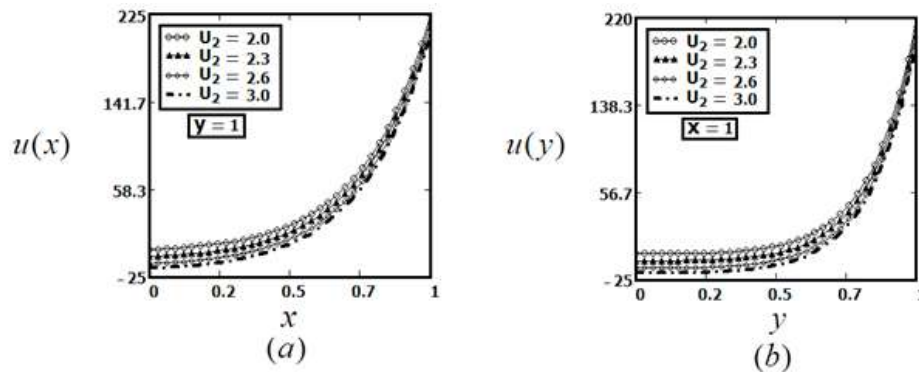


Figure 19. Velocity Profiles $u(x,y)$ for viscous fluid given by equations (52), for $B_1 = 0.002, B_2 = 0.003, a = 2, b = 3, m=2, U_1 = 0.1$ and differing values of U_2 .

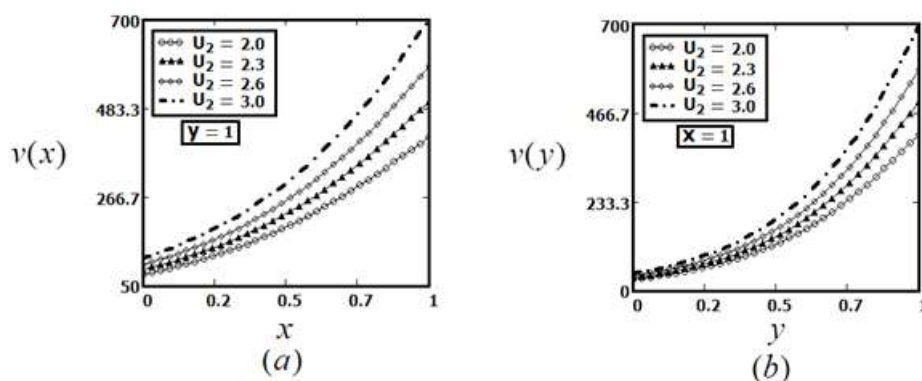


Figure 20. Velocity Profiles $v(x,y)$ for viscous fluid given by equations (53), for $B_1 = 0.002$, $B_2 = 0.003$, $a = 2$, $b = 3$, $m=2$, $U_1 = 0.1$ and differing values of U_2 .

5 Conclusion

This study focuses on analyzing the motion of an incompressible fluid with variable viscosity, which is conducted to a finite extent. The analysis takes into account the presence of an attractive field and heat transfer over an infinite plane. The purpose of this study is to determine the general solutions for flows where the vorticity distribution is directly proportional to the stream function, which is perturbed by uniform and exponential streams. The mathematical equation used to represent this flow is $\nabla^2 \psi = K [\psi - U_1(ax + by) - U_2 e^{(ax+by)}]$. In order to find the exact solutions, the flow equations are first written in terms of the stream function ψ , the vorticity function ω , and the generalized energy function J . Employing the integrability condition on the generalized energy function J , an equation is determined that must be satisfied by the perturbed stream function Ψ and the viscosity μ for the flows under consideration. The solutions are obtained with the help of transformation variable ξ that converted all governing equations in terms of simple linear ordinary differential equations. The exact solutions are then obtained for u, v, ψ, μ, H, T and p , written in terms of elementary functions, satisfy all governing equations and the compatibility condition $J_{xy} = J_{yx}$ ($\rho_{xy} = \rho_{yx}$). The solutions in case-I represent a sine stream plus a perturbation that is directly proportional to uniform and exponential functions in x and y . The solution in case-II, in general, represents a hyperbolic (sine or cosine) stream plus a perturbation that is again directly proportional to uniform and exponential function in x and y . The solution in case-II for $B_{11} = 0$ represents an exponential flow in the region $x > 0$ $y > 0$ perturbed by apart which is proportional to uniform and exponential stream in x and y . Likewise, we can provide an explanation for flow in further regions. It is also remarked that the transverse magnetic field H holds the same behavior in all cases. Finally the effects of various parameters of concentration on the velocity components $u(x,y)$ and $v(x,y)$ are represented and discussed.

6 Credit Author Statement

Iftikhar Ahmed Bhutto: Conceptualization, Methodology, **Afaq Ahmed Bhutto:** Writing- Original draft preparation **Rahim Bux Khokhar:** Writing -Reviewing,, **Muhammad Aslam Soomro:** Suggestions and ideas, **Fozia Shaikh** Software, Investigation, Writing -Reviewing.

7 Compliance with Ethical Standards:

It is said that there are no conflicts of interest among any of the authors. Additionally, it is stated that none of the authors of this paper conducted any experiments using human or animal subjects. Furthermore, each participant who took part in the study gave their free, informed consent.

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9 Author Information

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