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GLOBAL STABILITY ANALYSIS OF AN SIR MODEL WITH BILINEAR INCIDENCE RATE

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Abstract. *In this research work, we derive and analyse an SIR model with bilinear incidence rate. and We prove both global and local stability of the disease-free steady state and endemic equilibrium state. Using the idea of Lyapunov direct aproach by combining composite quadratic and linear functions we prove that the system equilibria are locally and globally asymptotically stable for any parameter regime.*

Keywords: SIR model; Linear stability analysis, System's equilibria, Lyapunov functional.

1. **Introduction.** In recent years there has been made a significant progress in understanding different scenarios for disease transmissions and behaviour of epidemics. Many models in the literature represent dynamics of diseases by systems of ordinary differential equations. In mathematical models of physical, engineering and biological systems it is important to consider both local and global stability of the system equilibria. The stability of epidemic models has been studied in many papers. But most of them are concerned with local stability of equilibria. The fraction of papers that obtain global stability of these models is relatively few. In order to understand the mechanism of disease transmission, many authors have paid attention to the stability analysis of the equilibria for various kinds of epidemic models. In most of these models, the classical assumptions are that the total population is divided into a number of classes according to their epidemiological status, and that the transmission of the infection in the population is modelled by incidence terms. Many forms are possible for the incidence term in epidemiological models and in this paper we consider the bilinear incidence rate.

Let $S(t)$, $I(t)$, $R(t)$ be the population densities at time t of the Susceptible, Infected and Recovered respectively. From these assumptions, we derive the following SIR delay epidemic model that incorporates two time delays (for details see [10]):

$$\begin{aligned} \tfrac{dS}{dt} &= b - \beta S(t)I(t) - \gamma S(t) + rI(t) \\ \tfrac{dI}{dt} &= \beta S(t)I(t) - (\mu + r + \gamma)I(t) \\ \tfrac{dR}{dt} &= rI(t) - \gamma R(t) \end{aligned} \tag{1}$$

with given non-negative initial conditions:

$$\begin{aligned} &S(0) > 0 \\ &I(0) \geq 0, \quad \text{and} \\ &R(0) \geq 0 \end{aligned}$$

The model parameters are defined as follows: b the birth rate, β the transmission rate, γ the natural death rate, μ the disease-induced death rate, r the recovery rate.

Since the first two equations in the model (1) above are independent of the R variable, we can ignore the third equation without loss of generality. Therefore, the model 1 becomes:

$$\begin{aligned} \frac{dS}{dt} &= b - \beta S(t)I(t) - \gamma S(t) + rI(t) \\ \frac{dI}{dt} &= \beta S(t)I(t) - (\mu + r + \gamma)I(t), \end{aligned} \tag{2}$$

with the above initial conditions and $S + I = N$. From now on we concentrate in our analysis on model 2.

2. **Linear stability analysis.** Here, we analyse the system's equilibria. We will analyse both the trivial and the non-trivial steady states of the system (2). We will analyse the model equilibria both locally and globally using both Routh-Hurwitz criterion and Lyapunov direct approach. The model 2 has a trivial steady state $E^0 = (S^0, I^0) = (b/\gamma, 0)$ and for $\mathcal{R}_0 > 1$, the non-trivial solution E^* exists and is unique where,

$$\mathcal{R}_0 = \frac{b\beta}{\gamma(\mu + r + \gamma)} \tag{3}$$

$$\begin{aligned} S^* &= \left(\frac{(\mu+r+\gamma)}{\beta}\right) = \frac{b}{\gamma\mathcal{R}_0}, \\ I^* &= \frac{b - \frac{\gamma(\mu+r+\gamma)}{\beta}}{\mu+\gamma} = \frac{b}{\mu+\gamma}\left(\frac{\mathcal{R}_0 - 1}{\mathcal{R}_0}\right). \end{aligned} \tag{4}$$

Linearising the system (2) near the steady state $E^* = (S^*, I^*)$, we have the following characteristic equation:

$$\begin{aligned} \Delta(\lambda) = \lambda^2 + \lambda[\gamma + (\gamma + r + \mu) + \beta I^* - \beta S^*] + (\gamma + r + \mu)(\gamma + \beta I^*) + \\ -\beta\gamma S^* - \beta r I^*. \end{aligned} \tag{5}$$

Introducing the notation

$$\begin{aligned} \hat{p}_0 &= (\mu + r + 2\gamma) + \beta I^*, \\ \hat{p}_1 &= -\beta S^*, \\ \hat{q}_0 &= (\mu + r + \gamma)(\gamma + \beta I^*), \\ \hat{q}_1 &= -\beta\gamma S^* \text{and} \\ \hat{q}_2 &= -\beta r I^*, \end{aligned} \tag{6}$$

the characteristic equation becomes:

$$\lambda^2 + \lambda(\hat{p}_0 + \hat{p}_1) + (\hat{q}_0 + \hat{q}_1 + \hat{q}_2) = 0. \tag{7}$$

This section presents and analyses the system's equilibria for the model (2). We present the following conditions which will be proved later:

$$\hat{p}_0 + \hat{p}_1 > 0 \tag{A1}$$

$$\hat{q}_0 + \hat{q}_1 + \hat{q}_2 > 0 \tag{A2}$$

We will prove that conditions A1 and A2 hold and are valid throughtout this paper. The following theorem forms the basis for the analysis.

Theorem 2.1.
The characteristic equation 7 has eigenvalues with negative real parts, and the following properties are true: for $\mathcal{R}_0 < 1$, the disease-free steady state E^0 is locally asymptotically stable and,
for $\mathcal{R}_0 > 1$, the non-trivial state E^* exists and is locally asymptotically stable.

Proof.
We begin the proof by recalling the characteristic equation 7

$$\lambda^2 + \lambda(\hat{p}_0 + \hat{p}_1) + (\hat{q}_0 + \hat{q}_1 + \hat{q}_2) = 0. \tag{8}$$

It is enough to show that the assumptions A1−A2 are satisfied, that is, to show that the coefficients $(\hat{p}_0 + \hat{p}_1)$ and $(\hat{q}_0 + \hat{q}_1 + \hat{q}_2)$ are non-negative, which would then imply that the roots of 8 have negative real part. To see this, we proceed as follows. At the disease-free steady state E^0, we have thus:

$$\begin{array}{ll} \hat{p}_0 + \hat{p}_1 & = (\mu + r + 2\gamma) + \beta I^0 - \beta S^0 = (\mu + r + 2\gamma)(1 - \mathcal{R}_0) > 0 \\ \text{if} & \mathcal{R}_0 < 1 \end{array}$$

and

$$\begin{array}{ll} \hat{q}_0 + \hat{q}_1 + \hat{q}_2 & = (\mu + r + \gamma)(\gamma + \beta I^0) - \beta\gamma S^0 - \beta r I^0 \\ & = \gamma(\mu + r + \gamma)(1 - \mathcal{R}_0) > 0 \\ \text{if} & \mathcal{R}_0 < 1. \end{array}$$

At the endemic steady state E^*, we have:

$$\begin{array}{ll} \hat{p}_0 + \hat{p}_1 & = (\mu + r + 2\gamma) + \beta I^* - \beta S^* \\ & = (\mu + r + 2\gamma) + \frac{\gamma(\mu+r+\gamma)}{\mu+\gamma}(\mathcal{R}_0 - 1) > 0 \\ \text{if} & \mathcal{R}_0 > 1 \end{array}$$

and

$$\begin{array}{ll} \hat{q}_0 + \hat{q}_1 + \hat{q}_2 & = (\mu + r + \gamma)(\gamma + \beta I^*) - \beta\gamma S^* - \beta r I^* \\ & = \gamma(\mu + r + \gamma)(\mathcal{R}_0 - 1) > 0 \\ \text{if} & \mathcal{R}_0 > 1. \end{array}$$

From the Routh-Hurwitz principle, we know that all eigenvalues of 8 will have negative real parts, and by implication, the disease-free steady state is locally asymptotically stable if $\mathcal{R}_0 < 1$. From the same argument it follows that the endemic steady state E^* is locally asymptotically stable for $\mathcal{R}_0 > 1$. Hence, the proof. □

2.1. **Global stability of the disease-free steady state.** Here we present global stability analysis of E^0 using Lyapunov direct method, by combining composite quadratic and linear functions.

Theorem 2.2.
Let $\mathcal{R}_0$ be defined as in 3, then the disease-free steady state E^0 of model 2 is globally asymptotically stable if $\mathcal{R}_0 \leq 1$, and unstable for $\mathcal{R}_0 > 1$.

Proof.
Define $U_1 : (S, I) \in \mathbb{D} : S > 0, I > 0 \to \mathbb{R}$ by

$$U_1(S, I) = \frac{1}{2}\left[(S - S^0) + I\right]^2 + \frac{(\mu + 2\gamma)}{\beta} I,$$

$U_1(S, I) \geq 0 \in \mathbb{D}$. Then U_1 is C^1 on the interior of $\mathbb{D}$, E^0 is the global minimum of U_1, and $U_1(S^0, I^0) = 0$. The time derivative of the functional U_1 can be computed along solutions of 2 as follows

$$\begin{array}{ll} \frac{dU_1}{dt} & = [(S - S^0) + I]\left(\frac{dS}{dt} + \frac{dI}{dt}\right) + \frac{(\mu+2\gamma)}{\beta}\frac{dI}{dt}, \\ & = [(S - S^0) + I]\left(b - \gamma(S + I) - \mu I\right) + \frac{(\mu+2\gamma)}{\beta}(\beta SI - (\gamma + r + \mu)I), \end{array}$$

which $b = \gamma S^0$, can be transformed into

$$\begin{array}{ll} \frac{dU_1}{dt} & = [(S - S^0) + I](\gamma S^0 - \gamma(S + I) - \mu I) + \frac{(\mu+2\gamma)}{\beta}(\beta SI - (\gamma + r + \mu)I) \\ & = -\gamma(S - S^0)^2 - (\mu + \gamma)I^2 - (\mu + 2\gamma)\left(\frac{(\gamma+r+\mu)}{\beta} - S^0\right) I \\ & = -\gamma(S - S^0)^2 - (\mu + \gamma)I^2 - \frac{(\mu+2\gamma)(\gamma+r+\mu)}{\beta}(1 - \mathcal{R}_0)I. \end{array}$$

We can see from the above that whenever $\mathcal{R}_0 \leq 1 \implies \dot{U}_1 \leq 0$, with $\dot{U}_1 = 0$ only if $S = S^0$ and $I = I^0$. Hence, E^0 is the largest compact invariant. Now by LaSalle's invariant principle [3], [4], [5], [8], [9] the disease-free solution E^0, is globally asymptotically stable. □

2.2. Global stability of the endemic steady state. It has already been established that when $R_0 > 1$, the system 2 has a unique endemic steady state E^*. Now, using Lyapunov direct method by combining linear and composite functions, we shall show that E^* is globally asymptotically stable.

Theorem 2.3.
For $\mathcal{R}_0 > 1$, the endemic equilibrium is globally asymptotically stable.

Proof.
We define a Lyapunov functional

$$U_2 : (S, I) \in \mathbb{D} : S > 0 \to \mathbb{R}, \tag{9}$$

as follows

$$U_2(S, I) = \frac{1}{2}[\nu_1 + \nu_2]^2 + \frac{(\mu + 2\gamma)}{\beta}\left(\nu_2 - I^* \ln \frac{I}{I^*}\right), \tag{10}$$

where,

$$\nu_1 = S - S^*, \ \ \nu_2 = I - I^*.$$

Then U_2 is C^1, E^* is the global minimum of U_2 on $\mathbb{D}$, and $U_2(S^*, I^*) = 0$.
Now differentiating U_2 with respect to time, we have thus.

$$\begin{aligned}\frac{dU_2}{dt} &= [\nu_1 + \nu_2]\left(\tfrac{dS}{dt} + \tfrac{dI}{dt}\right) + \tfrac{(\mu+2\gamma)}{\beta}\tfrac{\nu_2}{I}\left(\tfrac{dI}{dt}\right) \\ &= [\nu_1 + \nu_2](R - \gamma(S + I) - \mu I) + \tfrac{(\mu+2\gamma)}{\beta}\tfrac{\nu_2}{I}(\beta SI - (\gamma + r + \mu)I),\end{aligned}$$

using $R = \gamma(S^* + I^*) + \mu I^*$ and $\beta S^* = (\gamma + r + \mu)$, we have

$$\begin{aligned}\frac{dU_2}{dt} &= [\nu_1 + \nu_2]\{-\gamma[\nu_1 + \nu_2] - \mu\nu_2\} + (\mu + 2\gamma)\nu_1\nu_2 \\ &= -\gamma\nu_1^2 - (\gamma + \mu)\nu_1^2\end{aligned}$$

Therefore, $\dot{U}_2 \leq 0$, $\dot{U}_2 = 0$ only if $S = S^*$ and $I = I^*$. Hence, E^* is the largest invariant. By applying LaSalle's invariant principle [3], [4], this means the endemic steady state is globally asymptotically stable. □

3. Conclusion. In summary, we have proved that the system equilibria both disease-free and endemic of the model 2 are globally asymptotically stable using Lyapunov direct functionals.

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