VFAST Transactions on Mathematics
http://vfast.org/journals/index.php/VTM@ 2023, ISSN(e): 2309-0022, ISSN(p): 2411-6343
Volume 11, Number 1, January-June, 20223 pp: 84-95

Simulation of 13 points Laplacian operator in cylindrical mesh system by using explicit finite difference technique

Rabnawaz Mallah[1], Inayatullah Soomro[1], Altaf Ahmed[1], Dost Muhammad[1], Sarang Latif[1], Munwar Jameel[1], Irshad Ali[1], Shakeel Ahmed[1]

[1]Shah Abdul Latif University, Khairpur, Pakistan

Keywords: *Finite Difference Method, Discretization, Laplacian operator, 13 points stencils, Cell dynamic simulation, Explicit functions, cylindrical mesh system.*
Subject Classification: *35J05, 35J10, 35K05, 35L05*

Journal Info:
Submitted:
January 15, 2023
Accepted:
March 20, 2023
Published:
March 31, 2023

Abstract In the recent period of time a considerable and remarkable work have been done on the graphic design, artificial intelligence, optimization theory, solving numerical approximation of ordinary as well as Partial differential equations by some numerical recipes. Discrete Laplacian operator is the one of the leading tool being involved and is used in various mathematical models and theories. It is the most demanding mathematical operator in Cell Dynamic simulation (CDS), Self-Consistent Field Theory (SCFT) to study advanced materials, and image processing. Researchers are working for novel numerical scheme to acquire in addition to accuracy and stability, isotropic numerical results. Researchers are facing challenging tasks in physically motivated discretization on curved surfaces/boundaries. In this work a numerical study is carried out to discretize Laplacian operator on 13-point stencil in cylindrical mesh system. The discretization process on Laplacian operator is carried out using finite difference method which allows fast computation competencies in the simulation processes. This numerical study explores new formulations for the discretization of partial differential equations which are verified against analytic solutions and give consistent, convergent, accurate and stable numerical results for 1 st order, 2 nd order partial derivatives and Laplacian operator.

***Correspondence Author Email Address:**
Rabnawaz.Mallah@iba-suk.edu.pk

1 Introduction

In mathematical and computational modeling, recreation of models, artificial intelligence and in so many other computer based applications, the discrete Laplacian operator plays an essential role specially he represents the surfaces by applying mesh or gird system. These gird or mesh system have been made by Cartesian system and it can be operated, replicated and analyze. Recently, many novel methods have been developed for the solution of differential Geometry of the surfaces of the discrete Laplacian operator by researchers. These novel methods have been applied in several areas such as image processing, artificial intelligence, graphic designing, two and three dimension animation, in robotics and also in shape representations. In the pore geometry and the solution of surfaces, heat and diffusion equation discrete Laplacian operator studies provide novel accuracy and stability in results. So many schemes have been made on different coordinate system especially on Cartesian, spherical, polar as well as cylindrical coordinate system. The usages of finite difference scheme especially based on the co-tangent scheme, which keeps high efficiency in his novel outcomes. In this scheme high accuracy and stability of the scheme has been discussed with improvements.

In this research paper, finite difference scheme has been used for discretiztion of discrete Laplacian operator into cylindrical gird system by applying 13-points stencils. This novel technique is extensively applied in the numerical approximation of physical, computational, chemical models which may belong to several braches of the physics and science. The stability , efficiency and consistency of the developed 13-points schemes have been discussed in this research paper.

1.1 Aims and Objectives of the cylindrical thirteen points stencil

- To calculate five (5-point) approximations of the partial derivatives of the first and second orders in the cylindrical coordinate system.
- To use a finite difference approach to develop the Laplacian operator in a cylinder-mesh structure containing Thirteen-points stencil. It will confirm the outcomes system's stability about developed scheme.

1.2 Survey of the Finite difference scheme including the partial difference scheme.

There are many facets to the study of the Partial Differential equations;. The traditional approaches to solving PDEs in the 19th century focused on providing strategies and abilities to identify concrete solutions. Each mathematical development that enabled a solution to the most recent kind of the Partial Differential equations brought about effective growth in Physics because PDEs have enormous usefulness in many different branches of physics. In this approach, Hamilton's characteristics technique aided in important advances in analytical mechanics and optics, while Fourier's technique helped to explain heat transfer and wave propagation. The concept of electromagnetic advanced greatly as a result of Green's strategy. Since the previous sixty or seventy (60 to 70) years, the concept of numerical approaches have caused a tremendous progress in Partial differential equation.

In general setups and with additional subjective circumstances, these numerical techniques assisted by computer use have been successful in solving PDEs of practically all sorts; nonetheless, there are still many implementation challenges that need to be overcome or at least mitigated [1],[2]. Although the method of splitting the entire domain to work out is fairly effective, it is not possible to solve the entire model domain at once. A key technique that bridges the discrete and continuous worlds is called discretization [3]. For a mathematical model to produce results that are suitable, discretization is essential.

Discretization has a lot of applications for continuous problems. These methods that are typically used for discretization are: the Spectral Method for approximating the solution, the (Finite Elements Method) used for approximating the clarification, the (Finite volume Method) intended for approximating the maintenance principle the (Finite Difference Method) applied for approximating the differential operator. The use of Taylor series expansion and polynomial interpolation is extensively utilized in developed discretization scheme which is used for the Whole differential in the (FDM) for creating high-order differences [4]. The development of differential operators must be done individually in order to solve the Partial Differential Equations linked to continuous theory or to derive quantitative approximations related to events of classical or quantum science [5]. Partial differential equation (PDES) frequently utilize differential operators including gradient, divergence, curl, and Laplacian[6]. Among them, the Laplacian operator plays a crucial role in many multidimensional linear mathematical physics models, as it does when it describes many classical and quantum wave events, as well as when it appears in mathematical models of heat diffusion and viscosity effects [7].

Laplacian operators are used in many engineering disciplines, including electricity, fluid movement, and constant heat conduction [8]. Discrete Laplace operators have been frequently used, especially in a variety of geometric processing assignments. Grid editing, model-hape description, and shape interpolation can all benefit from grid processing methods based on the Laplacian [9].

The Laplacian by Robin boundary value problem, whose globular geometry may be found throughout the cosmos in both the greatest configurations and the tiniest particles, is one of the most important boundary value problems in many disciplines of science [10].For elliptic Partial differential equation and for development of the discrete Laplacian operator central difference technique of finite difference method is used. The first and second partial derivative of spatial domain at node can be shown by linear patterned of three functioned valued. Moreover, for the spatial partial derivative of higher order domain there is need of extra nodes. Higher order finite differences scheme contains consistency and accuracy better than in finite difference. There are several techniques used for the evaluation of the PDES and also used for the discretization of the ordinary as well as partial differential equations explicit finite difference scheme, implicit finite difference method and Crank Nicolson method are among them. These mentioned techniques contain very high accuracy, stability and consistency in the novel results of numerical approximation of the developed schemes [11].

In fair cases it so difficult and challenging to practices these techniques in Cartesian coordinate system. Furthermore in the cylindrical, spherical as well as in polar mesh system these techniques provides charming results in formation and simulation of mathematical modeling, computational and numerical approximation when choosing Cartesian mesh system. The starling a mathematician developed the Laplacian operator by finite difference technique in cylindrical mesh system [12].cylindrical mesh system is most adjustable system for the novel results of numerical approximation and also for various physical as well as chemical structures. Such as polymerizations, Di-block copolymer self-assembly, climatologically and geophysical, fluid flow (laminar as well as turbulent), and etc.

2 Methodology

So many mathematical physical, biological and chemical models can be simulated by partial differential equation. The study of partial differential equation has shown quite novelty in simulation results of models which may belong to economics, financial forecasting, image processing, artificial intelligence and so many biological aspects [13]. The most important point in partial differential equation is that it cannot be solved analytical. So due to the non-analytical results partial differential equation will consider under numerical approximation solution. In these solution most of cases, the numerical approximation results characterizes with the useful principles of the mesh points or gird points. For the solution, evaluation and simulation of numerical approximation so many numerical models, operator and numerical methods are presents in the literature such as finite difference method (FDM), Finite Volume method (FVM) and Finite Element Method (FEM).

2.1 FINITE DIFFERENCE METHODS (FDM)

It is numerical approximation scheme depend upon the replacing or exchanging the approximated numerical differential equation into finite difference scheme. The algebraic equation will be generated in finite difference method but in it provide the approximated results which will appear in form of mesh or gird system. Finite difference scheme depends upon major three important steps which are. 1) Forward Difference Method 2) Backward Difference Method 3) Central Difference Method

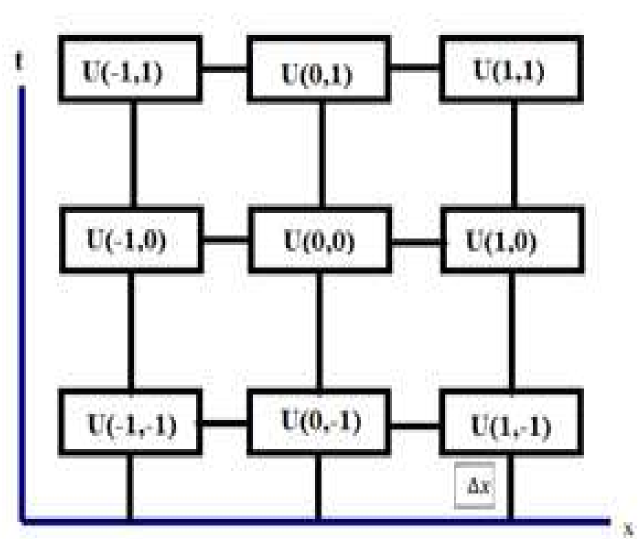

Figure 1. Proposed plane representing the points in x and t direction [8]

2.2 Forward Difference Method

It is Taylor series expansion from right to left of the origin which shows the common difference in r-axis. It can be seen in figure. Here r represents the change in points. Here partial derivative of u with respect to r-axis is the operator of the equation.

$$\frac{\partial u}{\partial r} = \frac{1}{(\Delta r)}\left[u_{(1,0,0)} - u_{(0,0,0)}\right] \tag{1}$$

2.3 Backward Difference Method (BDM)

It is Taylor series expansion from left to right of the origin which shows the common difference. It can be seen in figure. Here r represents the change in points.Here partial derivative of u with respect to r-axis is the operator of the equation.

$$\frac{\partial u}{\partial r} = \frac{1}{(\Delta r)}\left[u_{(0,0,0)} - u_{(1,0,0)}\right] \tag{2}$$

2.4 Central Difference Method (CDM)

It is Taylor series expansion from left to right as well as right to left of the origin which shows the common difference. It can be seen in figure. Here r represents the change in points.

$$\frac{\partial u}{\partial r} = \frac{1}{(2\Delta r)}\left[u_{(1,0,0)} - u_{(2,0,0)}\right] \tag{3}$$

Three-Dimensional Cartesian coordinate system, the discrete Laplacian operator can be written in standard form as in equation (4).Here partial derivative of u with respect to r-axis is the operator of the equation.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{4}$$

The standard formula of discrete Laplacian operator in cylindrical mesh structure is defined as in equation no (5). Here in equation (3), r is representing i-axis (x-axis in Cartesian coordinate system), is representing j-axis but in Cartesian coordinate system it is(y-axis), and last one is which is also k-axis, it can been be written as a z-axis in Cartesian Coordinate system. Furthermore, the and will be considered in radian angle and same type of Laplacian can also explained in polar coordinate system as [7],

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \tag{5}$$

Now first, second order partial derivative is required w.r.t to Using Tailor series, the first and second order partial derivative will develop. This developed equation may put into equation. Now obtained equation is called laplacian operator containing Thirteen-point stencils in cylindrical mesh system with weighing factor.

3 Procedure for developing Scheme.

Finite difference scheme plays an essential role in the mathematical, Computational methods and so many fluid dynamics models specially for it is used for solving the Laminar and turbulent flow models [14].

Due to his high accuracy in novel results many researcher have applied this technique in Crank Nicolson method, Navier Stoke method and Helmholtz equation [15].Furthermore, among highest application

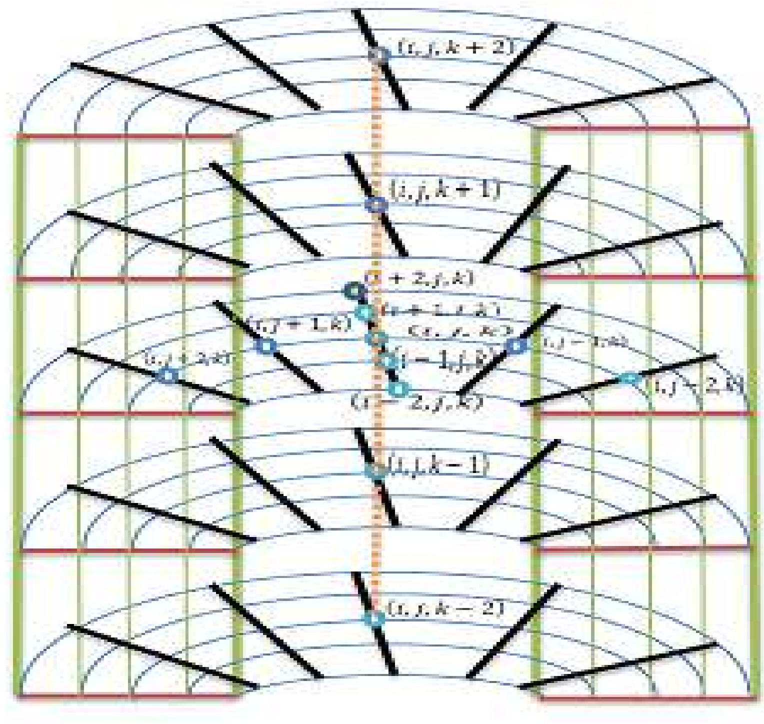

Figure 2. Proposed Molecule in 3- Dimension in cylindrical form [8]

of finite difference method to find the intensity of earth quick is the best and novel appication of finite difference method [16].

As the proposed molnamics and also in fluid dyecule in 3-dimensional so by taking help from Cartesian coordinate system, it can be drawn as figure(3).This mode has been adopted for discretization purpose here r-axis has been taken for second order partial derivative ,the developed second order partial derivative w.r.to r can be seen in equation no (6). In this equation (6) there is variation in r-axis, where j-axis and k-axis are zero. In equation no (7) variation in -axis has been obtained and due to the -axis, r-axis and z-axis remain static. This equation deal with the second order partial derivative w.r.to -axis containing five points stencil and equation (8) is being developed on second order partial derivative on z-axis here variation is in z-axis and r-axis and -axis will be in rest form and according the demand of discrete Laplacian operator first order partial derivative w.r.to r-axis is required. It has been investigated from the developed molecule which is containing thirteen points' stencils in cylindrical mesh system that there are five points are presents in r-axis so the first order partial derivative of the scheme can be notified by equation (9). These second order as well as first order partial derivative equation have been developed by using Taylor series expansion. All below developed equations are containing five points (5) stencil in cylindrical mesh system.

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{12(\Delta r)^2}\left[\, -u_{(-2,0,0)} + 16u_{(-1,0,0)} - 30u_{(0,0,0)} + 16u_{(1,0,0)} - 2u_{(2,0,0)} \,\right] \tag{6}$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{12(\Delta \theta)^2}\left[\, -u_{(0,-2,0)} + 16u_{(0,-1,0)} - 30u_{(0,0,0)} + 16u_{(0,1,0)} - 2u_{(0,-2,0)} \,\right] \tag{7}$$

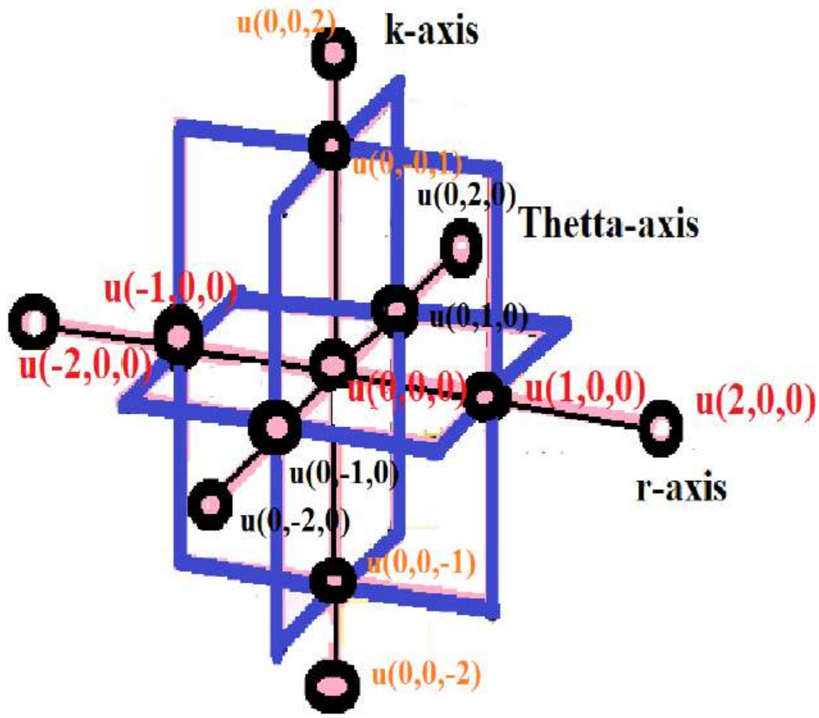

Figure 3. Three dimensional molecule contains 13 points stencils for r-axis. In this graph, three categories have been developed, a) the red points indicates the r-axis where -axis and z-axis will remain zero (r,0,0).b) The block points belongs to the -axis and r-axis and z-axis will be zero in this case (0, ,0). C) The range points are the z-axis points again where ra nd -axis becomes nothing (0.0.z) [8]

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{12(\Delta z)^2}\left[\; -u_{(,0,0)} + 16u_{(-1,0,0)} - 30u_{(0,0,0)} + 16u_{(1,0,0)} - 2u_{(2,0,0)} \;\right] \tag{8}$$

$$\frac{\partial u}{\partial r} = \frac{1}{12(\Delta r)}\left[u_{(-2,0,0)} - 8u_{(-1,0,0)} + 8u_{(1,0,0)} - u_{(2,0,0)}\right] \tag{9}$$

Putting above all the equation into equation no. 5.

$$\frac{\partial u}{\partial r} = \frac{1}{12(\Delta r)}\left[u_{(-2,0,0)} - 8u_{(-1,0,0)} + u_{(1,0,0)} - u_{(2,0,0)}\right] \tag{10}$$

$$\nabla^2 u_{(0,0,0)} = W\left[\begin{array}{l} \frac{1}{12(\Delta r)^2}\left[-u_{-2,0,0} + 16u_{-1,0,0} + 16u_{1,0,0} - u_{2,0,0}\right] + \\ \frac{1}{12r(\Delta r)}\left[u_{-2,0,0} - 8u_{-1,0,0} + 8u_{1,0,0} - u_{2,0,0}\right] + \\ \frac{1}{12(\Delta\theta)^2}\left[-u_{0,2,0} + 16u_{0,-1,0} + 16u_{0,1,0} - u_{0,2,0}\right] + \\ \frac{1}{12r^2(\Delta Z)^2}\left[-u_{0,0,-2} + 16u_{0,0,-1} + 16u_{0,0,1} - u_{0,0,1}\right] \end{array} \quad +\right]-$$

Here w is the weighting factor, equal to where

$$W = \left[\frac{3r^2.\Delta r^2.\Delta\theta^2}{5((\Delta z^2.\Delta\theta^2.r^2) + ((\Delta r^2.\Delta z^2)) + ((\Delta r^2.\Delta\theta^2.r^2))}\right]$$

The developed thirteen point's stencils equation is discrete Laplacian operator in cylindrical mesh system. Now for the numerical approximation of the developed scheme, examples is given below.

4 Example

Find the analytically and numerical approximation results of u(0,0,0) = r + z + cos by applying thirteen points stencils in cylindrical mesh system with error analysis and also discuss its stability and accuracy? For the discretization purpose there are some notation which will be used for calculating the analytical and approximated values. The notation are given below.

$$r_i = r_a = i(\Delta r) \ , \ \theta_j = j(\Delta\theta) \ \ and \ _k = k(\Delta\phi) \ \ so \ \ i = j = k = 0, 1, 2, 3 \ldots\ldots n, m$$

To execute the developed scheme made on thirteen point's stencils in cylindrical mesh system contains some values which will be taken as constantly in the developed scheme such as, the value of $r_a willbeoneunit, Thechangeinrorrwillbeconsideras 0.1 unitmeasurevalue. \Delta r = 0.1$ and $\Delta\theta = \frac{\pi}{180} and \Delta\phi = \frac{\pi}{180} radians$

5 Results and Discussions of developed scheme

For the verification of obtained discrete Laplacian by the use of (CFDS) central finite difference scheme containing Thirteen-Points stencil in cylindrical mesh system. For the development of mathematical and computational results, a FORTRAN code has been developed. The obtained results will show the accuracy and consistency in the results. Several fundamental numerals of have been involved under FORTRAN code developed on discretization scheme in cylindrical mesh system for obtaining the novel results. Applying the above table here are the radial, azimuthal (angle will represents in radian) as well as z-axis can be seen, these updated values will use in code for numerical approximation results.

r_i	angle	Radian	Analytical Value	Approximate value	Error of Scheme
1.1	1.7450E-02	0.1	2.199	2.199	-1.136E-05
1.1	1.745E-02	0.2	2.199	2.1995	-1.136E-05
1.1	3.4908E-02	0.1	2.199	2.199	-1.142E-05
1.1	3.490E-02	0.2	2.199	2.199	-1.142E-05
1.2	1.745E-02	0.1	2.299	2.299	-2.242E-05
1.2	1.7450E-02	0.2	2.299	2.299	-2.242E-05
1.2	3.490E-02	0.1	2.299	2.299	-2.247E-05
1.2	3.490E-02	0.2	2.299	2.299	-2.247E-05

Table 1 : a) ra, it is the radius value, b) j-axis value, c) z-axis value d) Analytical results of the developed scheme, d) Approximated results of developed scheme and e) Error analysis of developed thirteen point's stencils in cylindrical mesh system.

The numerical results are obtained for proposed isotropic discrete Laplacian operator in cylindrical coordinate system by taking novel examples. Intended for mathematical result estimated by DLO on cylindrical mesh scheme a code under FORTRAN scheme has been developed. The novel results obtained by using FORTRAN code has been discussed with the comparisons of analytical as well as approximated results of developed numerical scheme in the estimated table. The estimated table shows the six (6) decimal point accuracy in the scheme. For numerical schemes the stability is intimately connected with statistical error. A developed Finite Difference scheme is conditionally stable up-to the 6 points decimal error. In the above table, it can be seen that increasing the value of r, and the analytical as well as approximate value of the developed scheme is so closed and obtained error is up-to 5 (five) decimal places, which is quite accuracy and efficiency in novel result of the developed scheme under thirteen points stencils in cylindrical mesh system. Furthermore, on the behalf of the developed novel results of the discrete laplacian operator, it is stable scheme on thirteen point's stencils in cylindrical mesh system in finite difference method

6 Stability Analysis of the developed scheme.

For computational and numerical approximation results of developed schemes, the stability is strongly associated with approximated numerical error. If one time step error occurred in numerical and computational approximation, which is not actually effectible on error, therefore due to his non effectible error the finite difference scheme considered as a stable scheme. The error analysis and stability analysis tables indicates that the change in r is directly put an impact on the change in error, which can be seen in the table. The investigated table shows that if r 0.1 increasing the error of the scheme will be decrease even though if r 0.00000001 then error will be zero, this shows the convergence and stability of developed scheme on thirteen points stencil in cylindrical mesh system.

r_i	**angle**
0.00000001	-2.168E-19
0.0000001	-4.336E-19
0.000001	-6.114E-17
0.00001	-6.4919E-15
0.0001	-1.008E-12
0.001	4.584E-10
0.01	-2.989E-07
0.1	-1.136E-05

Table 2 : Table shows the error analysis of developed thirteen points stencils. Here 0.1 to 0.0000001 values have been taken, the obtained results are satisfied due to decreasing the value of r_a and the error become near to zero, it is considered as the scheme is convergent.

In this table we can identify the logic of error and consistency as well as stability, when by applying the 0.01 in, the novel results of schemes obtained up-to 5 exponents, as decreasing the value of, we can analyze that novel results of developed scheme reaches very closed to zero, this shows the convergence of the developed scheme. Now it can be seen and write that the convergence of the above examples is.

The analyzed results of the developed scheme can be visualized in below graph.

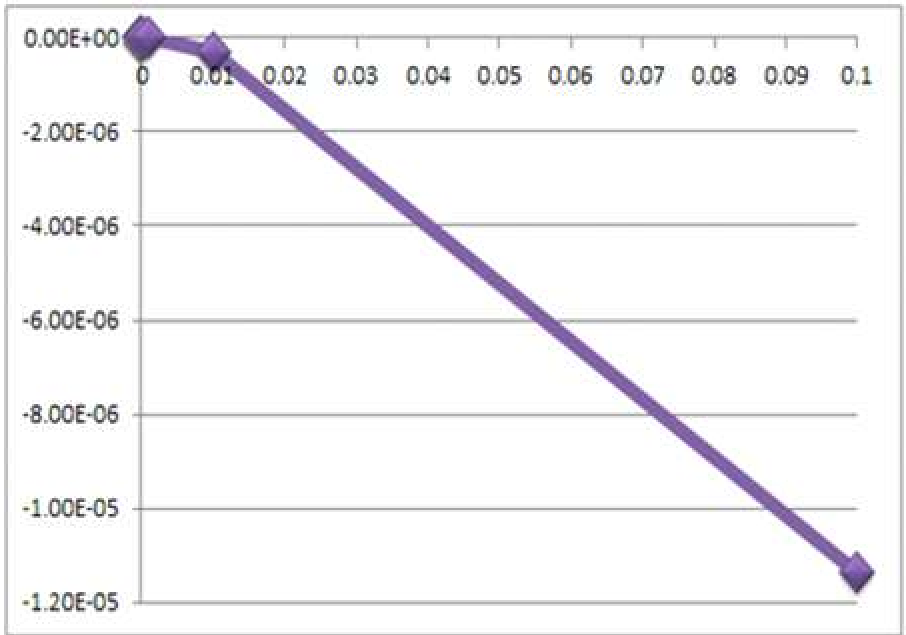

Figure 4. This graph has been developed by taking thirteen (13-points) stencil in cylindrical mesh system and graph made by (GNUPLOT 5.4 PATCHLEVEL 4) software. In this graph, it can be seen that decreasing the value of r error has reached on zero, this shows the accuracy and efficiency of the scheme and due to his accuracy and stability, it can be said that it is a stable scheme.

7 Conclusion

The Laplacian operator under thirteen (13) points stencil in cylindrical mesh system is developed by using Taylor's series and the novel numerical formulation developed in this study is verified by developed Fortran code. The numerical results obtained by the numerical formulations are visualized by GNUPLOT 5.4 PATCHLEVEL 4. The new numerical formulations devised for Laplacian operator/partial differential equations show accuracy, consistency and stability. The obtained numerical formulations can be used in solutions of partial differential equations involved in 1 st order and 2 nd order partial derivatives on cylindrical mesh system. Due to the good accuracy of numerical results, it is upright method may be used in modern mathematical, scientific chemical and biological fields. In this research article may be able to discrete the

Laplacian operator on thirteen point stencil in cylindrical coordinate system, which may help in simulation study of di-block copolymers system confined into cylindrical mesh system, simulation of hollow cylindrical form as well as in pattern of block copolymers.

8 Future Work

- Use thirteen-points stencil in Crank-Nicholson's method to generate discrete Laplacian operator in cylindrical mesh system, it is most accurate, efficient, stable method and having convergence criteria.
- Discretize the Laplacian operator for reduce the computational cost by applying thirteen (13) points in cylindrical mesh system

- Applying Cell Dynamic Simulation model on thirteen points stencil to obtained novel results in cylindrical mesh system.
- Increasing the number of stencils will be effects on the approximation numerical results.

Author Contributions

Rabanwaz Mallah: Conceptualization, Methodology, Software **Rabnawaz Mallah and Inayatullah Soomro**: Data curation, Writing- Original draft preparation. **Altaf Ahmed ,Sarang Latif and Irshad Ahmed**: Visualization, Investigation. **Dost Muhammad and Munwar Jameel**: Supervision.: **Inayatullah Soomro**: Software, Validation. **Rabnawaz Mallah, Shakeel Ahmed and Altaf Ahmed**: Writing- Reviewing and Editing

Compliance with Ethical Standards

Individually author has personally and substantially contributed to the research work that was done in order to bring about the publishing of the article, and they will each bear public responsibility for the contents of the work that they have produced.

Funding Information

This research is not funded from individual, institution or from any funding agency.

References

[1] Momoh, o.d. and sadiku, m.n.o., 2009. an easy approach to numerical computation of spherical shell capacitance using finite difference method. international review on modelling and simulations, 2(2), pp.196-200.

[2] Gitis, v., rubinstein, i., livshits, m. and ziskind, g., 2010. deep-bed filtration model with multistage deposition kinetics. chemical engineering journal, 163(1-2), pp.78-85.

[3] Ramadugu, r., thampi, s.p., adhikari, r., succi, s. and ansumali, s., 2013. lattice differential operators for computational physics. europhysics letters, 101(5), p.50006.

[4] Reimer, a.s. and cheviakov, a.f., 2013. a matlab-based finite-difference solver for the poisson problem with mixed dirichlet–neumann boundary conditions. computer physics communications, 184(3), pp.783-798.

[5] Bruno-alfonso, a., cabezas-gómez, l. and navarro, h.a., 2012. alternate treatments of jacobian singularities in polar coordinates within finite-difference schemes. world journal of modelling and simulation, 8(3), pp.163-171.

[6] Mallah, r. and soomro, i., 2022. comparative study of numerical approximation schemes for laplacian operator in polar mesh system on 9-points stencil including mixed partial derivative by finite difference method. journal of mathematical sciences computational mathematics, 3(4), pp.516-525.

[7] Mykhaylyk, t.a., dmitruk, n.l., evans, s.d., hamley, i.w. and henderson, j.r., 2007. comparative characterisation by atomic force microscopy and ellipsometry of soft and solid thin films. surface and interface analysis: An international journal devoted to the development and application of techniques for the analysis of surfaces, interfaces and thin films, 39(7), pp.575-581.

[8] *Esmaeilzadeh, M., 2016. A Cartesian Cut-Stencil Method for the Finite Difference Solution of PDEs in Complex Domains (Doctoral dissertation, University of Windsor (Canada)).* PhD thesis.

[9] Peres, n., poncet, s. and serre, e., 2012. a 3d pseudospectral method for cylindrical coordinates. application to the simulations of rotating cavity flows. journal of computational physics, 231(19), pp.6290-6305.

[10] Latif, s., mallah, r. and soomro, i., 2021. discretization of laplacian operator in polar coordinates system on 9-point stencil with mixed pde's derivative approximation using finite difference method. journal of mathematical sciences computational mathematics, 2(3), pp.387-394.

[11] Kozdon, j.e., dunham, e.m. and nordström, j., 2013. simulation of dynamic earthquake ruptures in complex geometries using high-order finite difference methods. journal of scientific computing, 55, pp.92-124.

[12] Mittal, r.c. and gahlaut, s., 1991. high-order finite-differences schemes to solve poisson's equation in polar coordinates. ima journal of numerical analysis, 11(2), pp.261-270.

[13] Ahimed, m., 2018. existence and uniqueness of solutions of systems of odes and some application.

[14] Weiler, m., forré, p., verlinde, e. and welling, m., 2021. coordinate independent convolutional networks–isometry and gauge equivariant convolutions on riemannian manifolds. arxiv preprint arxiv:2106.06020.

[15] Yu, h., hou, j., qin, z., saleh, m., shugurov, i., wang, k., busam, b. and ilic, s., 2022. riga: Rotation-invariant and globally-aware descriptors for point cloud registration. arxiv preprint arxiv:2209.13252.

[16] Mahdi Saleh, Yige Wang, Nassir Navab, Benjamin Busam, and Federico Tombari. Cloudattention: Efficient multi-scale attention scheme for 3d point cloud learning. In *2022 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, pages 1986–1992. IEEE, 2022.