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DELAYED SIR MODEL WITH LATENCY AND SATURATED INCIDENCE RATE

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Abstract. *In this paper, we derive and analyse a delayed SI model with saturated incidence rate and latent or infectious period τ. We prove local stability of the system's steady states in the absence and the presence of the time delay. We discover the the disease-free steady state is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$. While, the endemic steady state is always stable for any parameter values and for all $\tau \geq 0$.*

Keywords: Delayed model; Latency period; Stability Analysis, Disease-free steady state, endemic steady state.

1. **Introduction.** It is a well known fact that the spread of diseases involve disease-related factors which include; mode of transmission, infectious agent, infectious periods, incubation periods, resistance, and susceptibility [2]. Communicable disease models describing a directly transmitted viral or bacterial agent in a closed population and consisting of susceptibles (S), infectives (I), and recovered (R) were considered by Kermack and McKendrick (1927). In modeling disease transmission, it is usually convenient to subdivide the population being considered into compartments or classes of susceptible, infective and recovered individual populations, with sizes at time t denoted by $S(t)$, $I(t)$, $R(t)$, respectively. Each of the classes comprises of cohort of individuals and each of these cohorts of individuals are assumed to have the same characteristic features. In many diseases (for instance, influenza, tuberculosis, measles e.t.c), on adequate interaction or contact with infective individuals, the susceptible individuals become infected but not yet infectious. These individuals remain in this status for a certain infectious/latent period before finally becoming infectious [5], [12].

In this paper, we derive a model which includes a saturated incidence rate with latency (or infectious) period of the infectives and susceptibles. We prove local stability of the disease-free and endemic steady states. Now, consider the following model describing the interaction between susceptible and infected individuals with latent period τ as follows.

$$\begin{aligned} \frac{dS(t)}{dt} &= b - \gamma S(t) - \frac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)} + rI(t) \\ \frac{dI(t)}{dt} &= \frac{\beta e^{-\gamma\tau} S(t-\tau)I(t-\tau)}{1+\alpha_1 S(t-\tau)+\alpha_2 I(t-\tau)} - (\gamma + r + \alpha)I(t) \end{aligned} \tag{1}$$

where β is the transmission rate, b is the natural birth rate, γ is the natural death rate, α is the diseases-induced death rate of infected individuals, α_1 and α_2 are the parameter that measure the inhibitory effect,r is the treatment rate of the infected individuals, τ is the latent period, S is the density of susceptible individuals, I is the density of infected individuals.

2. **Local Stability Analysis.**
This section presents the local stability analysis of the system steady state solution for the system (1). The system (1) has a diseases-free steady state $E^0 = (S^0, I^0) = (b/\gamma, 0)$ and a unique positive endemic steady state $E^* = (S^*, I^*)$ when $\mathcal{R}_0 > 1$, where

$$\mathcal{R}_0 = \frac{b\beta e^{-\gamma\tau}}{(\alpha b+\gamma)(\gamma+\alpha+r)} \tag{2}$$

$$\begin{aligned} S^* &= \frac{b[(\gamma+\alpha+r)+\alpha_2 b e^{-\gamma\tau}]}{(\gamma+\alpha+r)[\alpha_1 b(\mathcal{R}_0-1)+\gamma\mathcal{R}_0]+\alpha_2 b e^{-\gamma\tau}} \\ I^* &= \frac{b(\mathcal{R}_0-1)e^{-\gamma\tau}(\alpha_1 b+\gamma)}{(\gamma+\alpha+r)[\alpha_1 b(\mathcal{R}_0-1)+\gamma\mathcal{R}_0]+\alpha_2 b e^{-\gamma\tau}} \end{aligned} \tag{3}$$

2.1. **Stability of the diseases-free steady state.**
In this subsection, we analyse the diseases-free steady state solution E^0 for the system (1). The linearization of the syetem (1) about the steady state has the following characteristic quasi-polynomial equation

$$(\lambda+\gamma)\left[\lambda+(\gamma+\alpha+r)-\frac{\beta b e^{-\gamma\tau}}{\gamma+\alpha_1 b}e^{-\lambda\tau}\right]=0. \tag{4}$$

Theorem 2.1.
The diseases-free steady state E^0 is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.

Proof.
For $\tau = 0$, the characteristic equation (4) reduces to thus:

$$(i\xi+\gamma)\left[i\xi+(\gamma+\alpha+r)-\frac{\beta b e^{-\gamma\tau}}{\gamma+\alpha_1 b}(\cos(\xi\tau)-i\sin(\xi\tau))\right]=0 \tag{5}$$

We can clearly see that, the characteristic equation (5), has two eigenvalues with negative real parts if $\mathcal{R}_0 < 1$ and eigenvalue with positive real parts if $\mathcal{R}_0 > 1$. The roots are $\lambda_1 = -\gamma$ and $\lambda_2 = (\gamma+\alpha+r)(\mathcal{R}_0 - 1)$. Therefore, the disease-free steady state E^0 is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$. Hence, if there is no lantency then the system has no purely imaginary roots and disease-free steady state is always stable for any parameter values for $\mathcal{R}_0 < 1$. Now, to have instability, we require one of the roots to cross the imaginary axis or from the left half plane to the right half plane as the latent period τ varies.
Now, if $\tau \neq 0$, let $\lambda = i\xi$, $(\xi > 0)$, then substituting into characteristic equation (4) we have the following

$$(i\xi+\gamma)\left[i\xi+(\gamma+\alpha+r)-\frac{\beta b e^{-\gamma\tau}}{\gamma+\alpha_1 b}(\cos(\xi\tau)-i\sin(\xi\tau))\right]=0 \tag{6}$$

expanding and seperating into real and imaginary parts, we have the following

$$\begin{aligned} (\gamma+\alpha+r)-\tfrac{\beta b e^{-\gamma\tau}}{\gamma+\alpha_1 b}\cos(\xi\tau) &= 0 \\ \xi+\tfrac{\beta b e^{-\gamma\tau}}{\gamma+\alpha_1 b}\sin(\xi\tau) &= 0. \end{aligned} \tag{7}$$

Squaring and adding both sides of (7) we have thus:

$$\xi^2-(\gamma+\alpha+r)(\mathcal{R}_0-1)\left[(\gamma+\alpha+r)+\frac{\beta b e^{-\gamma\tau}}{\gamma+\alpha_1 b}\right]=0. \tag{8}$$

Now, we know that if $\mathcal{R}_0 < 1$ then equation (8) has roots with negative real parts and therefore, the disease-free steady state is locally asymptotically stable for all $\tau \geq 0$ and if $\mathcal{R}_0 > 1$ then E^0 is unstable for all $\tau \geq 0$. Hence the proof. □

2.2. **Stability of the positive endemic steady state.**
This subsection presents the local stability analysis of the endemic steady state of the system (1). To begin with, we linearise the system about the positive steady state E^* and make the following substitution. Let $S = u_1 + S^*$ and $I = u_2 + I^*$, then system (1) becomes thus

$$\begin{aligned}\frac{du_1}{dt} &= \left(-\gamma - \tfrac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2}\right) u_1(t) - \frac{\beta S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2} u_2(t)\\ \frac{du_2}{dt} &= \tfrac{\beta I^*(1+\alpha_2 I^*)e^{-\gamma\tau}}{(1+\alpha_1 S^*+\alpha_2 I^*)^2} u_1(t-\tau) + \frac{\beta S^*(1+\alpha_1 S^*)e^{-\gamma\tau}}{(1+\alpha_1 S^*+\alpha_2 I^*)^2} u_2(t-\tau).\end{aligned} \tag{9}$$

The characteristic equation at the endemic state E^* for the above system is thus

$$\lambda^2 + \lambda(p_0 + p_1 e^{-\lambda\tau}) + q_0 + q_1 e^{-\lambda\tau} = 0. \tag{10}$$

where, we have
$p_0 = (2\gamma + \alpha + r) + \frac{\beta I^*(1+\alpha_1 I^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2}$,
$p_1 = -\frac{\beta S^*(1+\alpha_1 S^*)e^{-\gamma\tau}}{(1+\alpha_1 S^*+\alpha_2 I^*)^2}$,
$q_0 = \left[\gamma + \frac{\beta I^*(1+\alpha_1 I^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2}\right](\gamma + \alpha + r)$,
$q_1 = -\frac{\gamma\beta S^*(1+\alpha_1 S^*)e^{-\gamma\tau}}{(1+\alpha_1 S^*+\alpha_2 I^*)^2}$.
If $\tau = 0$, the characteristic equation (10) becomes

$$\lambda^2 + \lambda(p_0 + p_1) + q_0 + q_1 = 0. \tag{11}$$

Since $\mathcal{R}_0 > 1$, we have

$$p_0 + p_1 = \frac{(\gamma + \alpha + r)^2(\gamma + \alpha_1 b)(\mathcal{R}_0 - 1)}{\beta b[(\gamma + \alpha + r) + \alpha_2 b]}[\alpha_2 b + \alpha_1 b(\mathcal{R}_0 - 1) + \gamma\mathcal{R}_0] > 0$$

and

$$q_0 + q_1 = \frac{(\gamma + \alpha + r)^2(\gamma + \alpha_1 b)(\mathcal{R}_0 - 1)}{\beta b[(\gamma + \alpha + r) + \alpha_2 b]}[\alpha_2\gamma b + (\gamma + \alpha + r)(\alpha_1 b(\mathcal{R}_0 - 1) + \gamma\mathcal{R}_0)] > 0$$

Thus, since $(p_0 + p_1) > 0$ and $(q_0 + q_1) > 0$, this implies that the characteristic equation (11) has roots with negative real parts. Therefore, by Rough-Hurwitz criteria the endemic steady state is locally asymptotically stable for $\tau = 0$.
Now, if $\tau \neq 0$, we check if any root of the characteristic equation (10) crosses the imaginary axis and therefore becomes positive as τ varies. The main result of this paper is contained in the following theorem which we shall prove. Before the theorem, we state the following condition which we shall also prove.

$$\begin{aligned}&p_0^2 - p_1^2 - 2q_0 > 0, \text{ and } q_0^2 - q_1^2 > 0,\\ &\text{or } [p_0^2 - p_1^2 - 2q_0]^2 < 4[q_0^2 - q_1^2].\end{aligned} \tag{A}$$

Theorem 2.2.
If (A) holds, then all roots of equation (10) have negative real parts for all $\tau \geq 0$. Furthermore, the endemic steady state E^* of the system (1) is locally asymptotically stable for all $\tau \geq 0$.

Proof.
We want to check if the real parts of some roots increase to reach zero and eventually become non-negative as τ varies. Let $\lambda = i\xi(\xi > 0)$ be a purely imaginary root of the characteristic equation (10). It suffices to seek solutions with $\xi > 0$, since $\lambda = 0$ is not a root and since complex roots occur in conjugate, then ξ satisfies the following equation:

$$-\xi^2 + i\xi p_0 + i\xi p_1(\cos(\xi\tau) - i\sin(\xi\tau)) + q_0 + q_1(\cos(\xi\tau) - i\sin(\xi\tau)) = 0. \tag{12}$$

Seperating into the real and imaginary parts we have thus

$$q_0 - \xi^2 = p_1\xi\sin(\xi\tau) - q_1\cos(\xi\tau),$$
$$p_0\xi = q_1\sin(\xi\tau) - p_1\xi\cos(\xi\tau). \tag{13}$$

Squaring and summing up both sides of equations (13) we have the following

$$\xi^4 + [p_0^2 - p_1^2 - 2q_0]\xi^2 + q_0^2 - q_1^2 = 0. \tag{14}$$

If we let $z = \xi^2$ then we have thus:

$$z^2 + [p_0^2 - p_1^2 - 2q_0]z + q_0^2 - q_1^2 = 0. \tag{15}$$

By condition (A), $p_0^2 - p_1^2 - 2q_0 > 0$ and $q_0^2 - q_1^2 > 0$. This means if z_1 and z_2 are roots of (15) then $z_1 + z_2 = p_0^2 - p_1^2 - 2q_0 > 0$ and $z_1 z_2 = q_0^2 - q_1^2 > 0$. This implies that equation (15) has no positive roots, and therefore, the characteristic equation (10) has no purely imaginary roots. Hence, all roots of (10) have negative real parts.

Now, using the system parameters, we can explicitly show that condition (A) holds as follows: we can see that $q_0 - q_1 > 0$, this implies $q_0^2 - q_1^2 > 0$ since $\mathcal{R}_0 > 1$. So, we only need to show that $p_0^2 - p_1^2 - 2q_0 > 0$.

$$p_0^2-p_1^2-2q_0 = \frac{\alpha_2(\gamma+\alpha_1 b)^2(\mathcal{R}_0-1)}{\beta[(\gamma+\alpha+r)+\alpha_2 b]e^{-\gamma\tau}}\left[(\gamma+\alpha+r)+\frac{\beta S^*(1+\alpha_1 S^*)e^{-\gamma\tau}}{(1+\alpha_1 S^*+\alpha_1 I^*)^2}\right]+\left[\gamma+\frac{\beta I^*(1+\alpha_2 I^*)e^{-\gamma\tau}}{(1+\alpha_1 S^*+\alpha_1 I^*)^2}\right]^2 > 0 \tag{16}$$

since $\mathcal{R}_0 > 1$.

□

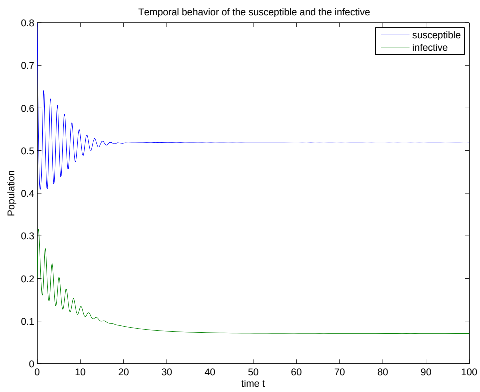

FIGURE 1. Stable solution profiles for $\tau = 2$.

3. **Conclusion.** In this paper, we have showed that if condition (A) holds then none of ξ_+^2 and ξ_-^2 is positive, that is equation (15) does not have positive roots. Therefore, the characteristic equation (10) does not have purely imaginary roots and hence, the endemic steady state E^* is locally asymptotically stable if $\mathcal{R}_0 > 1$ for all $\tau \geq 0$. We conclude that for any values of the latency the system is always stable.

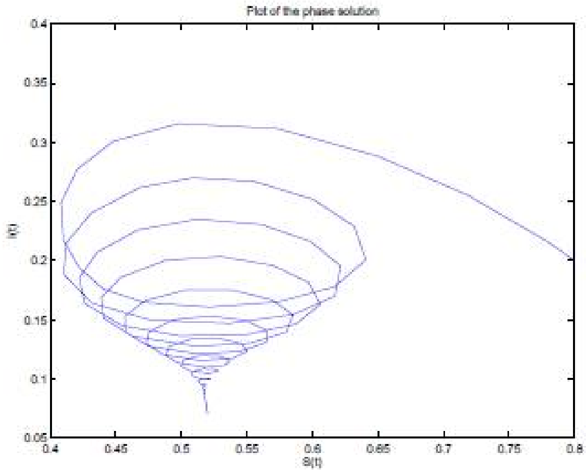

FIGURE 2. Phase solution for $\tau = 2$.

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