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Exact solution on the impact of slip condition for unsteady tank drainage flow of Ellis fluid

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Abstract In this paper, we look into the effect of slip condition on isothermal and incompressible Ellis fluid of an unsteady tank drainage flow. The non-linear PDE (partial differential equation) is solved exactly by applying the governing continuity and momentum equations, subject to the proper boundary condition, using the separation of variables approach. Unique situations this model put out by Ellis fluid is used to develop concepts like Newtonian, Power law model, and Bingham Plastic model solution. On setting the slip parameter $\beta = 0$exact solution for Ellis fluid flow is retrivred as well as Newtonian solution is bring back, which was done through Bernoulli's equation. Expressions for velocity field, pipe shear stress, volume flux, velocity average, depth of fluid in the tank at different times and also the relationship between length of the time be different with depth of the tank and the length of time required to complete the drainage is determined. Graphical representation is given of the effects of various development factors on the velocity field V_z and fluid depth $H(t)$. The tank can empty faster for Ellis fluid compared to its special situations, according to the analogy of Ellis, Power law, Newtonian, and Binghan plastic fluids for the relation of depth with respect to time.

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1 Introduction

Non-Newtonian fluids have been the subject of current research due to their wide-ranging biological, industrial, and technological uses, including in clay coatings, polymer melts, polymer melts, blood, drilling mud, paints, greases, dental paste etc. Contrary to Newtonian fluids, which include the Navier-Stokes equation, non-Newtonian fluids are a varied family of fluids and lack a single model that can account for all of their features [6]. Numerous constitutive equations, such as the Power law fluid model, the second order fluid model, the third order fluid model, the Sisco fluid model, the eyring fluid model, and the Phan-Thien-Tanner fluid model, have been proposed to predict the physical structure and behavior of various types of non-Newtonian fluids. [10][17][20]. There are some problems and causes whose precise solutions are deemed essential. [24].

Because of the numerous applications in the engineering and industrial sectors in the current era, studies of the slip flow of an Ellis fluid have inspired a lot of researchers. Numerous applications, including as small scale channels or Nano directs and in usage anywhere a thin coating of "light oils" is connected toward the moving-plates while at the same time the surface is covered with distinctive covering, exhibit fluid amazing slippage put at stable bounds [18]. Navier investigated a limit state of liquid slip at strong surface such that $V_z = -\beta S_{rz}|_{r=R}$ where β is the slip coefficient, Srz is a component of the extra stress tensor, and V_z is the velocity along the z-axis. This limit state was studied in the context of the historical flow of liquid through channels. In the case when $\beta = 0$ appears, there is no slide at the limit.

Tanks and vessels are frequently utilized to deliver a range of engineering services. It is necessary to measure the flow or drainage when filling or emptying a tank. It's a long-standing yet nonetheless challenging issue when fluid drains down a tank's pipe due to gravity. The tank can either be emptied through an attached pipe or only a hole in the "orifice scenario." The pipe could have a full directing system with vertical drop, even enlargement with valve and fittings, etc., or it could be flat or vertical. The bottom of a tank might be flat, conical, hemispherical, or any other shape; nevertheless, tanks often have a cylindrical shape with a vertical wall. Many different industries employ the classification of gravity drainage fluids; a few examples include: drainage condensate into storage, water distribution, waste water management, and dams. For their analysis, there have been numerous study endeavours. [19] and [4] have used the Newtonian fluid for tank drainage flow and power law fluid, respectively. Additionally, accurate solutions for tank drainage for Newtonian fluid under slip conditions, tank drainage for electrically conducting fluid using the Bessel Function, and analytical tank drainage for electrically conducting power law fluid are provided in [18][20] and [22]. Passive Rotor in [3][11] drains a water tank using the Runge-Kutta method and another approach with a pipe output. The theory outlining a tank's efflux time has been developed by [7][4] and subsequently extended to systems with installed fittings by [14]. Additionally, an exact solution for tank drainage through a circular pipe for a pair stress fluid has also been developed by [15]. It is a well-known fact that when a tank is drained through a hole, Torricelli's equation is used to describe the discharge velocity and flow rate [8]. In addition, slow drainage for a large tank under gravity is briefly explained in [23], and unsteady drainage flows from a rectangular tank are given by [12], along with three-dimensional drainage for circular tanks.
The exact solution for the Ellis fluid tank drainage problem, as well as unstable fluid drainage from circular tanks, were studied in this publication. The ensuing non-linear partial differential equations with exact solutions [13][16] are obtained analytically. We have also recovered the exceptional instances. The expressions for velocity profile, pipe shear stress, flow rate, average velocity, fluid depth in the tank, and drainage time are then determined.

2 Basic Equations

Without taking into account thermal effects, the basic governing equations for an incompressible Ellis fluid are:

$$0 = \nabla \cdot \mathbf{V}, \tag{1}$$

$$\rho \mathbf{f} + \nabla \cdot \mathbf{T} - \nabla p = \frac{\rho D\mathbf{V}}{Dt}, \tag{2}$$

The letters ρ stand for density which is constant, $\mathbf{f}$ for body fore, $\mathbf{T}$ is the extra stress tensor, p be dynamic pressure, $\mathbf{V}$ denotes velocity vector and the material derivate is indicated by the operator $\frac{D}{Dt}$. For Ellis model, the extra stress tensor is describe by [1][2][5][7][16],

$$\mathbf{T} = \eta \mathbf{A}_1, \tag{3}$$

where the parameter η is defined as:

$$\frac{1}{\eta} = \frac{1}{\eta_0}\left[1 + \left|\sqrt{\frac{\frac{1}{2}(\mathbf{T}:\mathbf{T})}{T_{\frac{1}{2}}}}\right|^{-1+\alpha}\right], \tag{4}$$

$$\mathbf{A}_1 = (\nabla \mathbf{V})^T + \nabla \mathbf{V}. \tag{5}$$

Here in euation (4), the symbol η be the scalar quantity and holds three constants $\eta_0, T_{\frac{1}{2}}, \alpha$, that can be determined experimentally for each fluid, in equation (5) $\mathbf{A}_1$ represent for first Rivilin-Erickson tensor and $\mathbf{T}:\mathbf{T}$ can be characterized as:

$$\mathbf{T}:\mathbf{T} = trace(T^2) \tag{6}$$

The Ellis fluid model requires a specific case since it governs the Newtonian fluid on substitution of low shear stress $\left(T_{\frac{1}{2}} \to \infty\right)$, whereas the proposed model becomes the Power law model at high shear rates.

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3 Problem Statement

With a conduit of diameter d connecting to the middle of the tank's bottom, imagine a cylindrical storage tank filled with incompressible, isothermal, Unsteady Ellis fluid. Assume the fluid is initially H_0 deep. A tube of length L is used to drain the liquid from the tank. The pipe radius is R, and the radius of the tank is R_T. Additionally, the liquid flow in the pipe is caused by the hydrostatic pressure and gravity of the liquid in the tank, assuming that the liquid depth in the tank at any given moment t is *H(t)* in the tank.

Our goal is to calculate the volumetric flow rate, average fluid velocity, velocity profile, tube shear stress, link between discharge time and fluid depth in cylindrical tank, and time needed for full discharge using slip condition on circular pipe connected to tank. In this case, we work with cylindrical coordinates *(r, θ, z)*, where the *r* axis is perpendicular to the tube and the *z* axis runs vertically through the middle of the tube. Since the velocity vector V , r and θ components are zero and the z-direction flow is independent, accordingly

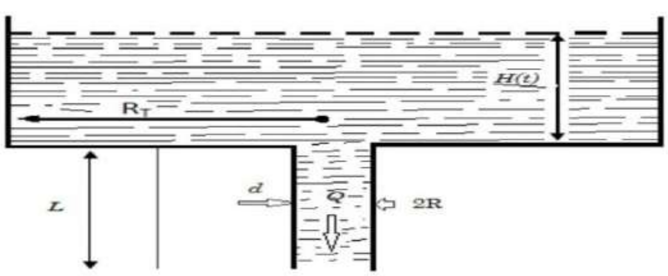

Figure 1. drainage of fluid from circular tank down through a circular pipe [19].

$$\mathbf{V} = \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ v_z(r,t) \end{bmatrix}, \qquad \mathbf{T} = T(r,t) \tag{7}$$

Since we neglected the convected part of the acceleration through equation (7), the continuity equation (1) is satisfied in exactly the same way, and the momentum equation (2) is reduced to

$$\frac{1}{r}\left(\frac{\partial\left(rT_{rr}\right)}{\partial r}-T_{\theta\theta}\right)=\frac{\partial p}{\partial r}, \tag{8}$$

$$\frac{\partial\left(r^{2}T_{r\theta}\right)}{\partial r}=r\frac{\partial p}{\partial\theta}, \tag{9}$$

$$r^{-1}\frac{\partial\left(rT_{rz}\right)}{\partial r}-\frac{\partial p}{\partial z}+\rho g=\rho\frac{\partial v_{z}}{\partial t} \tag{10}$$

Slow draining causes the almost constant velocity retention in the pipe flow with time t, making it possible to ignore the time derivative $\frac{\partial v_z}{\partial t}$ and use the pressures at the pipe entrance and exit from Eq (10). For this combined flow, the momentum equation is:

$$\frac{\partial(rT_{rz})}{\partial r}=-r\rho g\left(\frac{H(t)}{L}+1\right). \tag{11}$$

At the drain pipe, the related boundary conditions are

$$T_{rz}=0 \quad \text{at} \quad r=0, \qquad \text{(At center of the pipe)} \tag{12}$$

$$v_{z}=-\beta T_{rz}\big|_{r=R} \quad \text{at} \quad r=R \qquad \text{(slip condition)} \tag{13}$$

Since the fluid's velocity is maximised in the middle of the pipe, "shear stresses" there are zero according to the (centerline) boundary condition in Eq. (12). By integrating equation (11) with respect to r while maintaining a fixed value for *t*, we obtain

$$T_{rz}=f_{1}(t)-\frac{\rho g r}{2L}\left(L+H(t)\right). \tag{14}$$

Using the centerline boundary condition in Eq. (12), where $f_1(t)$ is any function of *t*, we obtain

$$T_{rz}=-\frac{\rho g r}{2L}\left(L+H(t)\right). \tag{15}$$

The arrangement of equations (3) and (4) is now achieved by incorporating elements of the stress tensor equation (15) into the constitutive equations of Ellis fluid.

$$\left[\left|\frac{T_{rz}}{T_{\frac{1}{2}}}\right|^{-1+\alpha}+1\right]\frac{T_{rz}}{\eta_{0}}=\frac{\partial v_{z}}{\partial r} \tag{16}$$

Due to the drainage flow issue, we view $||T_{rz}|$as positive; as a result, by relaxing to the absolute requirement and applying Eq. (15) to Eq. (16), we obtain

$$\frac{\partial v_z}{\partial r} = -\frac{r\rho g}{2\eta_0 L}\left(L + H(t)\right) - \frac{T_{\frac{1}{2}}^{1-\alpha}}{\eta_0}\left\{\frac{\rho g r}{2L}\left(L + H(t)\right)\right\}^{\alpha} \tag{17}$$

By holding *t* constant while integrating Eq. (17) with regard to *r*, we arrive at:

$$v_z = f_2(t) - \frac{\rho g r^2}{4L\eta_0}\{L + H(t)\} - \frac{T_{\frac{1}{2}}^{1-\alpha} r^{\alpha+1}}{\eta_0(\alpha+1)}\left\{\frac{\rho g}{2L}\left(L + H(t)\right)\right\}^{\alpha}. \tag{18}$$

here $f_2(t)$ function of time *t* ,which after utilizing equation (13) in (18), we have

$$f_2(t) = \frac{\beta\rho g R}{2L}\left(H(t) + L\right) + \frac{\rho g R^2}{4L\eta_0}\{L + H(t)\} + \frac{T_{\frac{1}{2}}^{1-\alpha} R^{\alpha+1}}{\eta_0(\alpha+1)}\left\{\frac{\rho g}{2L}\left(L + H(t)\right)\right\}^{\alpha} \tag{19}$$

Hence Eq. (18) reduces to

$$v_z = \frac{\beta\rho g R}{2L}\left(H(t) + L\right) + \frac{\rho g}{4\eta_0 L}\left(R^2 - r^2\right)\left(L + H(t)\right) + \frac{T_{\frac{1}{2}}^{1-\alpha}}{\eta_0(1+\alpha)}\left\{\frac{\rho g}{2L}\left(L + H(t)\right)\right\}^{\alpha}\left(R^{\alpha+1} - r^{\alpha+1}\right) \tag{20}$$

The "flow rate Q" per unit width is specified through the formula

$$\boldsymbol{Q = \int_0^{2\pi}\int_0^{R} r(v_z(r,t))\,dr\,d\theta} \tag{21}$$

Using velocity profile Eq. (20) in Eq. (21), the rate of flow will be

$$Q = \frac{\beta\pi\rho g R^3}{2L}\left(L + H(t)\right) + \frac{\rho g\pi R^4}{8L\eta_0}\left(L + H(t)\right) + \frac{T_{\frac{1}{2}}^{1-\alpha}\pi R^{\alpha+3}}{\eta_0(\alpha+3)}\left\{\frac{\rho g(L + H(t))}{2L}\right\}^{\alpha}. \tag{22}$$

Using the following formula, we can get the average velocity, $\bar{V}$. (23)

$$\bar{V} = \frac{Q}{\pi R^2} \tag{23}$$

In order to express the average velocity, Eq. (22) and Eq. (23), respectively, can be used.

$$\bar{V} = \frac{\beta\rho g R}{2L}\left(L + H(t)\right) + \frac{\rho g R^2}{8\eta_0 L}\left(L + H(t)\right) + \frac{T_{\frac{1}{2}}^{1-\alpha} R^{\alpha+1}}{\eta_0(\alpha+3)}\left\{\frac{(L + H(t))\rho g}{2L}\right\}^{\alpha}. \tag{24}$$

At the circular pipe, shear stress is

$$\frac{-\rho g\left(L + H(t)\right)R}{2L} = T_{rz} \quad at \quad r = R \tag{25}$$

This yields the same outcomes as earlier research for both the Power-law and Newtonian fluid models. For mass balance throughout the entire tank, modify the differential equation as follows by substituting the flow rate from equation (22) into equation (26).

$$\frac{d}{dt}[\pi R_T^2 H(t)] = -Q(t) \tag{26}$$

$$\frac{dH(t)}{\frac{-\beta\rho g R^3}{2LR_T^2}\left(H(t)+L\right)-\frac{\rho R^4 g}{8R_T^2\eta_0 L}\left(L+H(t)\right)-\frac{T_{\frac{1}{2}}^{1-\alpha}R^{\alpha+3}}{\eta_0 R_T^2(\alpha+3)}\left\{\frac{\rho g(L+H(t))}{2L}\right\}^{\alpha}}=dt. \tag{27}$$

Integrate on both sides to Eq. (27) separatly, we get:

$$\frac{-8LR_T^2\eta_0}{\rho g R^3(4\beta\eta_0+R)(1-\alpha)}\ln\left(-\frac{\rho g R^3(4\beta\eta_0+R)}{8LR_T^2\eta_0}\left(L+H(t)\right)^{1-\alpha}-\frac{R^{3+\alpha}T_{\frac{1}{2}}^{1-\alpha}}{(3+\alpha)\eta_0 R_T^2}\left(\frac{\rho g}{2L}\right)^{\alpha}\right)=C_0+t. \tag{28}$$

Constant of intigration will be after using initial depth of the tank drainage condition, at time $t = 0$ *depth will be* $H(t) = H_0$, we get,

$$-\frac{8\eta_0 R_T^2 L}{\rho g R^3(4\beta\eta_0+R)(1-\alpha)}\ln\left(-\frac{\rho g R^3(4\beta\eta_0+R)}{8LR_T^2\eta_0}\left(H_0+L\right)^{1-\alpha}-\frac{T_{\frac{1}{2}}^{1-\alpha}R^{\alpha+3}\left(\frac{\rho g}{2L}\right)^{\alpha}}{\eta_0(\alpha+3)R_T^2}\right)=C_0. \tag{29}$$

After Eq. (29) is substituted for Eq. (28), the fluid level in the tank then progressively declines in accordance with

$$H(t)=\left[-\frac{8LR_T^2\eta_0}{\rho g R^3(4\beta\eta_0+R)}e^{\frac{-\rho g R^3(4\beta\eta_0+R)(1-\alpha)t}{8LR_T^2\eta_0}}\left\{\begin{array}{l}-\frac{\rho g R^3(4\beta\eta_0+R)}{8LR_T^2\eta_0}\left(H_0+L\right)^{1-\alpha}\\-\frac{T_{\frac{1}{2}}^{1-\alpha}R^{\alpha+3}\left(\frac{\rho g}{2L}\right)^{\alpha}}{\eta_0(\alpha+3)R_T^2}\end{array}\right\}-\frac{4T_{\frac{1}{2}}^{1-\alpha}R^{\alpha}\left(\frac{\rho g}{2L}\right)^{\alpha-1}}{(4\beta\eta_0+R)(\alpha+3)}\right]^{\frac{1}{1-\alpha}}-L. \tag{30}$$

After applying naturat logirtihm on both sides of euation. (30), we can obtain the association between time t and the level of the fluid in the circular tank after rearranging the equation.

$$\frac{-8LR_T^2\eta_0}{\rho g R^3(4\beta\eta_0+R)(1-\alpha)}\ln\left(\frac{-\frac{\rho g R^3(4\beta\eta_0+R)}{8LR_T^2\eta_0}\left(L+H(t)\right)^{1-\alpha}-\frac{R^{\alpha+3}\left(\frac{\rho g}{2L}\right)^{\alpha}T_{\frac{1}{2}}^{1-\alpha}}{(3+\alpha)\eta_0 R_T^2}}{-\frac{\rho g R^3(4\beta\eta_0+R)}{8LR_T^2\eta_0}\left(L+H_0\right)^{1-\alpha}-\frac{R^{\alpha+3}\left(\frac{\rho g}{2L}\right)^{\alpha}T_{\frac{1}{2}}^{1-\alpha}}{(3+\alpha)\eta_0 R_T^2}}\right)=t. \tag{31}$$

By assuming that $H(t)=0$ in Eq, the time needed for full tank draining (Time of efflux) can be calculated (31)

$$\frac{-8LR_T^2\eta_0}{\rho gR^3(4\beta\eta_0+R)(1-\alpha)}\ln\left(\frac{-\frac{\rho gR^3(4\beta\eta_0+R)L^{1-\alpha}}{8LR_T^2\eta_0}-\frac{T_{\frac{1}{2}}^{1-\alpha}R^{\alpha+3}\left(\frac{\rho g}{2L}\right)^{\alpha}}{\eta_0(\alpha+3)R_T^2}}{-\frac{\rho gR^3(4\beta\eta_0+R)}{8LR_T^2\eta_0}\left(H_0+L\right)^{1-\alpha}-\frac{T_{\frac{1}{2}}^{1-\alpha}R^{\alpha+3}\left(\frac{\rho g}{2L}\right)^{\alpha}}{\eta_0(\alpha+3)R_T^2}}\right)=t_{eff}. \quad (32)$$

Remark: By using $\beta=0$, these solutions retrieves the Ellis fluid flow under the no-slip condition [16].

4 Special Cases for Ellis Fluid:

4.1 Newtonian Fluids

Ellis solutions theoretically reduce to the Newtonian fluid for $\left(T_{\frac{1}{2}}\to\infty\right)$ [16], that is, already mentioned in [18],

$$v_z=\frac{\rho g}{4L\eta_0}\left(L+H(t)\right)\left(2\beta R\eta_0+R^2-r^2\right) \quad (33)$$

$$Q=\frac{\rho g\pi}{8\eta_0}\left(\frac{H(t)}{L}+1\right)\left(4\beta R^3\eta_0+R^4\right) \quad (34)$$

$$\bar{V}=\frac{\rho g}{8\eta_0}\left(\frac{H(t)}{L}+1\right)\left(4\beta R\eta_0+R^2\right). \quad (35)$$

$$H(t)=-L+\left(L+H_0\right)e^{-\frac{\rho gt}{8\eta_0LR_T^2}\left(R^4+4\eta_0R^3\beta\right)}, \quad (36)$$

$$t=-\frac{8R_T^2\eta_0L}{\rho R^3g\left(R+4\beta\eta_0\right)}\ln\left(\frac{L+H(t)}{L+H_0}\right). \quad (37)$$

$$t_{eff}=\frac{8\eta_0LR_T^2}{\rho g(4\beta R^3\eta_0+R^4)}\ln\left(\frac{H_0}{L}+1\right). \quad (38)$$

Remark: By using $\beta=0$ in equation (38), Bernoulli's equation retrieves the Newtonian solution under the no-slip condition [25].

4.2 Power law Fluid

Ellis's model yields the Power law fluid's conclusion for substitution $\left(T_{\frac{1}{2}}=\eta_0,\ \frac{1}{n}=\alpha\right)$ [16],

$$v_z=\frac{R\rho\beta g}{2L}\left(L+H(t)\right)-\frac{n}{1+n}\left[\frac{\rho g}{2L\eta_0}\left(L+H(t)\right)\right]^{n^{-1}}\left(r^{1+\frac{1}{n}}-R^{1+\frac{1}{n}}\right). \quad (39)$$

$$Q=\frac{\beta\pi\rho gR^3}{2L}\left(L+H(t)\right)+\left(\frac{\rho gR\left(L+H(t)\right)}{2\eta_0L}\right)^{n^{-1}}\left(\frac{\pi R^3n}{3n+1}\right). \quad (40)$$

$$\bar{V}=\frac{\beta\rho gR}{2L}\left(L+H(t)\right)+\left(\frac{\rho g(L+H(t))R}{2L\eta_0}\right)^{n^{-1}}\left(\frac{nR}{1+3n}\right). \quad (41)$$

For the use of Eq. (40) into the Eq. (26), we acuire

$$H(t) = -L + \left[\left(\left(L + H_0\right)^{\frac{n-1}{n}} + \frac{Rn}{\eta_0\beta\left(3n+1\right)}\left(\frac{\rho g R}{2\eta_0 L}\right)^{\frac{1-n}{n}} \right) e^{\frac{\beta\rho g R^3(1-n)t}{2nLR_T^2}} - \frac{Rn}{\eta_0\beta\left(1+3n\right)}\left(\frac{\rho R g}{2\eta_0 L}\right)^{\frac{1-n}{n}} \right]^{\frac{n}{n-1}}, \quad (42)$$

Separately, the relationship between time and fluid depth in the tank can be expressed as follows:

$$t = \frac{2nLR_T^2}{\beta\rho g R^3(1-n)} \ln\left[\frac{(H(t)+L)^{\frac{n-1}{n}} + \frac{Rn}{\eta_0\beta\left(1+3n\right)}\left(\frac{\rho g R}{2\eta_0 L}\right)^{\frac{1-n}{n}}}{(H_0+L)^{\frac{n-1}{n}} + \frac{Rn}{\eta_0\beta\left(1+3n\right)}\left(\frac{\rho g R}{2\eta_0 L}\right)^{\frac{1-n}{n}}} \right]. \quad (43)$$

We assume *H(t) = 0* to be the amount of time needed to completely empty the tank.

$$t_{eff} = \frac{2nLR_T^2}{\beta\rho g R^3(1-n)} \ln\left[\frac{1 + \frac{Rn}{\eta_0\beta\left(1+3n\right)}\left(\frac{\rho g R}{2\eta_0}\right)^{\frac{1}{n}-1}}{\left(\frac{H_0}{L}+1\right)^{\frac{n-1}{n}} + \frac{Rn}{\beta\eta_0\left(1+3n\right)}\left(\frac{R\rho g}{2\eta_0}\right)^{\frac{1-n}{n}}} \right]. \quad (44)$$

Remark: By using $\beta = 0$, these solutions retrieves the Power law fluid model under the no-slip condition [16].

4.3 Bingham Plastic Fluid

The model defines this fluid as follows:

$$\eta_0\left(\frac{\partial v_z}{\partial r}\right) + T_0 = T_{rz}, \qquad \text{conerning} \qquad T_0 < T_{rz} \quad (45)$$

$$\left(\frac{\partial v_z}{\partial r}\right) = 0, \qquad \text{conerning} \qquad T_0 \ge T_{rz} \quad (46)$$

Thus, Eq. (16) is simplified to Eq. (45) on behalf of stresses greater than T_0 by setting $\left(T_{\frac{1}{2}} = T_0,\ \alpha = 0\right)$ while for $T_{rz} \le T_0$, due of the extremely high yield stress and excessively viscous material, Eq. (46) is derived from Eq. (3). Using Eq. (15), this results in the radius of circular pipe r_1 elsewhere which Eq. (46) holds when T_0 is substituted for T_{rz}:

$$r_1 = -\left(\frac{2LT_0}{\rho g}\right)\left(\frac{1}{L+H(t)}\right). \quad (47)$$

$$v_{z_1} = \frac{\rho g}{\eta_0 L}\left(\frac{L+H(t)}{4}\right)\left[2\beta\eta_0 R + 2r_1 r - 2R - r^2 + R^2\right], \quad \text{with} \quad r_1 > r > 0 \quad (48)$$

$$v_{z_2} = 0. \ \text{with} \quad R \ge r \ge r_1 \quad (49)$$

For this proposed model flow rate can be specified by

$$Q = 2\pi\left\{\int_0^{r_1} r\left(v_{z_1}(r)\right)dr\right\} + 2\pi\left\{\int_{r_1}^{R} r\left(v_{z_2}(r)\right)dr\right\}. \quad (50)$$

By using Eq. (48) and Eq. (49), in Eq. (50), We'll determine the overall flow rate.

$$Q = r_1^2\rho\pi g\left\{\frac{L+H(t)}{24L\eta_0}\right\}\left[12\beta R\eta_0 + 3(2R^2 - r_1^2) + 4(2r_1 - 3R)\right]. \quad (51)$$

The average drainage velocity will be, using Eq. (51) in Eq. (23),

$$\bar{V} = \frac{\rho g r_1^2}{24\eta_0 LR^2}\{L+H(t)\}\left[12\beta R\eta_0 + 3(2R^2 - r_1^2) + 4(2r_1 - 3R)\right]. \quad (52)$$

Making the use of euation (51) in (26), we acquire

$$H(t) = \{L+H_0\}e^{\frac{-r_1^2 g\rho}{24R_T^2 L\eta_0}\left[12\beta R\eta_0 + 6R^2 - 3r_1^2 + 8r_1 - 12R\right]t} - L. \quad (53)$$

When you rearrange equation Eq. (53), you get a conclusion that shows how time relates to the level of the fluid into the circular tank independently.

$$t = \frac{-24LR_T^2\eta_0}{\rho g r_1^2\left\{12\beta R\eta_0 + 3(2R^2 - r_1^2) + 4(2r_1 - 3R)\right\}}\ln\left(\frac{H(t)+L}{H_0+L}\right). \quad (54)$$

By putting $H(t) = 0$ in Equation (54), one may determine the amount of time needed for full tank drainage (Time of efflux), which is

$$t_{eff} = \frac{24LR_T^2\eta_0}{\rho g r_1^2\left\{12\beta R\eta_0 + 3(2R^2 - r_1^2) + 4(2r_1 - 3R)\right\}}\ln\left(\frac{H_0}{L}+1\right). \quad (55)$$

Remark: By using $\beta = 0$, these solutions retrieves the Bingham Plastic fluid mdel under the no-slip condition [16].

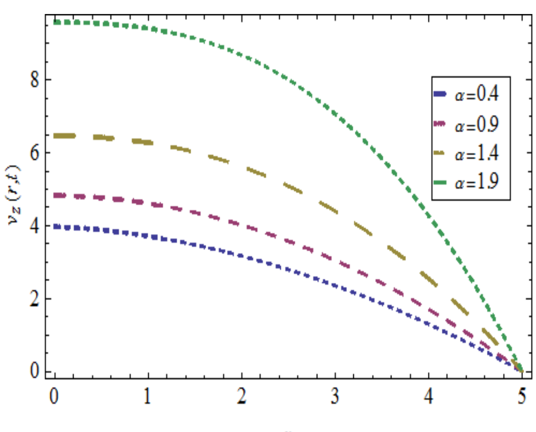

Figure 2. Impact of $\boldsymbol{\alpha}$ on velocity field, when $\rho = 1.38\ \frac{g}{cm^3}, \eta_0 = 100$ poise, $R = 5cm, \beta = 0.01$ $T_{\frac{1}{2}} = 21.554\ dyn/cm^2,\ H(t) = 20\ cm, L = 10\ cm.$

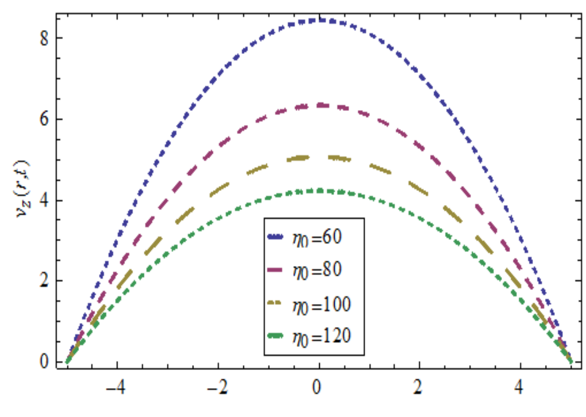

Figure 3. Impact of η_0 on velocity field, when $\alpha = 1$ poise, $R = 5cm,\ \rho = 1.38\ \frac{g}{cm^3}, \beta = 0.01$ $H(t) = 20\ cm, T_{\frac{1}{2}} = 21.555\ \frac{dyn}{cm^2},\ L = 10\ cm.$

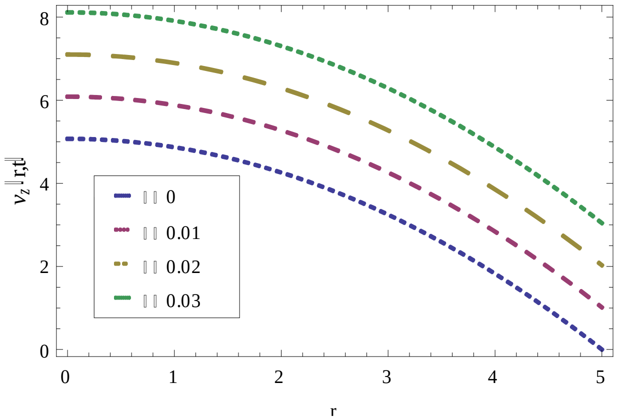

Figure 4. **Impact of β on the velocity, when**

$\rho = 1.38 \frac{g}{cm^3}, \alpha = 1, T_{\frac{1}{2}} = 21.554 \frac{dyn}{cm^2},$

$\eta_0 = 100\,\text{poise}, R = 5\,cm, L = 10\,cm, H(t) = 20cm.$

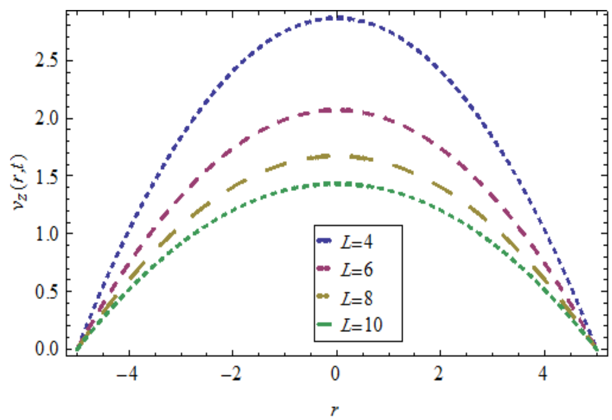

Figure 5. Impact of length of L on velocity field, when

$\rho = 0.78 \frac{g}{cm^3}, \alpha = 1, \eta_0 = 200\,\text{poise}, \beta = 0.01,$

$T_{\frac{1}{2}} = 21.554 \frac{dyn}{cm^2}, H(t) = 20\,cm, R = 5\,cm.$

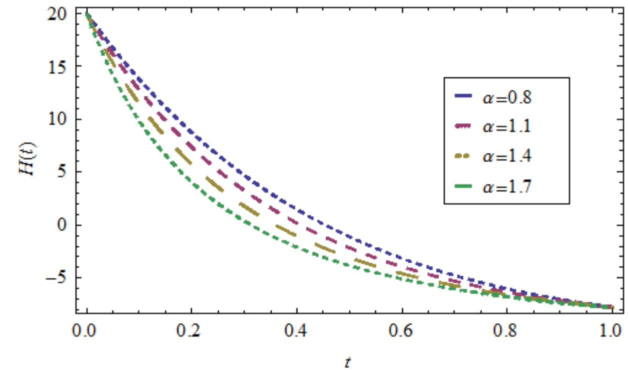

Figure 6. **α impact on depth of the tank, when**

$\rho = 1.38 \frac{g}{cm^3}, H_0 = 20\,cm,, \eta_0 = 0.13\,poise, R_T = 25\,cm$

$T_{\frac{1}{2}} = 21.554 \frac{dyn}{cm^2}, L = 10\,cm, R = 5\,cm, \beta = 0.01.$

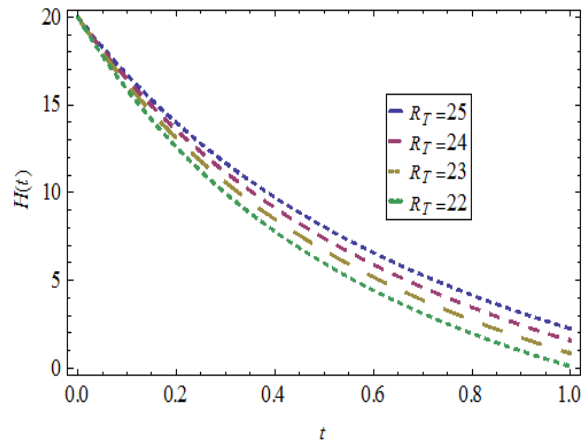

Figure 7. **R_T impact on depth of the tank, when**

$H_0 = 20\,cm, \rho = 1.38 \frac{g}{cm^3}, \eta_0 = 0.6\,\text{poise}, \alpha = 1.9,$

$, L = 10\,cm, T_{\frac{1}{2}} = 21.554 \frac{dyn}{cm^2}, R = 5\,cm, \beta = 0.01$

Table 1. Ellis fluid's relationship between depth and time, as well as some of its particular cases

t	Bingham Plastic	Power Law	Newtonian	Ellis
0	20	20	20	20
2	19.8596	19.7364	19.5056	19.2279
4	19.7198	19.474	19.0193	18.4805
6	19.5806	19.2129	18.5411	17.7568
8	19.4422	18.953	18.0707	17.0559
10	19.3043	18.6943	17.6081	16.3769

$$L = 10\,cm, \alpha = 1.7, \rho = 0.78 \frac{g}{cm^3}, \eta_0 = 11.5\,\text{poise}, T_{\frac{1}{2}} = 100.554 \frac{dyn}{cm^2}, R = 5\,cm, H_0 = 20\,cm, R_T = 25\,cm, n = 1.9, r_1 = 2.4\,cm, \beta = 0.01$$

5 RESULTS AND DISCUSSION

This study looked at unsteady tank drainage problems using an isothermal, incompressible Ellis fluid to get precise findings for the non-linear differential equation. On various parameters, the circular tank's depth *H(t)* and velocity field v_z have been explored. Through Figures 2 through 5, it is possible to observe the impacts of the pipe's length *L*, the Ellis index α, the viscosity η_0, and the slip parameter β on the velocity profile. Figures 6 and 7 demonstrate the impact of the tank's radius R_T and the Ellis index α on the depth *H(t).* It can be shown in Fig. 2 that as the Ellis index α rises, so does the velocity's magnitude. This explains why the magnitude of velocity decreases as fluid viscosity decreases. Figs. 3 and 4 depict how η_0 and β affect velocity profile v_z. These figures demonstrate how the magnitude of the velocity distribution increases as η_0 and β do, and vice versa. It is discovered that the viscosity, velocity distribution, tank depth, and pipe radius are all connected to one another. As shown in Fig. 5, the length of the pipe is also discovered to be inversely related to the velocity profile. In Figures 6 and 7, it is depicted how the radius of the tank R_T and the Ellis index α affect the height of the fluid in the tank *H(t)*. The depth of the fluid *H(t)* decreases as grows α , while the depth rises as tank radius increases. Table 1 compares the depth of the Ellis fluid with some of its special cases using fixed parameters that are listed in the table's caption. The outcomes shown in Table 1 were calculated quantitatively. The depth of Ellis fluid is less than the depth of its special cases, according to tabulated data that shows depth at different points in time. This also explains why Ellis fluid's velocity profile is higher than that of its special instances.

6 CONCLUDING REMARKS

Exact solutions for the suggested model have been found using the equation for unsteady tank drainage flow for incompressible and isothermal Ellis fluid. Additionally, we precisely retrieved the Ellis model's special cases, including Newtonian, Power law, and Bingham plastic. Eqs. (30) and (31), as well as the change in time and depth, are related in this equation. It is crucial to keep in mind that Ellis fluid has a higher apparent viscosity than other fluids, which causes the fluid's velocity to increase. It is also crucial to keep in mind that Ellis fluid drains more quickly than other fluids with similar properties. In addition, we have noted that a tank containing Ellis fluid will empty more quickly under slip conditions than they would under normal circumstances.

Credit Author Statement

Naina Salar Shaikh: Conceptualization, Methodology, Writing- Original draft perparation **K. N. Memon**: Supervision, **Muhammad Suleman Sial**: Software, Investigation, Writing -Reviewing **A. M. Siddiqui***:* Suggestions and ideas

Compliance with Ethical Standards:

It is said that there are no conflicts of interest among any of the authors. Additionally, it is stated that none of the authors of this paper conducted any experiments using human or animal subjects. Furthermore, each participant who took part in the study gave their free, informed consent.

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