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Development of an Explicit Iterative Numerical Scheme Over the Modified Euler's Method

Tuljaram Meghwar1[1], **Rahim Bux Khokhar**[1], **Asif Ali Shaikh**[1,2], **Evren Hincal**[2], **Muhammad Anwar Solangi**[1], **Prem Kumar**[1,3*]

[1]Department of Basic Sciences & Related Studies, Mehran Univeristy of Engineering and technology, Jamshoro, Sindh, Pakistan; [2]Department of Mathematics, Near East University, 99138 Mersin, Turkey; [3]Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Canada

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Abstract Major goal of this study, to propose a new explicit iterative numerical scheme that can substitute the modified Euler's method (MEM) for initial value issues in ordinary differential equations. This iterative strategy has been suggested by substituting the main slope of the MEM and the slope of the explicit forward Euler's method for the inner slope $h(\mu_i, v_i)$ of the MEM. Accuracy and stability are used to determine uniqueness in work. There are a few low accuracy and stability modified Euler technique variants that may be found in the literature. This study presents an innovative approach with the goal of obtaining greater accuracy and stability. The suggested iterative scheme's performance was evaluated by examining various numerical factors connected to the numerical method, and it was discovered that it is more accurate and consistent with the numerical order of convergence two. The proposed scheme's derived stability region and interval of stability region are greater than those technique which were present in literature previously, such as the Euler's method (EM), improved Euler's method (IEM), modified Euler's method (MEM), modified improved modified Euler's method (MIME), and improved modified Euler's method (IMEM). The proposed scheme generally showed greater accuracy in each initial value problem (IVPs) after being evaluated with several open literature initial value problems using various step sizes.

***Correspondence Author Email Address:**
prem.kumar@faculty.muet.edu.pk

1 Introduction

Initial value problems (IVPs) in ordinary differential equations (ODEs) of the type

$$v' = h(\mu, v); v(\mu_0) = v_0, \quad (1)$$

$v, h \in \mathbb{R}^n$ and $\mu \in [a, b]$ where $a, b \in \mathbb{R}$; are very important in the field of science and engineering for solving real-world problems [5, 12, 13]. The real world problems not only appear in the form of initial value problems but, sometimes, they appear in boundary value problems (BVPs). The initial value problems are those problems in which all conditions or restrictions are defined at a single independent variable whereas the boundary value problems are those problems in which conditions are defined at two or more than two independent variables involved. Due to their broad applications, many researchers and scientists have generated number of numerical methods with gradual improvement for solving different application problems appearing in the field of fluid dynamics, population growth, nuclear decay, temperature problems, electrical circuits' problems, chemical reaction equations and the motion of projectile, rocket, satellite etc. [4, 9, 10]. There are various reasons for developing new numerical methods. When the existing methods show weak performance in terms of accuracy, consistency, order of convergence, stability regions and required more work out for evaluation of slope per iteration step and computational time. Sometimes, numerical methods also used when the analytically methods fail to give the exact answers or solutions of given problems. In such cases, we use some approximations or numerical methods. A single numerical method is not enough for solving all kind of problems. Every numerical method is used for particular types of problems in generally. That is why; new techniques and algorithms have been made by the different researchers and scholars for the time. From the study of different areas; it is observed that the obtained initial value problems or boundary value problems appear in different forms such as autonomous, non-autonomous, Cauchy initial value problems, singular problems, fuzzy problems, delay problems and so on [15, 23]. Different numerical methods and algorithms have been developed on each type of above initial and boundary value problems on the basis of basic knowledge of Euler's method, modified Euler's method and Taylor series. Similarly, using ground knowledge of Euler's and Modified Euler's methods [11], a new numerical scheme has been introduced in this researcher paper which is more applicable as compared to similar type's methods. In order to achieve improved accuracy and stability, this study proposes an initial strategy. It was found that the recommended iterative scheme is more accurate and compatible with the numerical order of convergence two after evaluating the performance of different numerical factors related to the numerical approach. The stability region and interval of stability region of the proposed scheme are larger than those of earlier techniques, such as the Euler's method (EM), improved Euler's method (IEM), modified Euler's method (MEM) [17, 25], modified improved modified Euler's method (MIME), and improved modified Euler's method, that were published in the literature (IMEM). After being assessed with multiple open literature first value problems using different step sizes, the suggested technique typically demonstrated improved accuracy in each initial value problem (IVP).

2 The Proposed Scheme

Many researchers have brought a lot of modification in the Euler's method. Some of those are used in this research methodology. The most simplest form of the Euler's method (EM) is presented by the Leonhard Euler's in 1768 [1, 6, 8]. That is:

$$v_{i+1} = v_i + \beta h\left(\mu_i, v_i\right), \quad (2)$$

where $i = 1, 2, 3, \ldots$ and β is a step size. Another, simplest method is an implicit Backward Euler's method which is given by (3).

$$v_{i+1} = v_i + \beta h\big(\mu_{i+1}, v_{i+1}\big). \tag{3}$$

It can be written as

$$v_{i+1} = v_i + \beta h\Big(\mu_i + \beta, v_i + \beta h\big(\mu_i, v_i\big)\Big). \tag{4}$$

Other modifications of Euler's method are given by the following equations

$$v_{i+1} = v_i + \beta h\big(\mu_i + \frac{\beta}{2}, v_i + \frac{\beta}{2} h(\mu_i, v_i)\big). \tag{5}$$

This is known as modified Euler's method (MEM) [5], and this can be written as

$$v_{i+1} = v_i + \beta \rho, \tag{6}$$

where

$$\rho = h\Big(\mu_i + \frac{\beta}{2}, v_i + \frac{\beta}{2} h\big(\mu_i, v_i\big)\Big). \tag{7}$$

Equation (8) is an improved Euler's method (IEM) which is generated by taking the mean of the forward Euler's method and an implicit backward Euler's method. That is

$$v_{i+1} = v_i + \frac{\beta}{2}\Big(h\big(\mu_i, v_i\big) + h\big(\mu_i + \beta, v_i + \beta h(\mu_i, v_i)\big)\Big). \tag{8}$$

Using more slopes at per integration step, other modifications has been developed such as modified improved modified Euler's method (MIME) which is given by equation (9)

$$v_{i+1} = v_i + \beta h\Big(\mu_i + \frac{\beta}{2}, v_i + \frac{\beta}{2} h\big(\mu_i + \frac{\beta}{2}, v_i + \frac{\beta}{2} h(\mu_i, v_i)\big)\Big), \tag{9}$$

and iterative technique proposed by [20] which is given by equation (10)

$$v_{i+1} = v_i + \beta h\Big(\mu_i + \frac{\beta}{2}, v_i + \frac{\beta}{4} h\big(\mu_i, v_i\big) + h\big(\mu_i + \beta, v_i + \beta h(\mu_i, v_i)\big)\Big). \tag{10}$$

Similarly, for developing new proposed scheme, use the mean of $h(\mu_i, v_i)$ and ρ given by (7) at the place of inner slope $h(\mu_i, v_i)$ of (5).
Now, the proposed scheme will be in the form

$$v_{i+1} = v_i + \beta h\Big(\mu_i + \frac{\beta}{2}, v_i + \frac{\beta}{4} h\big(\mu_i, v_i\big) + h\big(\mu_i + \frac{\beta}{2}, v_i + \frac{\beta}{2} h(\mu_i, v_i)\big)\Big), \tag{11}$$

(11) is known as an explicit iterative numerical scheme over the modified Euler's method, which is new proposed scheme.

2.1 Stability Analysis

Stability of any numerical method can be analyzed by using following linear model [7, 24]

$$v' = \lambda v; \lambda \in C. \tag{12}$$

Using (12) on the proposed scheme we have

$$v_{i+1} = v_i + \beta\lambda\Big(v_i + \frac{\beta}{4}\big(\lambda v_i + \lambda(v_i + \frac{\beta}{2}\lambda v_i)\big)\Big),$$

$$v_{i+1} = v_i + \beta\lambda v_i + \frac{\beta^2\lambda^2 v_i}{2} + \frac{\beta^3\lambda^3 v_i}{8},$$

$$v_{i+1} = v_i + \beta\lambda v_i + \frac{\beta^2\lambda^2 v_i}{2} + \frac{\beta^3\lambda^3 v_i}{8},$$

let $z = \lambda\beta$,

$$v_{i+1} = \left(1 + z + \frac{z^2}{2} + \frac{z^3}{8}\right)v_i,$$

$$\psi(z) = \left(1 + z + \frac{z^2}{2} + \frac{z^3}{8}\right)v_i. \tag{13}$$

This is a stability polynomial function of developed an explicit numerical scheme. Geometry of stability polynomial function of proposed scheme is compared with the geometry of other methods such as Euler's method (EM), Improved Euler's method (IEM), Modified Euler's method (MEM), MIME method and iterative technique proposed by [20].

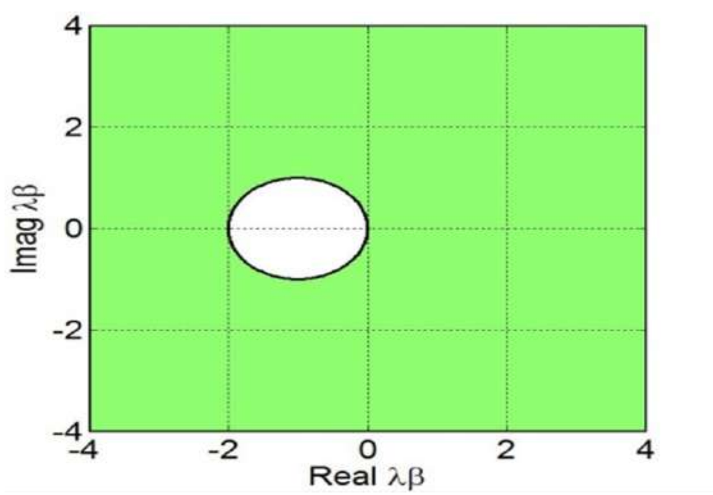

Figure 1. Stability Region of Forward EM.

Figure 1 shows the stability region of Euler's method (EM). In this figure, the white circular type region is the stable region of the Euler's method and shaded green region shows the unstable region. The numbers label on horizontal axis and vertical axis show the real values and imaginary value respectively of the complex plane.

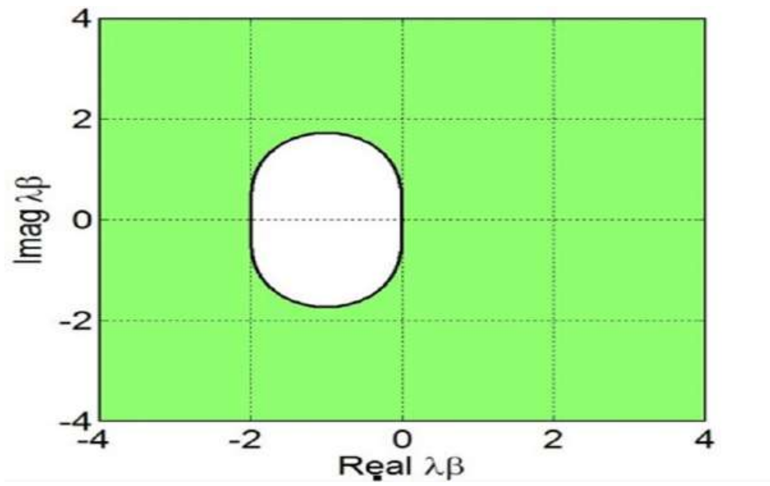

Figure 2. Stability Region of MEM & IEM.

Figure 2 shows the stability region drawn by using stability polynomial function in the complex plane of MEM and IEM. In this figure the white portion without color represent the stable region of MEM and IEM. Both methods have same stability region. The horizontal and vertical axes are labeled as real part and imaginary part of complex numbers respectively. These real and imaginary values labeled on x-axis and y-axis help in identifying the interval of stability region of concerned methods. The stability regions of MEM and IEM are greater than the stability regions of Euler's method (EM).

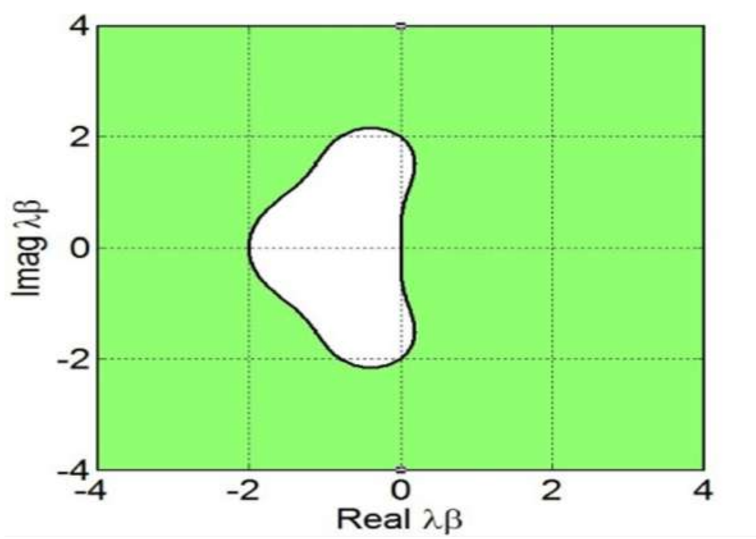

Figure 3. Stability Region of MIME Method and Method of [20].

Figure 3 show the stability regions of MIME method and an iterative method proposed by [20]. In this figure the white portion shows the stable region of both methods whereas remaining shaded region shows unstable region. The stability regions of both methods (MIME and iterative technique proposed by [20]. are same. These regions are greater than the stability regions of Euler's method and MEM. The horizontal and vertical axes show the real and imaginary values of complex plane. These axes help in identifying the interval of stability regions of methods.

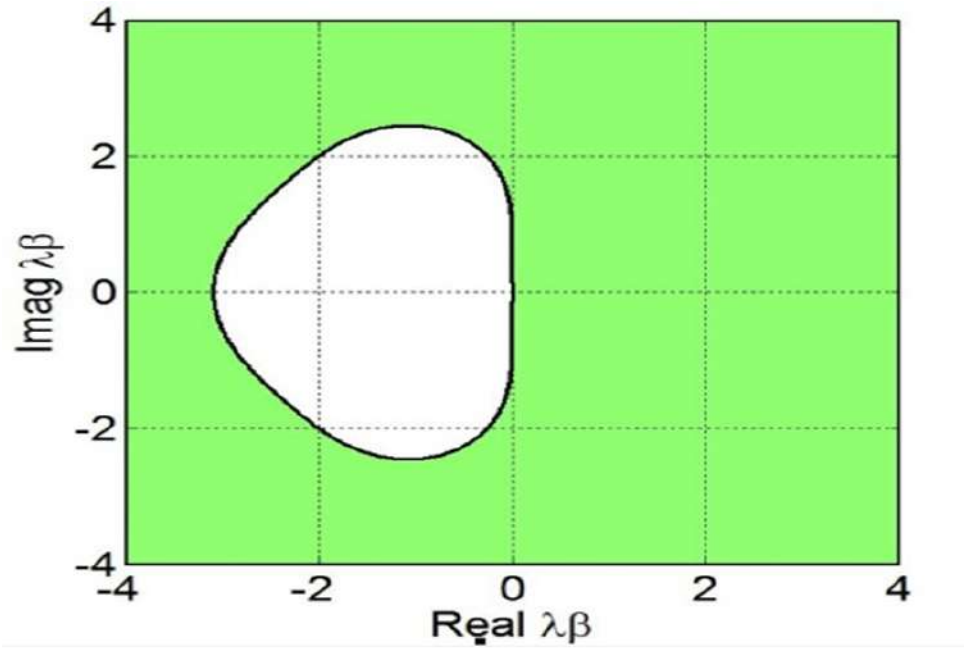

Figure 4. Stability Region of proposed Scheme.

Figure 4 shows the stability region of the proposed scheme. In this figure the white portion is stable region of the proposed method and remaining portion is unstable region. From figure 1, 2, 3 and figure 4. It observed that the stable region of proposed scheme shown by white portion in figure 4 is greater than the stable regions of other methods shown in figure 1, 2, and figure 3 by white portions. The horizontal axis and vertical axis of Fig 4 shows the real and imaginary axes of the complex plane. Using these axes; one can identify the interval of stability region. From figure 4, it is observed that the interval of stability region of proposed scheme lie in (-3, 0). From the above analysis of stability regions, it is clear that, proposed scheme is more stable than all others schemes which are discussed here in this paper.

Table 1. Stability Polynomials & Intervals of Absolute Stability

Algorithm	Stability polynomial	Absolute stability Interval
EM	$1+z$	(-2, 0)
MEM	$1+z+\frac{z^2}{2}$	(-2, 0)
IEM	$1+z+\frac{z^2}{2}$	(-2, 0)
MIME	$1+z+\frac{z^2}{2}+\frac{z^3}{4}$	(-2, 0)
[20]	$1+z+\frac{z^2}{2}+\frac{z^3}{4}$	(-2, 0)
Proposed Scheme	$1+z+\frac{z^2}{2}+\frac{z^3}{8}$	(-3, 0)

The above table 1 shows the comparison of the stability zone that has been studied in earlier literature [2] with the present study's proposed technique and it shows the grater stability.

2.2 Consistency

Definition 1. *If the incremental function of proposed scheme with the step size approaching zero, agree with IVPs (1) such that*

$$\lim_{\beta\to 0}\phi(\mu_i, v_i, \beta) = h(\mu_i, v_i). \tag{14}$$

Then the proposed scheme is said to be consistent [16]. Now, the incremental function of proposed scheme is

$$\phi(\mu_i, v_i, \beta) = h\Big(\mu_i + \frac{\beta}{2}, v_i + \frac{\beta}{4}(h(\mu_i, v_i) + h(\mu_i + \frac{\beta}{2}, v_i + \frac{\beta}{2}h(\mu_i, v_i)))\Big).$$

Using 14 to proposed scheme, we have From the above analysis of stability regions; it is clear that, developed scheme is more stable than all others schemes w hich are discussed here in this paper.

$$\lim_{\beta\to 0}\phi(\mu_i, v_i, \beta) = \lim_{\beta\to 0} h\Big(\mu_i + \frac{\beta}{2}, v_i + \frac{\beta}{4}(h(\mu_i, v_i) + h(\mu_i + \frac{\beta}{2}, v_i + \frac{\beta}{2}h(\mu_i, v_i)))\Big),$$

$$\lim_{\beta\to 0}\phi(\mu_i, v_i, \beta) = h(\mu_i, v_i).$$

This proves that proposed scheme is consistent

2.3 Analysis of Local Truncation Error

The local truncation error is defined as

$$L.T.E = C\beta^{p+1}v^{p+1}(\mu) + O(\beta^{p+1}), \tag{15}$$

Where the coefficient 'C' is a constant value which is known as an error constant and 'p' is known as an order of accuracy [3, 21]. Local truncation error can be found by taking the difference of exact Taylor series for $v(\mu+\beta)$ and Taylor series generated from composed scheme v_{i+1}. Taylor series for $v(\mu+\beta)$ is expanding as follow [12, 14, 19].

$$\begin{aligned} v(\mu+\beta) = v(\mu) + \beta h + \frac{\beta^2}{2!}(h_\mu + hh_v) + \frac{\beta^3}{3!}(h_{\mu\mu} + 2hh_{\mu v} + h^2 h_{vv} + hh_v{}^2 + h_\mu h_v) \\ + \frac{\beta^4}{4!}(h_{\mu\mu\mu} + 3hh_{\mu\mu v} + 3h^2 h_{\mu vv} + 5hh_v h_{\mu v} + 3h_\mu h_{\mu v} + h^3 h_{vvv} + \\ 4h^2 h_v h_{vv} + 3hh_\mu h_{vv} + hh_v{}^3 + h_\mu h_v{}^2 + h_{\mu\mu} h_v) + O(\beta^2). \end{aligned} \tag{16}$$

Now expanding the slopes of 11 by 16, we have

$$K_1 = h(\mu_i, v_i) = h, K_2 = h(\mu_i + \frac{\beta}{2}, v_i + \frac{\beta}{2})$$
$$K_2 = h + \frac{\beta}{2}\{h_\mu + hh_v\} + \frac{\beta^2}{8}\{h_{vv}h^2 + 2hh_{\mu v} + h_{\mu\mu}\}$$
$$+\frac{\beta^3}{48}\{h_{vvv}h^3 + 3h_{\mu vv}h^2 + 3h_{\mu\mu v}h + h_{\mu\mu\mu}\} + O(\beta^4), \tag{17}$$

$$K_3 = h(\mu_i + \frac{\beta}{2}, v_i + \frac{\beta}{4}(K_1 + K_2)),\; K_3 = h + \frac{\beta}{2}\{h_\mu + hh_v\} + \frac{\beta^2}{8}\{h_{\mu\mu} + 2hh_v + h^2h_{vv} + hh_v{}^2 + h_\mu h_v\}$$
$$+\frac{\beta^3}{48}\{h_{\mu\mu\mu} + 3hh_{\mu\mu v} + h^2h_{\mu vv} + 6h_vh_{\mu v} + h^3h_{\mu vv} + \frac{9}{2}h_vh^2h_{vv} + 3h_\mu hh_{vv} + \frac{3}{2}h_vh_{\mu\mu}\} + O(\beta^4). \tag{18}$$

Substitute K_3 in composed scheme at the place of incremental function we have

$$v_{i+1} = v_i + \beta h + \frac{\beta^2}{2}\{h_\mu + hh_v\} + \frac{\beta^3}{8}\{h_{\mu\mu} + 2hh_v + h^2h_{vv} + h_v{}^2 + h_\mu h_v\} + \frac{\beta^4}{48}\{h_{\mu\mu\mu}+$$
$$3hh_{\mu\mu v} + 3h^2h_{\mu vv} + 6hh_vh_{\mu v} + h^3h_{vvv} + \frac{9}{2}h_vh^2h_{vv} + 3h_\mu hh_{vv} + \frac{3}{2}hh_vh_{\mu\mu}\} + O(\beta^5). \tag{19}$$

Now by comparing (16) and (19), we found that both equations agree up to β^2. Hence the order of convergence of proposed scheme is 2 and local truncation error of proposed method is given by the following equation.

$$LTE = \beta^3\{\frac{1}{8}h_{\mu\mu} + \frac{1}{4}hh_v + \frac{1}{8}h^2h_{vv} + \frac{1}{8}hh_v{}^2 + \frac{1}{8}h_\mu h_v\} + O(\beta^4) \tag{20}$$

3 Results and discussion

In this section, the performance of proposed scheme has been measured on the basis of relative percentage error with the IEM, MIME and the method constructed by Qureshi et al (2013). The graphs of relative percentage error have been obtained by using MATLAB version 7.5.0 (R2007b). All problems have been taken from the open literature for checking the validity of results obtained from the proposed scheme.

3.1 Example 1

solve $\frac{dv}{d\mu} = cos2\mu + sin3\mu$, $v(0) = 1$; $\mu \in [0, 10]$ and $\beta = 0.1$. The theoretical solution is $v(\mu) = \frac{1}{2}sin2\mu - \frac{1}{3}cos3\mu + \frac{4}{3}$.
solution:

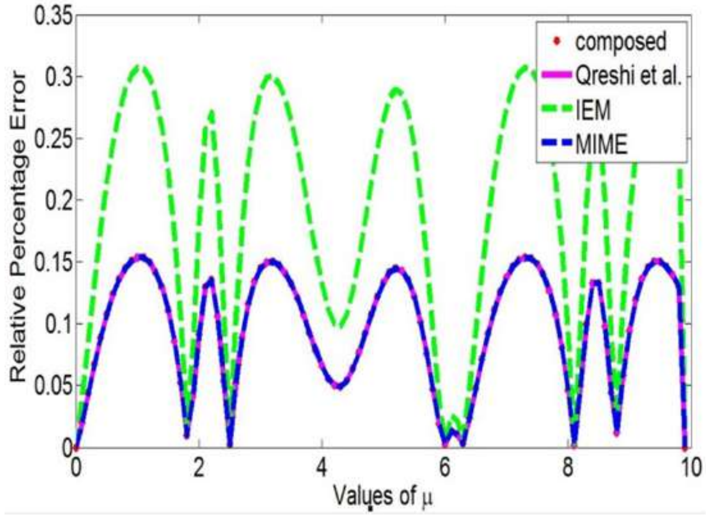

Figure 5. Relative percentage Errors of Example 1.

Figure 5 shows the graph of relative percentage errors obtained from different numerical methods. The label of horizontal axis shows the value of an independent variable μ and vertical axis represents the relative percentage error obtained from numerical methods at different value of μ. The graph of this problem shows that MIME, an iterative technique proposed by [20] and proposed scheme produce almost same error at each value of μ but, the IEM performance larger error than others at each value of μ.

3.2 Example 2

solve $v' = cos(\mu)$, $v(0) = 1$; $\mu \in [0, 1]$ and $\beta = 0.1$. The theoretical solution is $v(\mu) = \exp^{s}in(\mu)$ [22] solution:

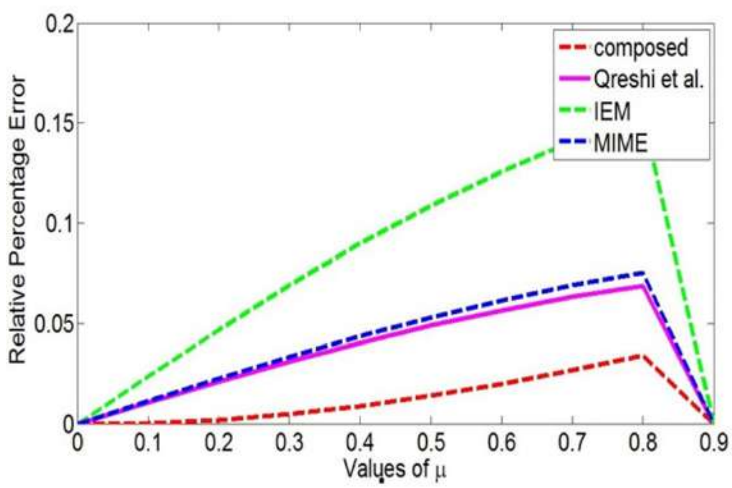

Figure 6. Relative percentage Errors of Example 2.

Figure 6 shows the graph of relative percentage errors of different methods which are written in legend at right top corner of the graph. From the curves of relative percentage errors; it is observed that the proposed scheme shows less error as compared to other method (IEM, MIME and iterative technique by [20]. Red dotted curve shows the relative percentage error of proposed scheme whereas light green, blue and purple colors show the relative percentage errors of IEM, MIME and iterative technique by [20]. In this figure the horizontal axis shows the values of μ and vertical axis shows the relative percentage errors.

3.3 Example 3

[18] solve $v' = 1 + 2v - v^2,\ v(0) = 0$. The theoretical solution is $v = 1 + \sqrt{2}tanh\{\sqrt{2}\mu + \frac{1}{2}ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\}$ [22]
solution:

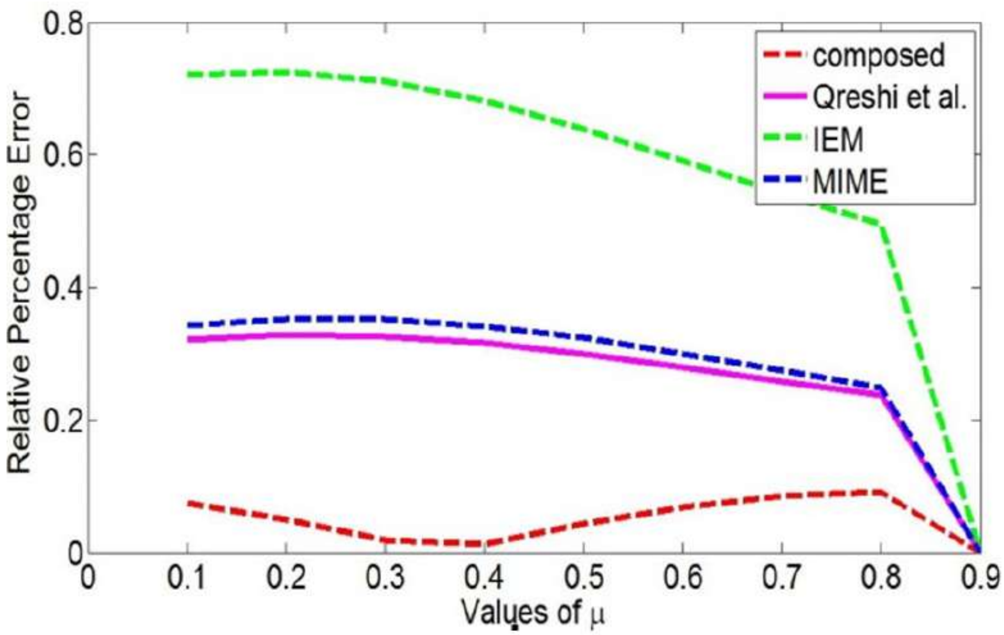

Figure 7. Relative percentage Errors of Example 3.

Figure 7 shows the graph of relative percentage errors of numerical methods written in legend at right top corner of the graph. The proposed scheme drawn by red dotted curve shows less error as compared to all others curves of different methods and an iterative technique by [20] drawn by the purple color shows less error as compared to other two methods (IEM and MIME). The light green curve of IEM shows larger error as compared to all others. The horizontal and vertical axes of figure 7 show the value of an independent variable and relative percentage error respectively.

3.4 Example 4

solve $v'(\mu) = 1 - v^2(\mu),\ v(0) = 0.$ in the interval $0 \leq \mu \leq 10$.The theoretical solution is $v(\mu) = tanh(\mu)$ and $\beta = 0.1$ [21]
solution:

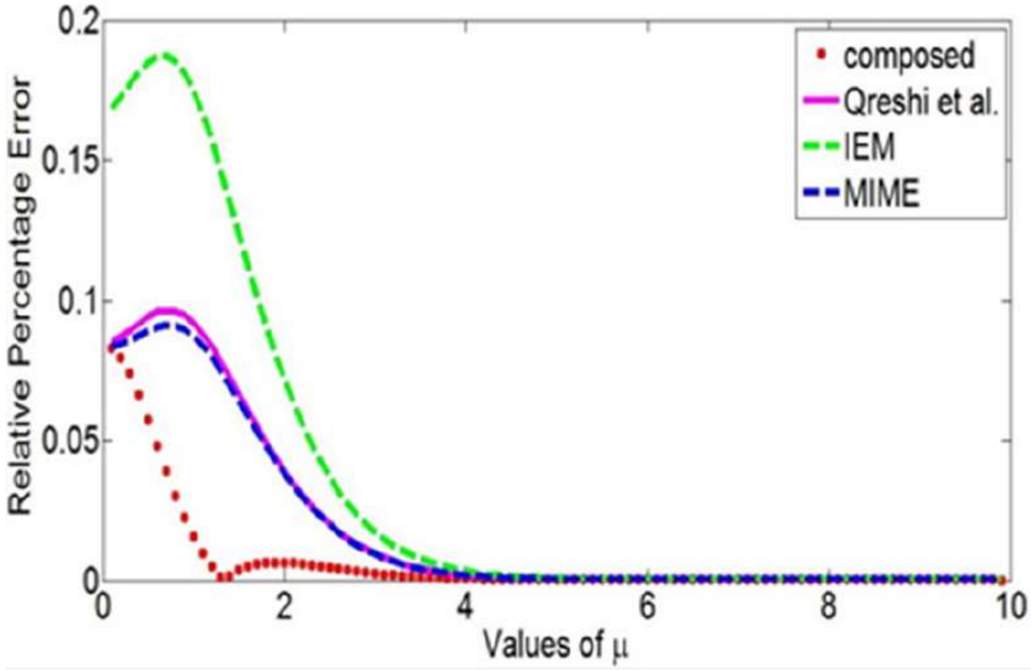

Figure 8. Relative percentage Errors of Example 4.

Figure 8 shows the graph of relative percentage errors of different numerical methods written in legend at right top of corner of the graph. The proposed scheme's relative percentage error drawn by the red dotted curve shows the less error as compared to all other mentioned methods in the defined interval. The relative percentage error in MIME method is less than the relative percentage error in an iterative technique by [20]. The IEM shows greater relative percentage error in entire interval.

In figure 5, the relative percentage error of the method constructed by [20] and MIME method almost show the same error as in proposed scheme. But, if we will zoom in the graphs, it will be clear that the proposed scheme converges faster as compared to others and modified Euler's method shows poor performance. The figure 6, 7 and figure 8 clearly present that the proposed scheme is showing less relative percentage error as compared to other methods which are shown in graphs. All these examples are solved at 0.1 step size but these examples are also performing more accurate result at small step size. The proposed scheme is also verified at different step sizes and different ranges of independent variable involved in the examples. These graphs show the validation of results of proposed scheme.

4 Conclusion

In this study, a new modification of a MEM has been introduced which is more efficient for solving IVPs in ODEs. This modified scheme has been proposed by replacing the inner slope $h(\mu_i, v_i)$ of the MEM by the mean of main slope of MEM and slope of explicit forward Euler's method. The proposed scheme has also been analyzed for the different numerical factors such as stability, consistency, order of accuracy and local truncation error. During the analysis, it is found that the proposed scheme shows large absolutely stability interval and more accuracy with same order of convergence as compared to others. The absolutely stability region of proposed scheme is (-3,0) whereas other methods have (-2, 0). The number of

IVPs in ODEs has been tested on the proposed scheme and on some existing methods (i.e. IEM, MIME and the method constructed by [20]) using MATLAB software and noted that the proposed method was showing better accuracy as compared to others. The solution results and relative percentage errors of proposed scheme and other methods has also been tested by taking different step size and different interval length on each problem, and found that the proposed scheme again showing more accurate results and less relative percentage error in over all the problems. Especially, the proposed scheme is best tool for solving autonomous Bernoulli's differential equations, logistic models, and linear differential equations. Overall, the novelty of this work is defined on the basis of accuracy and more stability; which is proved and analyzed through different derivations and numerical problems.

Author Contributions

Tuljaram Meghwar: Conceptualization, Methodology, Writing- Original draft preparation. **Prem Kumar**: Writing -Reviewing , **Rahim Bux Khokhar and Asif Ali Shaikh** : Software, Investigation, Writing -Reviewing **Muhammad Anwar Solangi and Evren Hincal**: Suggestions and ideas

Compliance with Ethical Standards

It is said that there are no conflicts of interest among any of the authors. Additionally, it is stated that none of the authors of this paper conducted any experiments using human or animal subjects. Furthermore, each participant who took part in the study gave their free, informed consent.

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Author Information

ORCID:
Asif Ali Shaikh: 0000-0002-3084-922X
Evren Hincal: 0000-0001-6175-1455
Prem Kumar: 0000-0002-7367-4431

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