

The effect of oscillating streams on heat transfer in viscous magnetohydrodynamic MHD fluid flow

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Abstract

This study focuses on developing and proving exact solutions for equations of motion involving a fluid with finite conductivity, variable viscosities, and heat transfer in the presence of a transverse magnetic field. By utilizing a transformation variable, the governing equation is transformed into a assortment of simple ordinary differential equations, enabling accurate solutions to be achieved for the problem. The solutions demonstrate that the distribution of vorticity is proportional to the stream function, which is disturbed by oscillating (sine or cosine) or even and exponential streams. This study compares the $u(x,y)$ profiles of steady fluid flow with changing viscosity and heat transfer travelling on a plane. The comparison is used to identify differences in the profiles of $u(x,y)$, providing insight into the changes that occur in interesting factors. These findings have significant implications for understanding fluid dynamics in complex systems and may have important applications in fields such as engineering and physics.

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1 Introduction

In recent years, there has been a growing interest in understanding the effect of oscillating streams on the heat transfer characteristics of MHD fluids. This research is particularly relevant in applications such as nuclear reactors, chemical processes, and cooling systems, where heat transfer is a critical factor in the design and operation of the system. Top of form the effect of oscillating streams on heat transfer in viscous magnetohydrodynamic (MHD) fluid flow is a topic of great interest in the field of fluid mechanics and heat transfer [1]. MHD refers to the study of the interaction between magnetic fields and electrically conductive fluids. The behavior of MHD fluids can be affected by a variety of factors, including the presence of magnetic fields, the viscosity of the fluid, and the flow rate [2]. The behaviour of MHD fluids in the presence of oscillating streams can be complex, and there are many factors that can influence the heat transfer process [3]. It is not possible to evaluate or validate the difficult flow problems and physical phenomena in their exactness until exact solutions are found [4,5]. In order to get the perfect solution in each sector where mathematical modelling is used, many researchers, scientists, and engineers are striving to find the correct solution in their particular discipline. The Navier-Stokes equations are crucial in the study of fluid mechanics. When it comes to fluid flow, the significance of Navier-Stokes' equations lies in its wide application for a variety of different types of fluid flow, ranging from thin layer to large-scale atmospheric and cosmic flows [6,8]. However, because of the nonlinear character of Navier-Stokes' equations, it is extremely difficult to find a precise solution to them [9,10]. To yet, no complete set of the general solution to Navier-equations Stokes's has been discovered, and this remains an unresolved subject. In order to overcome this difficulty, several researchers have proposed numerous transformations, inverse or semi-inverse approaches for the reformulation of insolvable form equations, among other things. Certain scholars were able to linearize Navier-equations Stokes's and successfully achieved some exact solutions using Martin's formulation [11], and the hodographs transformation [12,13]. Some authors [9,14,16,17,18] employed the inverse method and made some assumptions about the flow variable in order to arrive at more precise answers to their problems. However, with constant viscosity fluid flows, it has been possible to obtain solutions that are superior to all others. Viscosity no longer remains constant under a variety of physical circumstances, such as high pressure and temperature gradients, or in electrically conducting fluid flows where magnetic effects play a dominant role, in which the magnetic effects are dominant [19,20,21,22,23] When vorticity is treated as a variable, it becomes more difficult to establish the exact solutions to Navier-Stokes' equations. The accurate solution of the Navier-Stokes equations for a fluid with changing viscosity has only been published in a few articles in the literature so far. For incompressible fluid with varying viscosity, Naeem and Nadeem [24] modified Martin's technique, in which the system of equations is rewritten in curvilinear coordinates, to include nonlinear coordinates. Naeem [25], was able to translate the governing equations of an incompressible fluid with changing viscosity into a set of ordinary differential equations and achieve some accurate solutions by employing a one-parameter group of transformation. Moreover, using von-mises variables and various transformation approaches, Naeem and colleagues [26,27] discovered certain accurate solutions of steady planar incompressible fluid flows with varying viscosity using von-mises variables and various transformation techniques. Naeem and Jamil [28,29], defined a one-dimensional transformed variable $\xi = (x \cos\theta + y \sin\theta)$; $-\pi \leq \theta \leq \pi$, ordinary differential equations were derived from the governing equations, and a set of exact solutions for fluid flow with varying viscosity was obtained. These solutions reveal that the vorticity function is proportional to the stream function, which is perturbed by a uniform stream parallel and inclined to the x-axis. Jamil and Khan [30] used the same technique to consider an electrically conducting fluid with variable viscosity in the presence of a transverse magnetic field and the vorticity distribution proportional to the stream function perturbed by a uniform magnetic field. $U(x + y)$, where U is a constant that is inclined to the X-axis. In addition, employing the same techniques as described above, some additional work for Newtonian and non-Newtonian fluids with and without variable viscosity has been published in [31,32,33] and references cited. However, due to its widespread application in physics, engineering, and mathematics, its insignificant consideration has been focused on the hydrodynamic flow of electrically conducting viscous incompressible fluids with magnetic effects. It is the study of how conducting fluids interact with electromagnetic events [34] that is the focus of MHD research. A stationary frame of reference on Earth's surface that spins around an inertial frame while subjected to the earth's magnetic field is used in geophysics to determine and investigate the locations and velocities of a stationary frame of reference on Earth's surface in the presence of its magnetic field [35]. Magnetic field-induced fluid flow (MHD) is used extensively in a variety of fields including power production, MHD pumps, plasma, aerodynamics, nuclear engineering, earthquake, magnetic drug targeting, MHD sensors, and ion propulsion, among others. As a result, various academics have examined several fascinating flows in [36,37,38]. The main objective of this study is to obtain exact solutions for the governing equations that describe the flow of an electrically conducting fluid with variable viscosity, subjected to a transverse magnetic field $H = (0,0,H)$, while considering a vorticity distribution that is proportional to a stream function perturbed by a sinusoidal or cosine-like oscillating flow, i.e., $U \sin(ax + by)$ or $U \cos(ax + by)$.

2 Basic Magnetohydrodynamic (MHD) equations and problem formulation

Here are the fundamental dimensionless equations that govern a fluid with a variable viscosity is in motion in an incompressible state and finite conductivity in a steady state, under the influence of a magnetic field:

$$\nabla \cdot \mathbf{V} = 0 \quad (1)$$

$$\mathbf{V} \cdot \nabla \mathbf{V} = -\nabla p + \frac{1}{Re} \nabla \cdot [\mu(\nabla \mathbf{V} + (\nabla \mathbf{V})^T)] + R_H [(\nabla \times \mathbf{H}) \times \mathbf{H}] \quad (2)$$

$$\nabla \times (\mathbf{V} \times \mathbf{H}) = \frac{1}{R_\sigma} \nabla^2 \mathbf{H} \quad (3)$$

$$(\mathbf{V} \cdot \nabla) T = \frac{1}{RePr} \nabla^2 T + \frac{Ec}{Re} 2\mu[S:\nabla \mathbf{V}] + \frac{R_H Ec}{R_\sigma} (\nabla \times \mathbf{H})^2 \quad (4)$$

$$\mathbf{S} = \frac{1}{2}(\nabla \mathbf{V} + (\nabla \mathbf{V})^T) \quad (5)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (6)$$

Where, \mathbf{V} is the velocity, \mathbf{H} is the magnetic field, p the pressure, T the temperature, μ the viscosity all of these quantities are functions of x, y coordinates. Reynolds number (Re), Prandtl number (Pr), Eckert number (Ec), magnetic pressure number (RH), magnetic Reynolds number ($R\sigma$). Taken into consideration, the flow is a plane transverse flow. We have

$$\mathbf{V} = (u, v, 0), \mathbf{H} = (0, 0, H), (\cdot)_z = 0 \quad (7)$$

The system of equations (1-6), utilizing equations (7) becomes,

$$u_x + v_y = 0 \quad (8)$$

$$uu_x + vv_y = -\left[p + R_H \frac{H^2}{2}\right]_x + \frac{1}{Re} \left[(2\mu u_x)_x + (\mu(u_y + v_x))_y\right] \quad (9)$$

$$uv_x + vv_y = -\left[p + R_H \frac{H^2}{2}\right]_y + \frac{1}{Re} \left[(2\mu v_y)_y + (\mu(u_y + v_x))_x\right] \quad (10)$$

$$uH_x + vH_y = \frac{1}{R_\sigma} [H_{xx} + H_{yy}] \quad (11)$$

$$uT_x + vT_y = \frac{1}{RePr} [T_{xx} + T_{yy}] + \frac{Ec}{Re} \mu \left[2(u_x^2 + v_y^2) + (u_y + v_x)^2\right] + \frac{R_H Ec}{R_\sigma} (H_x^2 + H_y^2) \quad (12)$$

Equation (8) implies the existence of the stream function ψ such that.

$$u = \psi_y, v = -\psi_x \quad (13)$$

The number of equations (8 - 12) on employing equation (13), convert into the following equations.

$$J_x = -\psi_x \omega + \frac{1}{Re} [\mu(\psi_{yy} - \psi_{xx})]_y \quad (14)$$

$$J_y = -\psi_y \omega + \frac{1}{Re} [\mu(\psi_{yy} - \psi_{xx})]_x - \frac{4}{Re} [\mu\psi_{xy}]_y \quad (15)$$

$$\psi_y H_x - \psi_x H_y = \frac{1}{R_\sigma} [H_{xx} + H_{yy}] \quad (16)$$

$$\psi_y T_x - \psi_x T_y = \frac{1}{RePr} [T_{xx} + T_{yy}] + \frac{Ec}{Re} \mu \left[4(\psi_{xy})^2 + (\psi_{yy} - \psi_{xx})^2\right] + \frac{R_H Ec}{R_\sigma} [H_x^2 + H_y^2] \quad (17)$$

$$\omega = -(\psi_{xx} + \psi_{yy}) \quad (18)$$

$$J = p + R_H \frac{H^2}{2} + \frac{1}{2}(\psi_x^2 + \psi_y^2) - \frac{2\mu\psi_{xy}}{Re} \quad (19)$$

Because we're looking for a solution to the system of equations (14 - 17) in which the vorticity distribution is proportional to the stream function perturbed by a sine stream, the pressure p is calculated using equation (19) after the system of equations (14 - 17) is solved. As a result, we generate.

Type – I: When Vorticity Distribution Perturbed by Sine Oscillation

$$\psi_{xx} + \psi_{yy} = K[\psi - U \sin(ax + by)] \quad (20)$$

Where, $K, a, b \neq 0, a \neq b$ & U are real constant. On defining a new function

$$\Psi = \psi - U \sin(ax + by) \quad (21)$$

We call it perturbed stream function and employing equation (19), equation (18) becomes

$$\omega = -K \Psi \quad (22)$$

Equation (14) & (15), utilizing equation (21) & (22) becomes

$$J_x = \left[\frac{K\Psi^2}{2}\right]_x + a K U \Psi \cos(ax + by) + M_y \quad (23)$$

$$J_y = \left[\frac{K\Psi^2}{2}\right]_y + b K U \Psi \cos(ax + by) - \frac{4\mu}{Re} [\Psi_{xy} - U \sin(ax + by)]_y + M_x \quad (24)$$

Where,

$$M = \frac{1}{Re} [\mu\{\Psi_{yy} - \Psi_{xx} + U(a^2 - b^2)\sin(ax + by)\}]$$

Equation (23) & (24), using on the integrability condition $J_{xy} = J_{yx}$ provide motion

$$M_{xx} - M_{yy} + UK\cos(ax + by)(b\Psi_x - a\Psi_y) - \frac{4}{Re} [\mu(\Psi_{xy} - U\absin(ax + by))]_{yx} = 0 \quad (25)$$

Equation (25) illustrates the heat transfer motion of a finitely conducting incompressible fluid with variable viscosity in the presence of a transverse magnetic field. For this heat transfer motion to take place, the function Ψ and viscosity μ must satisfy certain conditions. An analogy can be made by contemplating vorticity distribution relational to the flow function flustered by a sine stream. Equation (21) is formed by combining Equation (15) and (16) results in the following equation:

$$[\Psi_y + bU\cos(ax + by)]H_x - [\Psi_x + aU\cos(ax + by)]H_y = \frac{1}{R_\sigma} [H_{xx} + H_{yy}] \quad (26)$$

$$[\Psi_y + bU\cos(ax + by)]T_x - [\Psi_x + aU\cos(ax - by)]T_y = \frac{1}{RePr} [T_{xx} + T_{yy}] + \frac{Ec}{Re}\mu \times \left[4(\Psi_{xy} - U\absin(ax + by))^2 + (\Psi_{yy} - \Psi_{xx} + U(a^2 - b^2)\sin(ax + by))^2\right] + \frac{EcR_H}{R_\sigma} [H_x^2 + H_y^2] \quad (27)$$

Equation (20) employing, equation (21) becomes.

$$\Psi_{xx} + \Psi_{yy} - K\Psi = U(a^2 + b^2)\sin(ax + by) \quad (28)$$

The transformation variable is introduced as follows:

$$\xi = ax + by \quad (29)$$

The equations (25 – 28) are transformed by introducing a new independent variable, ξ ,

$$\Psi_{\xi\xi} - \Lambda\Psi = U\sin\xi \quad (30)$$

$$\Lambda = \frac{K}{(a^2 + b^2)}, (\mu\Psi)_{\xi\xi} = 0, H_{\xi\xi} = 0, \& T_{\xi\xi} + EcPr\mu\Lambda^2(a^2 + b^2)\Psi^2 + \frac{EcR_H Re Pr}{R_\sigma} H_\xi^2 = 0 \quad (31)$$

3 Exact Solutions

We consider three specific cases in this portion to show some exact results to the equations (30 – 31):

Case I: $\Lambda = -n^2$ $n > 0$

Case II: $\Lambda = m^2$ $m > 0$

Case III: $\Lambda = 0$

The solution of the equations (33 – 34) is now considered independently for each case. Our technique is to first get Ψ from equation (33) and then apply to find μ, H, T, u, v and p from the of equations (19), (31) & (21).

Case I

When considering this particular instance, the physical-plane solution of equation (30) is given by

$$\Psi = A_{11}\sin(n(ax + by) + A_{12}) + \frac{U\sin(ax+by)}{n^2 - 1} \quad (32)$$

where A_{11} and A_{12} are constants. Equation (31) using (32) gives

$$\mu = \frac{A_{13}(ax+by)+A_{14}}{A_{11}\sin(n(ax+by)+A_{12})+\frac{U\sin(ax+by)}{n^2-1}} \quad (33)$$

where A_{12} and A_{14} are constants. The solution of equation (31) is

$$H = A_{15}(ax + by) + A_{16} \quad (34)$$

where A_{15} and A_{16} are constants. Equation (31), using equations (33) and (34), becomes

$$T_{\xi\xi} + EcPr\Lambda^2(a^2 + b^2)(A_3\xi + A_4)\Psi + \frac{R_H Ec Pr Re A_{15}^2}{R_\sigma} = 0 \quad (35)$$

the solution of equation (38)

$$T = \frac{Ec Pr \Lambda^2 (a^2 + b^2)}{n^3 (n^2 - 1)} [A_{13} \{A_{11} (n^2 - 1) \{n(ax + by) \sin(n(ax + by) + A_{12}) + 2\cos(n(ax + by) + A_{12})\} + n^3 U \{(ax + by) \sin(ax + by) + 2\cos(ax + by)\}\} + nA_{14} \{A_{11} (n^2 - 1) \sin(n(ax + by) + A_{12}) + n^2 U \sin(ax + by)\}] - \frac{Ec R_H Pr Re A_{15}^2}{2 R_\sigma} (ax + by)^2 + A_{17}(ax + by) + A_{18} \quad (36)$$

where A_{18} is constant.

Here we use, Stream function ψ

$$\psi = A_{11}\sin(n(ax + by) + A_{12}) + \frac{n^2 U \sin(ax+by)}{n^2 - 1} \quad (37)$$

It denotes a sin stream $\sin(n(ax + by) + A_{12})$ in the +ve x-direction, as well as trepidation that is periodic in both the x and y directions, velocity distribution and pressure components are:

$$u = nbA_{11}\cos(n(ax + by) + A_{12}) + \frac{bn^2 U \cos(ax+by)}{n^2 - 1} \quad (38)$$

$$v = -naA_{11}\cos(n(ax + by) + A_{12}) - \frac{an^2U\cos(ax+by)}{n^2-1} \tag{39}$$

$$p = \frac{nk}{2(n^2-1)^2} [2A_{11}\{(n^2-1)^2A_{11}\sin^2(n(ax+by) + A_{12}) + U\{\cos(n(ax+by) + A_{12})\times \cos(ax+by) + n\sin(n(ax+by) + A_{12})\sin(ax+by)\} - nU\{\sin(n(ax+by) + A_{12})\sin(ax+by) + n\cos(n(ax+by) + A_{12})\cos(ax+by)\}\} + nU^2\sin(ax+by)] - \frac{n^2(a^2+b^2)}{n^2-1} \left[\frac{bU}{aR_e} \{(n^2-1)A_{11}\sin(n(ax+by) + A_{12}) + U\sin(ax+by)\} + \frac{1}{2(n^2-1)} \{(n^2-1)A_{11}\cos(n(ax+by) + A_{12}) + nU\cos(ax+by)\} \right]^2 - R_H \frac{H^2}{2} + A_{19} \tag{40}$$

where A_{19} is constant.

Case II

In this case

$$\Psi = B_{11}e^{m(ax+by)} + B_{12}e^{-m(ax+by)} - \frac{U\sin(ax+by)}{m^2+1} \tag{41}$$

Equation (31) utilizing (41) gives.

$$\mu = \frac{B_{13}(ax+by)+B_{14}}{B_{11}e^{m(ax+by)}+B_{12}e^{-m(ax+by)}-\frac{U\sin(ax+by)}{m^2+1}} \tag{42}$$

The solution of Equation (31) is

$$H = B_{15}(ax + by) + B_{16} \tag{43}$$

Equation (31), using equations (42) and (43) becomes

$$T_{\xi\xi} + EcPr\Lambda^2(a^2 + b^2)(B_{13}\xi + B_{14})\Psi + \frac{R_H EcPr Re B_{15}^2}{R_\sigma} = 0 \tag{44}$$

the solution of equation (44) is

$$T = \frac{EcPr\Lambda^2(a^2+b^2)}{m^3(m^2+1)} [B_{13}\{(m^2+1)\{B_{11}e^{m(ax+by)}(2-m(ax+by)) - B_{12}e^{-m(ax+by)}\times(m(ax+by)+2)\} - m^3U\{(ax+by)\sin(ax+by) + 2\cos(ax+by)\} - B_{14}m \times \{(m^2+1)(B_{11}e^{m(ax+by)}+B_{12}e^{-m(ax+by)}) + m^2U\sin(ax+by)\}] - \frac{R_H EcPr Re B_{15}^2}{2R_\sigma} (ax+by)^2 + B_{17}(ax+by) + B_{18} \tag{45}$$

where B_{18} is constant.

Here we use Stream function ψ

$$\psi = B_{11}e^{m(ax+by)} + B_{12}e^{-m(ax+by)} + \frac{m^2U\sin(ax+by)}{m^2+1} \tag{46}$$

A hyperbolic (sine or cosine) stream $B_{11}e^{m(ax+by)} + B_{12}e^{-m(ax+by)}$ in the +ve x-direction, as well as a agitation in x and y , are represented in this situation, the component of velocity distribution and pressure is given by the expression

$$u = mbB_{11}e^{m(ax+by)} - mbB_{12}e^{-m(ax+by)} + \frac{m^2U\cos(ax+by)}{m^2+1} \tag{47}$$

$$v = -maB_{11}e^{m(ax+by)} + maB_{12}e^{-m(ax+by)} - \frac{m^2U\cos(ax+by)}{m^2+1} \tag{48}$$

$$p = \frac{k}{2(m^2+1)^2} [B_{11}e^{m(ax+by)}\{(m^2+1)^2B_{11}e^{m(ax+by)} - 2mU\{m\sin(ax+by) - \cos(ax+by)\} + 2m^2U\{\sin(ax+by) + m\cos(ax+by)\}\} + mB_{12}e^{-m(ax+by)}\{(m^2+1)B_{12}e^{-m(ax+by)} - 2U \times \{\cos(ax+by) + \sin(ax+by)\} + 2mU\{\sin(ax+by) - m\cos(ax+by)\}\} - m^2U^2\cos^2(ax+by)] + \frac{m^2(a^2+b^2)}{m^2+1} \left[\frac{bU}{aR_e} \{(m^2+1)\{B_{11}e^{m(ax+by)} + B_{12}e^{-m(ax+by)}\} - U\sin(ax+by)\} - \frac{1}{2} \{(m^2+1)\{B_{11}e^{m(ax+by)} - B_{12}e^{-m(ax+by)}\} + mU\} \right]^2 - R_H \frac{H^2}{2} + B_{19} \tag{49}$$

where B_{19} is constant.

Case III

In this case

$$\Psi = C_{11} + (ax + by)C_{12} - U\sin(ax + by) \tag{50}$$

$$\mu = \frac{C_{13}(ax+by)+C_{14}}{C_{11}+(ax+by)C_{12}-U\sin(ax+by)} \tag{51}$$

$$H = C_{15}(ax + by) + C_{16} \tag{52}$$

$$T = -\frac{EcPr\Lambda^2(a^2+b^2)}{12} [C_{13}\{(ax+by)^3(2C_{11} + (ax+by)C_{12}) + 12U \times \{(ax+by)\sin(ax+by) + 2\cos(ax+by)\}\} + 2C_{14}\{(ax+by)^2 \times (3C_{11} + (ax+by)C_{12} + 6U\sin(ax+by))\}] - \frac{R_H EcPr Re C_{15}^2}{2R_\sigma} (ax+by)^2 + (ax+by)C_{17} + C_{18} \tag{53}$$

where $C_{11}, C_{12} \dots \dots C_{18}$, are constant, $\Lambda = 0$.

$$\psi = C_{11} + (ax + by)C_{12} \tag{54}$$

Velocity distribution and pressure components are:

$$u = bC_{12} \quad (55)$$

$$v = -aC_{12} \quad (56)$$

$$P = \frac{C_{12}}{2} [K\{2C_{11}(ax + by) + C_{12}(ax + by)^2 + 2U \cos(ax + by)\} - (a^2 + b^2)C_{12}] + C_{19} \quad (57)$$

where C_{19} is constant.

Type II: When Vorticity Distribution Perturbed by Cosine Oscillation

When a cosine stream disrupts the vorticity distribution of equations (14 – 17). Therefore, we established:

$$\psi_{xx} + \psi_{yy} = K[\psi - U \cos(ax + by)] \quad (58)$$

Where $K, a, b \neq 0, a \neq b$, as well as U are constant. Applying disturbed stream function as

$$\Psi = \psi - U \cos(ax + by) \quad (59)$$

and employing equation (58), the equation (18) develops

$$\omega = -K \Psi \quad (60)$$

Equation (14) and (15), utilizing equation (59) and (60) develops

$$J_x = \left[\frac{K\Psi^2}{2} \right]_x - a K \Psi U \sin(ax + by) + M_y \quad (61)$$

$$J_y = \left[\frac{K\Psi^2}{2} \right]_y - b K \Psi U \sin(ax + by) - \frac{4}{Re} [\mu \{ \Psi_{xy} - U \cos(ax + by) \}]_y + M_x \quad (62)$$

Where $M = \frac{1}{Re} [\mu \{ \Psi_{yy} - \Psi_{xx} + U(a^2 - b^2) \cos(ax + by) \}]$

equation (61) and (62), on using the integrability condition $J_{xy} = J_{yx}$ provide

$$M_{xx} - M_{yy} + UK \sin(ax + by) [a \Psi_y - b \Psi_x] - \frac{4}{Re} [\mu \{ (\Psi_{xy} - U \cos(ax + by)) \}]_{yx} = 0 \quad (63)$$

Considering a transverse magnetic field and a fluid with variable viscosity that moves with heat transfer and has a vortex distribution proportional to the perturbed cosine stream function, it is crucial that the function Ψ and viscosity μ satisfy equation (63). The following equations can be obtained by combining equation (16) and (17) with equation (59):

$$[\Psi_y - b U \sin(ax + by)] H_x - [\Psi_x - a U \sin(ax + by)] H_y = \frac{1}{R\sigma} [H_{xx} + H_{yy}] \quad (64)$$

$$[\Psi_y - b U \sin(ax + by)] T_x - [\Psi_x - a U \sin(ax + by)] T_y = \frac{1}{RePr} (T_{xx} + T_{yy}) + \frac{Ec}{Re} \mu \times [4 \{ \Psi_{xy} - U \cos(ax + by) \}]^2 + [\Psi_{yy} - \Psi_{xx} + U(a^2 - b^2) \cos(ax + by)]^2 + \frac{EcR_H}{R\sigma} [H_x^2 + H_y^2] \quad (65)$$

equation (58) employing, equation (59) becomes

$$\Psi_{xx} + \Psi_{yy} - K\Psi = U(a^2 + b^2) \cos(ax + by) \quad (66)$$

Introducing the change variable such as

$$\xi = ax + by \quad (67)$$

Taking the equations (63 – 66) and converting them into new independent variables, we get

$$\Psi_{\xi\xi} - \Lambda \Psi = U \cos \xi \quad (68)$$

Where,

$$\Lambda = \frac{K}{(a^2 + b^2)}, (\mu \Psi)_{\xi\xi} = 0, H_{\xi\xi} = 0, T_{\xi\xi} + EcPr \mu \Psi^2 \Lambda^2 (a^2 + b^2) + \frac{EcR_H RePr}{R\sigma} H_{\xi}^2 = 0 \quad (69)$$

In this section, we consider the following three cases when presenting some exact solutions to the system of equations (69).

Case I: $\Lambda = -n^2 \quad n > 0$

Case II: $\Lambda = m^2 \quad m > 0$

Case III: $\Lambda = 0$

Consequently, we will analyze each of these scenarios individually and compute the solution for equations (69). Our approach is to initially obtain the value of Ψ from equation (68), which we can subsequently utilize to determine the solutions for H, T, ψ, u, v , and p through the system of equations (69), (59), and (60), respectively (19).

Case I

The physical plane solution of equation (68) is

$$\Psi = A_{21} \sin(n(ax + by)) + A_{22} + \frac{U \cos(ax + by)}{(n^2 - 1)} \quad (70)$$

where A_{22} is constants. Equation (69) utilizing (70) gives

$$\mu = \frac{A_{23}(ax + by) + A_{24}}{A_{21} \sin(n(ax + by)) + A_{22} + \frac{U \cos(ax + by)}{(n^2 - 1)}} \quad (71)$$

where A_{24} is constant. The solution of equation (68) is

$$H = A_{25}(ax + by) + A_{26} \tag{72}$$

Where A_{26} is constant. Equation (69), using equations (71) and (72), becomes

$$T_{\xi\xi} + EcPr\Lambda^2(a^2 + b^2)(A_{23}\xi + A_{24})\Psi + \frac{R_H EcPrReH_{\xi}^2}{R_{\sigma}} = 0 \tag{73}$$

The solution of equation (73)

$$T = \frac{EcPr\Lambda^2(a^2+b^2)}{n^3(n^2-1)} [A_{23} \{A_{21}(n^2 - 1) \{n(ax + by) \sin(n(ax + by) + A_{22}) + 2 \cos(n(ax + by) + A_{22})\} + n^3U\{(ax + by) \cos(ax + by) - 2 \sin(ax + by)\}\} + nA_{24}\{A_{21}(n^2 - 1) \sin(n(ax + by) + A_{22}) + n^2U \cos(ax + by)\}] - \frac{EcR_H PrReA_{25}^2}{2R_{\sigma}}(ax + by)^2 + A_{27}(ax + by) + A_{28} \tag{74}$$

This case is represented by the following stream function.

$$\psi = A_{21}\sin(n(ax + by) + A_{22}) + \frac{n^2U}{(n^2-1)}\cos(ax + by) \tag{75}$$

It represents a sine stream $\sin(n(ax + by) + A_{22})$ in the +ve x-direction as well as a perturbation that is again periodic in x and y. The velocity distribution and pressure are presented by

$$u = nbA_{21}\cos(n(ax + by) + A_{22}) - \frac{bn^2U\sin(ax+by)}{n^2-1} \tag{76}$$

$$v = -naA_{21}\cos(n(ax + by) + A_{22}) + \frac{an^2U\sin(ax+by)}{n^2-1} \tag{77}$$

$$p = \frac{k}{2(n^2-1)^2} [A_{21}\{A_{21}(n^2 - 1)^2\sin^2(n(ax + by) + A_{22}) + 2nU\{n \sin(n(ax + by)+A_{22})\cos(ax + by) - \cos(n(ax + by) + A_{22}) \sin(ax + by)\}\} - n^2U\{2\{\sin \times (n(ax + by) + A_{22}) \cos(ax + by) - n \cos(n(ax + by) + A_{22})\sin \times (ax + by)\} - U\sin^2(ax + by)\}] - \frac{n^2(a^2+b^2)}{n^2-1} \left[\frac{bU}{aRe} \{(n^2 - 1) \times A_{21}\sin(n(ax + by) + A_{22}) - U\cos(ax + by)\} + \frac{1}{2}\{(n^2 - 1)A_{21} \times \cos(n(ax + by) + A_{22}) - nU \sin(ax + by)\}^2] - R_H \frac{H^2}{2} + A_{29} \tag{78}$$

where A_{29} is constant.

Case II

In this case, we have

$$\psi = B_{21}e^{m(ax+by)} + B_{22}e^{-m(ax+by)} - \frac{U\cos(ax+by)}{m^2+1} \tag{79}$$

where B_{21} and B_{22} are constants equation (69) utilizing (79) gives

$$\mu = \frac{B_{23}(ax+by)+B_{24}}{B_{21}e^{m(ax+by)}+B_{22}e^{-m(ax+by)}-\frac{U\cos(ax+by)}{m^2+1}} \tag{80}$$

where B_{23} and B_{24} are constants. The solution of equation (68) is

$$H = B_{25}(ax + by) + B_{26} \tag{81}$$

where B_{25} and B_{26} are constants. Equation (68), using equations (80) and (81) becomes.

$$T_{\xi\xi} + EcPr\Lambda^2(a^2 + b^2)(B_{23}\xi + B_{24})\Psi + \frac{R_H EcPrReH_{25}^2}{R_{\sigma}} = 0 \tag{82}$$

The solution of Equation (82) is

$$T = \frac{EcPr\Lambda^2(a^2+b^2)}{m^3(m^2+1)} \left[B_{23} \left\{ (m^2 + 1) \{ B_{21}e^{m(ax+by)}(2 - m(ax + by)) - B_{22}e^{-m(ax+by)} \times (m(ax + by) + 2) \} - m^3U \{ (ax + by) \cos(ax + by) - 2 \sin(ax + by) \} - B_{24}m \{ (m^2 + 1) (B_{21}e^{m(ax+by)} + B_{22}e^{-m(ax+by)}) + m^2U \cos(ax + by) \} \right\} - \frac{R_H EcPrReB_{25}^2}{2R_{\sigma}}(ax + by)^2 + B_{27}(ax + by) + B_{28} \right] \tag{83}$$

where B_{27} and B_{28} are constants. For this case stream function

$$\psi = B_{21}e^{m(ax+by)} + B_{22}e^{-m(ax+by)} + \frac{m^2U\cos(ax+by)}{m^2+1} \tag{84}$$

It represents a hyperbolic (sine or cosine) stream $B_{21}e^{m(ax+by)} + B_{22}e^{-m(ax+by)}$ in the +ve x-direction as well as a agitation is again sporadic in x and y. Pressure and velocity components, is given by

$$u = m b B_{21} e^{m(ax+by)} - m b B_{22} e^{-m(ax+by)} - \frac{m^2U b \sin(ax+by)}{m^2+1} \tag{85}$$

$$v = -m a B_{21} e^{m(ax+by)} + m a B_{22} e^{-m(ax+by)} + \frac{m^2U a \sin(ax+by)}{m^2+1} \tag{86}$$

$$p = \frac{K}{2(1+m^2)^2} [B_{21}e^{m(ax+by)}\{B_{21}(1 + m^2)^2e^{m(ax+by)} - 2mU\{\sin(ax + by) + m\cos(ax + by)\} + 2m^2U\{\cos(ax + by) - m\sin(ax + by)\}\} + 2B_{22}e^{-m(ax+by)}\{(1 + m^2)^2 + mU\{\sin(ax + by) - m\cos(ax + by)\} + m^2U\{\cos(ax + by) + m\sin(ax + by)\}\} + m^2U^2\sin^2(ax + by)] + \frac{m^2(a^2+b^2)}{1+m^2} \left[\frac{bU}{aR_H} \{(1 + m^2)\{e^{m(ax+by)} + e^{-m(ax+by)}\} - U\cos(ax + by)\} - \frac{1}{2(1+m^2)} \{(1 + m)\{e^{m(ax+by)} - e^{-m(ax+by)}\} - mU\sin(ax + by)\}^2] - R_H \frac{H^2}{2} + B_{29} \tag{87}$$

where B_{29} is constant.

Case III

Here, we have

$$\Psi = C_{21} + (ax + by)C_{22} - U \cos(ax + by) \tag{88}$$

$$\mu = \frac{C_{23}(ax+by)+C_{24}}{C_{21}+(ax+by)C_{22}-U \cos(ax+by)} \tag{89}$$

$$H = C_{25}(ax + by) + C_{26} \tag{90}$$

$$T = -\frac{EcPrA^2(a^2+b^2)}{12} [C_{23}\{(ax + by)^3(2C_{21} + (ax + by)C_{22}) + 12U \times \{(ax + by)\cos(ax + by) + 2\sin(ax + by)\}\} + 2C_{24}\{(ax + by)^2 \times (3C_{21} + (ax + by)C_{22} + 6U\cos(ax + by))\}] - \frac{R_H EcPrRe C_{25}^2}{2R\sigma} \times (ax + by)^2 + (ax + by)C_{27} + C_{28} \tag{91}$$

where $C_{21}, C_{22} \dots \dots C_{28}$ are constants,

$$\psi = C_{21} + (ax + by)C_{22} \tag{92}$$

The factors of velocity and pressure, are granted by

$$u = bC_{22}, v = -aC_{22} \tag{93}$$

$$p = C_{22}k \left(C_1(ax + by) + \frac{(ax+by)^2 C_{22}}{2} - U\sin(ax + by) \right) - \frac{(a^2 + b^2)}{2} C_{22}^2 - R_H \frac{H^2}{2} + C_{29} \tag{94}$$

Where, C_{29} is a constant.

4 Result and Discussion

The present study focuses on obtaining exact solutions for the heat transfer and steady plane flow equations. These solutions are obtained by solving the equations, and the study reveals a proportionality relationship between the vorticity distribution and the stream function. A proportionality relationship was found between the vorticity distribution and the stream function, which was perturbed by an oscillating stream of the form $\nabla^2\psi = K[\psi - U\sin(ax + by)]$ or $\nabla^2\psi = K[\psi - U\cos(ax + by)]$. To better understand the physical aspects of the obtained results, a number of graphs were created and included in this study. The focus of this work was to identify and quantify the differences between the velocity profiles $u(x, y)$ and $v(x, y)$ of Plane flow that is steady for Heat transfer occurs between incompressible fluids of variable viscosity in one direction on a plane. The vorticity distribution was found to be proportional to the stream function perturbed by a sine oscillation, and the velocity components $u(x, y)$ and $v(x, y)$ were analyzed for Case-I ($\Lambda = n^2$) and Case-II ($\Lambda = -m^2$) respectively. In this section, we show the graphs of the velocity profile $u(x, y)$ and $v(x, y)$ have been drawn against x and y for different values of emerging parameters of interest a, b, n , and U . Keeping things simple, all graphs are plotted by taking, $A_{11} = 1, A_{12} = 2, B_{11} = 0.0002, B_{12} = 0.0003, a = 2, b = 3, n = 2$, and $U = 1$ and using computational software Mathcad. Figure. 1 to Figure. 8 is for case-I and Figure. 9 to Figure. 16 are for case-II. All Figure.1(a) to Figure.16(a), for the velocity components $u(x, y)$ and $v(x, y)$ are considered as a function of “ x ” only and “ y ” treat as constant and it is “ $y = 1$ or 2”. All Figure.1(b) to Figure.16(b), In the case of velocity components $u(x, y)$ and $v(x, y)$ are treated as a function of “ y ” only and “ x ” assume that is constant and it is “ $x = 1$ or 2”.

Case-I: In Figures 1 and 2 depict the velocity components $u(x, y)$ and $v(x, y)$ at three different values of parameter “ a ”. As “ a ” increases, both components exhibit an increase in amplitude. Furthermore, the absolute values of $u(x, y)$ are larger than those of $v(x, y)$ in all graphs. Figures 3 and 4 illustrate the effect of parameter “ b ” on the two velocity components. The first component $u(x, y)$ increases with increasing values of “ b ” while the second component $v(x, y)$ remains unaffected by changes in “ b ”. The impact of parameter “ n ” on the fluid motion is shown in Figures 5 and 6, which reveal that even small variations in “ n ” can significantly affect the amplitude of both velocity components. Finally, Figures 7 and 8 demonstrate that the amplitude of both velocity components $u(x, y)$ and $v(x, y)$ is an increasing function of the parameter “ U ”. This suggests that changes in the value of “ U ” can substantially affect the motion of the fluid.

Case-II: In Figures 9 and 10 demonstrate the effect of parameter “ a ” on the fluid motion. The two velocity components exhibit opposite behavior as “ a ” changes: the velocity component $u(x, y)$ increases as “ a ” increases, while $v(x, y)$ decreases as “ a ” increases. Figures 11 and 12 illustrate the influence of parameter “ b ” on the fluid motion, which has a similar effect on the profiles of both velocity components as parameter “ a ”. Figures 13 and 14 depict the behaviour of parameter “ m ”, which qualitatively resembles the effects of parameters “ a ” and “ b ” on the fluid motion. Both velocity components are increasing functions of parameter “ n ”. Finally, Figures 15 and 16 provide the profiles of the velocity components $u(x, y)$ and $v(x, y)$ for different values of parameter “ U ”. The velocity component $u(x, y)$ increases as U increases, while $v(x, y)$ decreases as U increases. All figures in this study use SI units for the material constants, which provides a standard framework for comparing the results of this study with those of other studies and for making predictions about the behaviour of the fluid in real-world applications.

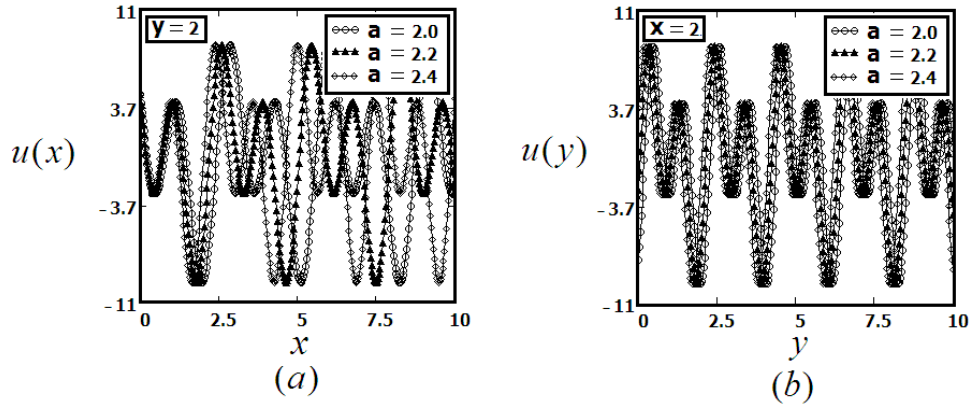


Figure 1. Profiles velocity field $u(x, y)$ for viscous fluid given by equation (38), for $A_{11} = 1, A_{12} = 2, b = 3, n = 2, U = 1$ and various values of a .

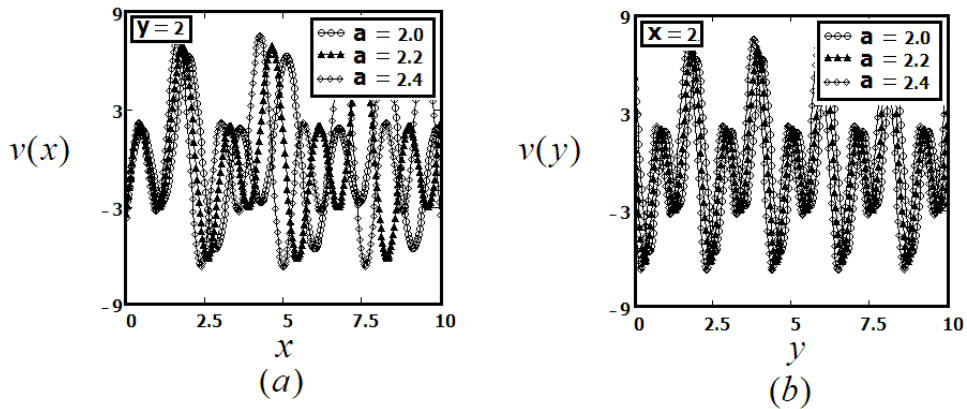


Figure 2. Profiles velocity field $v(x, y)$ for viscous fluid given by equation (39), for $A_{11} = 1, A_{12} = 2, b = 3, n = 2, U = 1$ and various values of a .

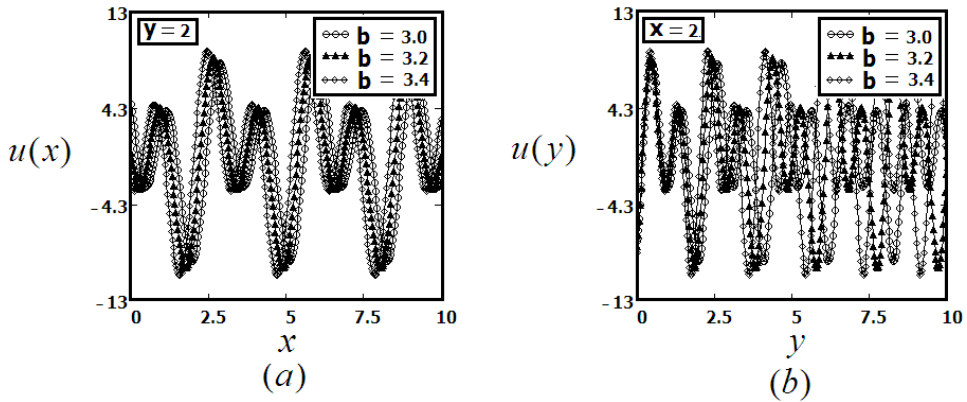


Figure 3. Profiles velocity field $u(x, y)$ for viscous fluid given by equation (38), for $A_{11} = 1, A_{12} = 2, a = 2, n = 2, U = 1$, and various values of b .

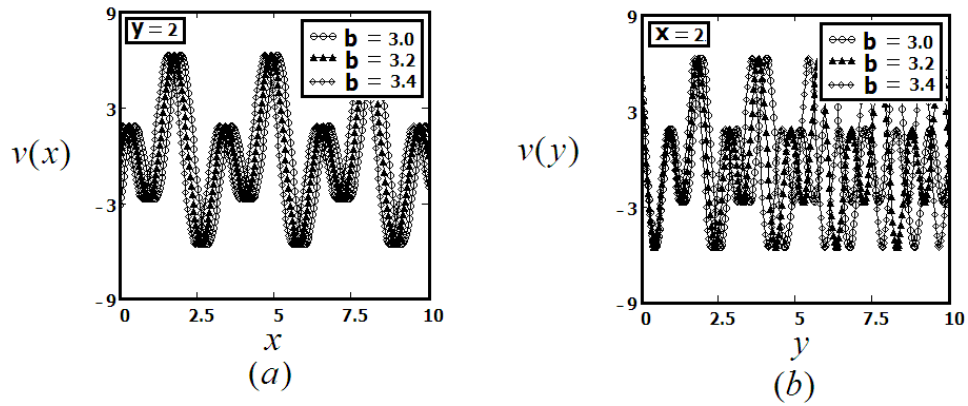


Figure 4. Profiles velocity field $v(x, y)$ for viscous fluid given by equation (39), for $A_{11} = 1, A_{12} = 2, a = 2, n = 2, U = 1$, and various values of b .

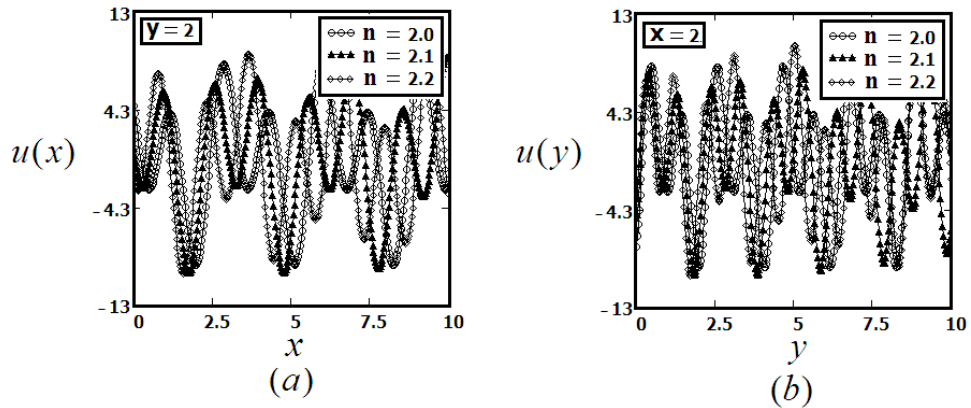


Figure 5. Profiles velocity field $u(x, y)$ for viscous fluid given by equation (38), for $A_{11} = 1, A_{12} = 2, a = 2, b = 3, U = 1$, and various values of n .

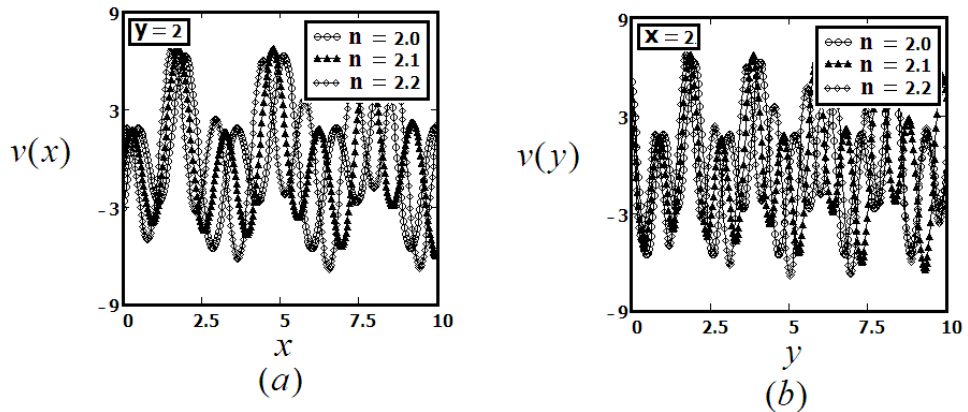


Figure 6. Profiles velocity field $v(x, y)$ for viscous fluid given by equation (39), for $A_{11} = 1, A_{12} = 2, a = 2, b = 3, U = 1$ and various values of n .

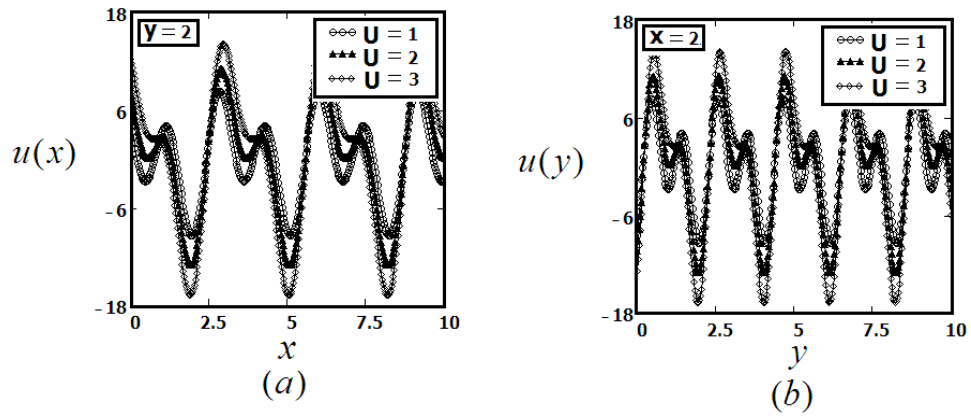


Figure 7. Profiles velocity field $U(x, y)$ for viscous fluid given by equation (38), for $A_{11} = 1, A_{12} = 2, a = 2, b = 3, n = 2,$ and various values of U .

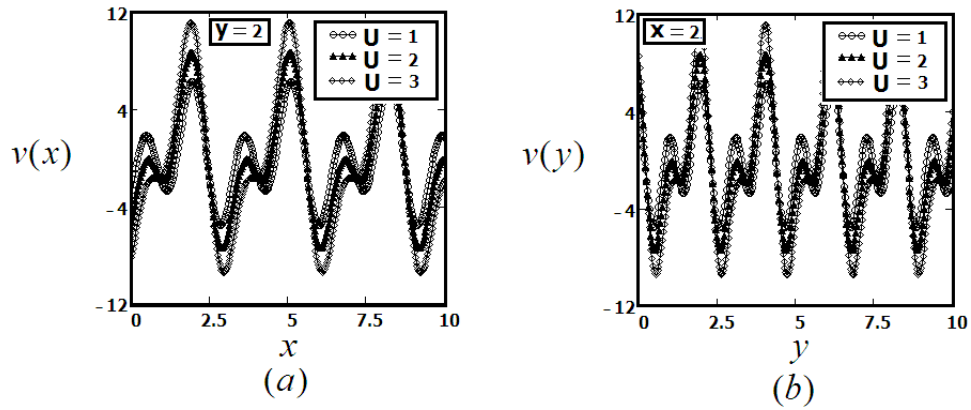


Figure 8. Profiles velocity field $v(x, y)$ for viscous fluid given by equation (39), for $A_{11} = 1, A_{12} = 2, a = 2, b = 3, n = 2$ and various values of U .

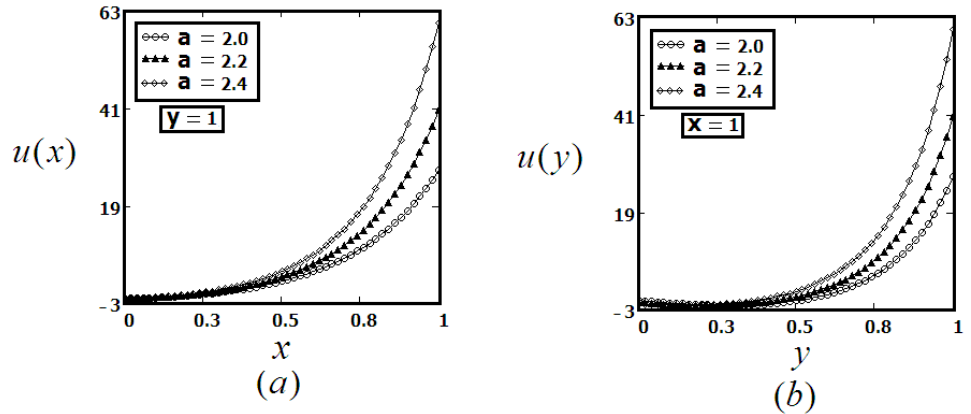


Figure 9. Profiles velocity field $U(x, y)$ for viscous fluid given by equation (47), for $B_{11} = 0.0002, B_{12} = 0.0003, b = 3, m = 2, U = 1$ and various values of a .

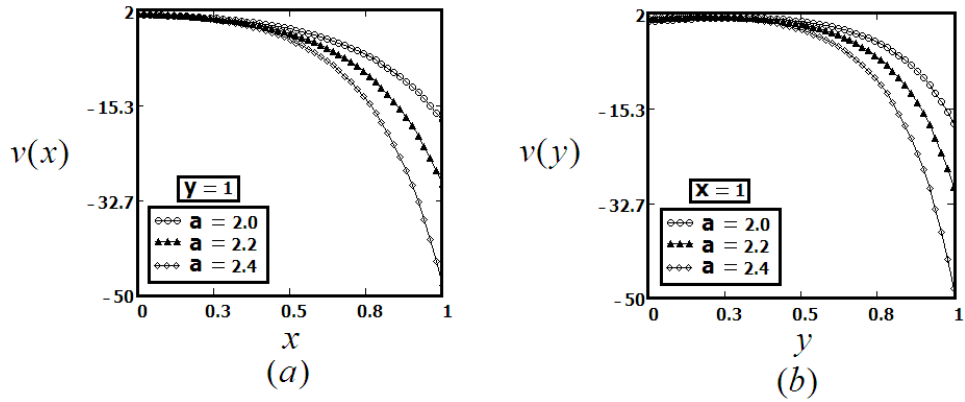


Figure 10. Profiles velocity field $v(x, y)$ for viscous fluid given by equation (48), for $B_{11} = 0.0002, B_{12} = 0.0003, b = 3, m = 2, U = 1$, and various values of a .

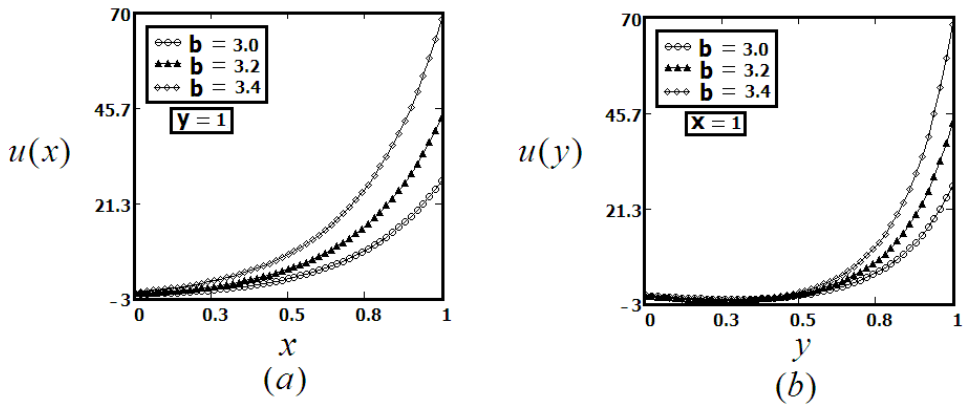


Figure 11. Profiles velocity field $U(x, y)$ for viscous fluid given by equation (47), for $B_{11} = 0.0002, B_{12} = 0.0003, a = 2, m = 2, U = 1$, and various values of b .

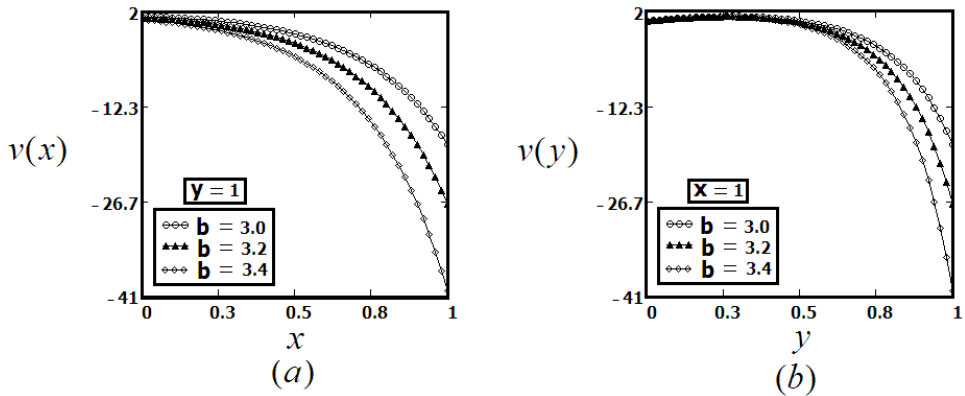


Figure 12. Profiles velocity field $v(x, y)$ for viscous fluid given by equation (48), for $B_{11} = 0.0002, B_{12} = 0.0003, a = 2, m = 2, U = 1$, and various values of b .

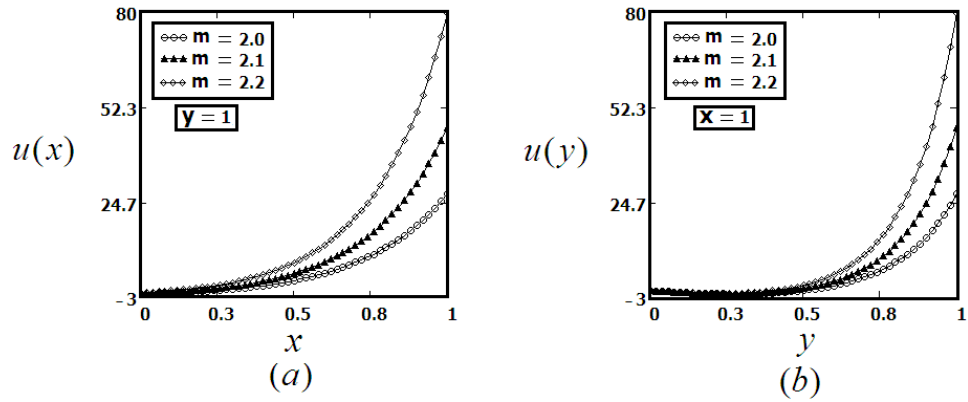


Figure 13. Profiles velocity field $U(x, y)$ for viscous fluid given by equation (47), for $B_{11} = 0.0002, B_{12} = 0.0003, a = 2, b = 3, U = 1$, and various values of m .

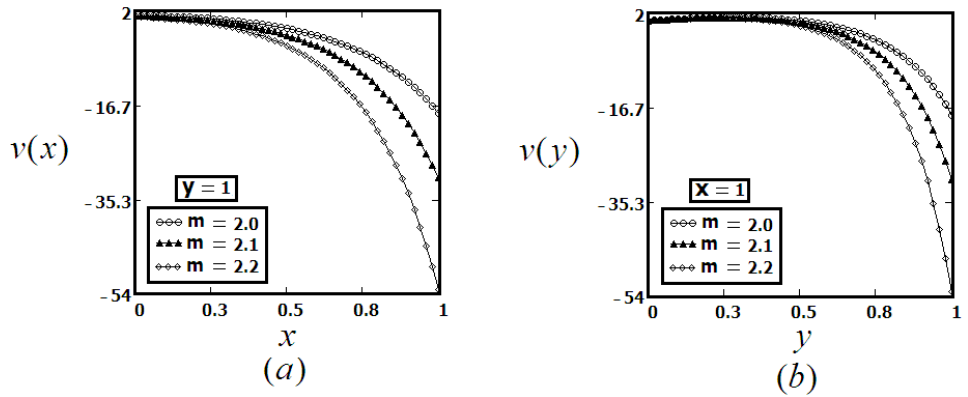


Figure 14. Profiles velocity field $v(x, y)$ for viscous fluid given by equation (48), for $B_{11} = 0.0002, B_{12} = 0.0003, a = 2, b = 3, U = 1$, and various values of m .

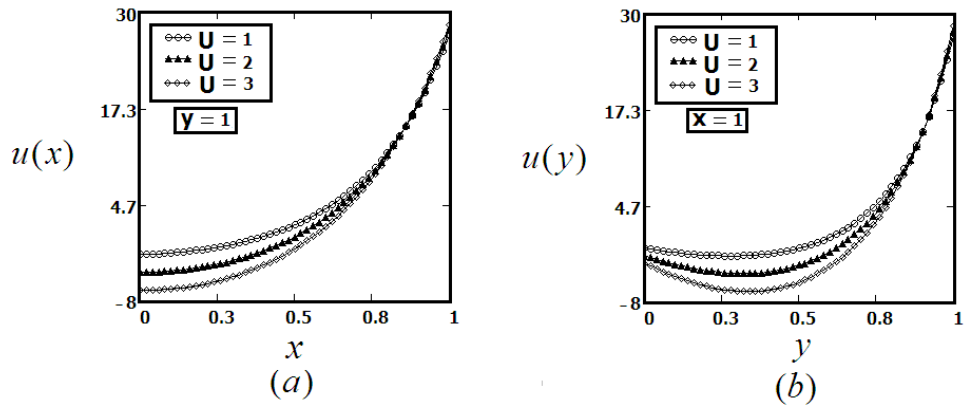


Figure 15. Profiles velocity field $U(x, y)$ for viscous fluid given by equation (47), for $B_{11} = 0.0002, B_{12} = 0.0003, a = 2, b = 3, m = 2$, and various values of U .

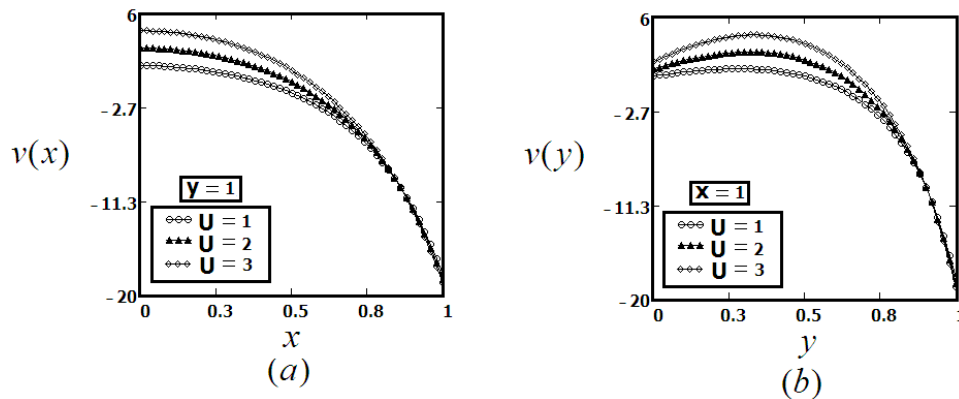


Figure 16. Profiles velocity field $v(x, y)$ for viscous fluid given by equation (48), for $B_{11} = 0.0002$, $B_{12} = 0.0003$, $a = 2$, $b = 3$, $m = 2$, and various values of U .

Overall, the graphs presented in this study provide valuable insights into the behaviour of fluid flow and highlight the importance of various parameters in determining the motion of the fluid. The findings of this study are expected to have important implications for future research in this area.

5 Conclusion

This research investigates the movement of an incompressible fluid over an infinite plane in the presence of a magnetic field and heat transfer, with varying viscosity and finite conductivity. Two general solutions are derived for the vorticity distribution proportional to a perturbed stream function, one with an oscillating stream in the form of $\nabla^2\psi = K[\psi - U\sin(ax + by)]$, and the other with an oscillating stream in the form of $\nabla^2\psi = K[\psi - U\cos(ax + by)]$. To obtain exact solutions, the fluid equations are first stated in terms of the stream function, vorticity function, and generalized energy function J . An equation that must be satisfied for the flows under consideration is determined by applying the integrability condition on the generalized energy function J and the perturbed stream function and viscosity. Solutions are obtained using a transformation variable that converts all governing equations into simple linear ordinary differential equations that can be solved. The exact solutions are obtained for variables u, v, ψ, μ, H, T , and p when represented in terms of elementary functions that fulfill all governing equations and compatibility requirements $J_{xy} = J_{yx}$ ($p_{xy} = p_{yx}$). The solutions in case-I represent a sine stream with an interrupted in x and y , while the solution in case-II generally symbolizes a hyperbolic (sine or cosine) stream with an intervallic perturbation in x and y . When $B_{11} = 0$ in the solution of case-II, the solution represents an exponential flow in the region $x > 0, y > 0$ perturbed by a part that swings as x and y increase. A discussion of the impact of numerous considerations on the velocity components $u(x, y)$ and $v(x, y)$ is also presented.

Credit Author Statement

Afaqe Ahmed Bhutto: Conceptualization, Methodology, Writing- Original draft preparation **Iftikhar Ahmed Bhutto:** Writing -Reviewing , **Saeed Ahmed Rajput:** Software, Investigation, Writing -Reviewing **Syed Asad Raza Shah:** Suggestions and ideas

Compliance with Ethical Standards:

It is said that there are no conflicts of interest among any of the authors. Additionally, it is stated that none of the authors of this paper conducted any experiments using human or animal subjects. Furthermore, each participant who took part in the study gave their free, informed consent.

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