VFAST Transactions on Mathematics
http://vfast.org/journals/index.php/VTM@ 2021, ISSN(e): 2309-0022, ISSN(p): 2411-6343
Volume 10, Number 2, July-December, 2022 pp: 189-200

Effect of SLIP Condition on Unsteady Tank Drainage Flow of The Third Order Fluid

Muneer Ahmed Mahar[1*], **K. N. Memon**[2*], **Syed Feroz Shah**[1], **Azam Ali Amur**[3], **A. M. Siddiqui**[4]

[1*]Department of Basic Science and Related Studies, Mehran University of Engineering and Technology Jamshoro, Pakistan; [2*]Department of Mathematics and Statistics, QUEST, Nawabshah, Pakistan; [3]Department of Basic Science and Related Studies, Mehran University of Engineering and Technology Jamshoro, Pakistan; [4]Pennsylvania State University, York Campus, Edgecombe 17403, USA

Keywords: *Slip condition, Perturbation method, third-order fluid, tank drainage.*
Subject Classification: *Third Order Fluid in the tank problem, Modeling for nonlinear equation.*

Journal Info:
Submitted:
April 13, 2022
Accepted:
May 15, 2022
Published:
June 30, 2022

Abstract Unsteady, isothermal, and in-compressible third-order fluid is considered in the tank, cylindrical coordinates are taken into account and the flow of the fluid is considered in the z-direction. The behavior of visco-elastic fluid is analyzed, flow of the fluid posses some velocity at the walls of the pipe so that the Slip condition is utilised. Perturbation technique is used to find out analytical solution in terms of velocity profile for Non-linear partial differential equation furthermore, flow rate, Average velocity, mathematical formulation of the mass equilibrium and mathematical relation between time depth of mass and time required to complete the drainage, etc are calculated. Different physical flow parameters are inspected graphically including slip and ϵ . A comparison of slip and no-slip is presented for third-order fluid also Newtonian and third-order fluid's velocity profiles are compared.

***Correspondence Author Email Address:**
muneerahmed7861996@gmail.com

1 Introduction

Every day is presenting the best outcomes in the field of fluid dynamics for research purposes which brings lots of ease in the engineering field as well in daily life. Not only in daily life but as well as in industrial areas fluid is the most important asset which can be defined as "any material that deforms under the action of the shearing force" there are two important brands of fluid acknowledged as Newtonian and non-Newtonian. Non-Newtonian fluids behavior is nonlinear with shear stress and rate of deformation[9, 11].The viscosity of the fluids varies when force is applied that's the reason it does not obey the law of viscosity. Not every fluid can possess all characteristics of non-Newtonian but there is a class of viscous and elastic fluid known as "n" ordered fluids, these are classified according to the order of truncation in their assumption Second-grade fluid, which is associated with second-order truncation, is an exceptional subclass. Although a second-grade fluid model exhibits normal stress impacts for constant flow, it appears to lack the shear thickening or shear thinning property that too many fluids have[1, 7]. Third-order fluid belongs to this family but has strong viscous and elastics characteristics which are the main cause of its great attention to a mathematician.

This manuscript contains third order fluid model due to its plentiful applications in industries such as chemical, bio-medical, pharmaceutical, petroleum industry, water management distribution, and supply[2, 8, 10].Third-order fluid problems produce strong nonlinear partial differential equations for such equations no exact or numerical technique is suitable. Here perturbation method is utilized to attain the analytical solution considering the slip condition[5, 16]. Slip condition is used for viscous fluids where a solid boundary fluid has some velocity.

Mooney conducted one of the first investigations of the slip at the wall, observing that when the stress surpassed a particular amount, the flow curves (shear stress vs nominal shear rate) of various fluids changed with the radius of the capillary. He deduced that slip was occurring near the wall and proposed an approximate method for calculating the slip velocity. The Mooney approach, sometimes known as the Mooney method, is still commonly used to determine slip velocity [14, 17].Here it is assumed as $V_z = -\beta\tau_{rz}$ at r = R because flow of fluid is normal to bottom of tank and is due to hydro-static pressure and gravity. Series solutions are obtained for the arising differential equations with slip boundary conditions and switching ($\epsilon = \frac{\beta_2+\beta_3}{\mu} = 0$), we compare results of third order fluid with Newtonian fluid correspondingly slip with no slip boundary conditions[4, 18].Likewise terminologies for velocity distribution , average velocity, flow-rate, and time efflux and time depth of the mass relation are calculated.

2 Governing Equations

The essential equations leading the motion of an in-compressible fluid, neglecting temperature influences and body forces, are provided as:

$$\nabla.V = 0 \tag{1}$$

$$\rho.b + \nabla.S - \nabla p = \rho\frac{DV}{Dt} \tag{2}$$

Here V characterizes to velocity vector, b be the body force, s represent to extra stress tensor, dynamic pressure is denoted as p and $\frac{DV}{Dt}$ implies to total derivative which is one and the same (equal) to $\frac{\partial V}{\partial t}+(\nabla.V)V$ relative the convective derivative and the local derivative.The extra stress tensors that describe third-order fluid models are specified by,

$$S = \sum_{i=0}^{3} T_i \tag{3}$$

where

$$T_1 = \mu A_1 \tag{4}$$

$$T_2 = \alpha_1 A_2 + \alpha_2 A_1^2 \tag{5}$$

$$T_3 = \beta_1 A_3 + \beta_2(A_1 A_2 + A_2 A_1) + \beta_3(tr(A_1^2))A_1 \tag{6}$$

where μ is the coefficient of viscosity and α_1, α_2, β_1, β_2 and β_3 are material constants. The Rivilin-Ericksen tensor, A_n are defined by

$$A_n = \frac{DA_{n-1}}{Dt} + (\nabla V)^T A_{n-1} + A_{n-1}(\nabla V) \qquad 1 \leq n \tag{7}$$

3 Problem Formulation

Consider a cylinder-shaped tank comprising of 3rd order, isothermal and in-compressible fluid. The diameter of the pipe that connects the tank's center and bottom most is d; radius of the tank is pretend to be R_T. At beginning, the fluid deepness in the tank is H_o.The fluid within the tank is drained via a pipe with a length of L and radius of R. Likewise; assume that the distance from the top to the bottom of fluid in the tank for all time t is H(t). The route of fluid flow is directed towards hydro-static pressure and gravity of the fluid within tank. The velocity profile, flow rate, average velocity, mathematical relationship between the draining time and depth of the fluid in the circular tank for total drainage of the fluid, the time of efflux for third-order fluid is going to be found. In this case we will use r, θ, z as cylindrical coordinates with the r-axis perpendicular to the axis of pipe and pipe's axis is along to the z-axis in the upright direction. After then, the fluid flow is exclusively in the z-axis direction. The components of velocity vector v in the direction of r and θ are both zero, therefore velocity field and shear stress can be assumed as:

$$V = [0, 0, V_z(r,t)] = [V_r, V_\theta, V_z], S = S(r,t) \tag{8}$$

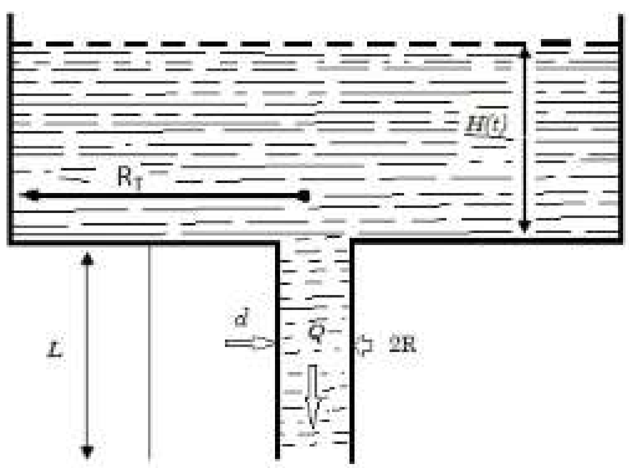

Figure 1. Tank drainage flow down via a round pipe [12]

By using Eq.(8) equation of continuity Eq.(1) is satisfied identically from Eq.(2) and Eq.(3) resulting components of momentum equation Eq. (2) are the form of:

r- component of momentum:

$$0 = \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\tau_{rr})}{\partial r} \tag{9}$$

θ - component of momentum

$$0 = \frac{\partial \rho}{\partial \theta} \tag{10}$$

z -component of momentum

$$\rho\frac{\partial V_z}{\partial t} = -\frac{\partial \rho}{\partial z} + \frac{1}{r}\frac{\partial (r\tau_{rz})}{\partial r} + \rho g \tag{11}$$

components of extra stress tensor will be

$$\tau_{\theta r} = \tau_{r\theta} = \tau_{z\theta} = \tau_{\theta} z = \tau_{\theta\theta} = 0 \tag{12}$$

$$\tau_{rr} = (2\alpha_1 + \alpha_2)\left(\frac{\partial v_z}{\partial r}\right)^2 + 2\beta_1\left(\frac{\partial v_z}{\partial r}\frac{\partial^2 v_z}{\partial r \partial t} + \frac{\partial}{\partial t}\left(\frac{\partial v_z}{\partial t}\right)^2\right) + 2\beta_2\left(\frac{\partial v_z}{\partial r}\frac{\partial^2 v_z}{\partial r \partial t}\right) \tag{13}$$

$$\tau_{rz} = \tau_{zr} = \mu\left(\frac{\partial v_z}{\partial r}\right) + \alpha_1\left(\frac{\partial^2 v_z}{\partial r \partial t}\right) + \beta_1\left(\frac{\partial^3 V_z}{\partial r \partial t^2}\right) + 2(\beta_2 + \beta_3)\left(\frac{\partial v_z}{\partial r}\right)^3 \tag{14}$$

$$\tau_{zz} = \alpha_2\left(\frac{\partial v_z}{\partial r}\right)^2 + 2\beta_2\left(\frac{\partial v_z}{\partial r}\frac{\partial^2 V_z}{\partial r \partial t}\right) \tag{15}$$

The velocity of the pipe flow remains almost constant throughout time due to very very slow draining, therefore time derivative have been ignore in the comparison of spatial derivatives.Flow of the fluid is in the direction of gravity and hydro-static pressure through pipe of radius R.Entrance and departure of the pressure at the pipe are Self governing [15]. Consequently

$$\frac{\partial p}{\partial z} = \frac{p_2 - p_1}{L} = -\frac{\rho g H(t)}{L} \tag{18}$$

the equation of the motion Eq.(11) therefore reduces to, after substituting the value of $\tau_r z$ and the value of $\frac{\partial p}{\partial z}$ from Eq.(18) , after simplification , we acquire

$$-\frac{\rho g r}{\mu}\left(\frac{H(t)}{L} + 1\right) = r\frac{\partial^2 v_z}{\partial r^2} + \frac{\partial v_z}{\partial r} + \frac{2(\beta_2 + \beta_3)}{\mu}\left[3r\left(\frac{\partial v_z}{\partial r}\right)^2\frac{\partial^2 v_z}{\partial r^2} + \left(\frac{\partial v_z}{\partial r}\right)^3\right] \tag{19}$$

For the proposed problem corresponding (center line) and slip boundary conditions are respectively:

$$r = 0, \quad \frac{\partial v_z}{\partial r} = 0 \tag{20}$$

$$r = R, \quad \mathrm{V}_z = -\beta\tau_{rz} \tag{21}$$

4 Perturbation solution of the Problem

Let's Suppose $\epsilon = \frac{\beta_2+\beta_3}{\mu}$ is a important parameter so that the the velocity profile $v_z(r,\epsilon)$ is expressed using a power series defined as[6]:

$$\mathrm{v}_z(r,\epsilon) = v_0(r) + \epsilon v_1(r) + \epsilon^2 v_2(r)\cdots \tag{22}$$

The set of problems given below at the same time their related boundary conditions are originated plugging Eq.(22) into Eqs.(19-21) and equating coefficients of related powers we have.

4.1 zeroth order problem

$$\epsilon^0 : -\frac{\rho g r}{\mu}\left(\frac{H(t)}{L}+1\right) = r\frac{\partial^2 v_o}{\partial r^2}+\frac{\partial v_o}{\partial r} \tag{23}$$

$$\frac{dv_o}{dr} = 0, \quad r = 0 \tag{24}$$

$$v_o = -\beta\mu\frac{\partial v_o}{\partial r}, \quad r = R \tag{25}$$

4.2 First order problem

$$\epsilon^1 : 0 = r\frac{\partial^2 v_1}{\partial r^2}+\frac{\partial v_1}{\partial r}+2\left[3r\left(\frac{\partial v_o}{\partial r}\right)^2\frac{\partial^2 v_o}{\partial r^2}+\left(\frac{\partial v_o}{\partial r}\right)^3\right] \tag{26}$$

$$\frac{dv_1}{dr} = 0, \quad r = o \tag{27}$$

$$v_1 = -\beta\mu\left[\frac{\partial v_1}{\partial r}+2\left(\frac{\partial v_o}{\partial r}\right)^3\right], \quad r = R \tag{28}$$

4.3 Second order problem

$$\epsilon^2 : 0 = r\frac{\partial^2 v_2}{\partial r}+\frac{\partial v_2}{\partial r}+2\left[3r\left(2\frac{\partial v_o}{\partial r}\frac{\partial v_1}{\partial r}\right)\frac{\partial^2 v_o}{\partial r^2}+3r\left(\frac{\partial v_o}{\partial r}\right)^2\frac{\partial^2 v_1}{\partial r^2}+3\left(\frac{\partial v_o}{\partial r}\right)^2\frac{\partial v_1}{\partial r}\right] \tag{29}$$

$$\frac{dv_2}{dr} = 0, \quad r = 0 \tag{30}$$

$$v_2 = -\beta\mu\left[\frac{\partial v_2}{\partial r}+6\left(\frac{\partial v_o}{\partial r}\right)^2\frac{\partial v_1}{\partial r}\right], \quad r = R \tag{31}$$

The Zeroth order solution of Eq.(23) while imposing boundary conditions from Eq.(24) and Eq.(25) is

$$v_o = \frac{\rho g\,(H(t)+L)}{2L}\left[\beta \mathrm{R}+\frac{1}{2\mu}\left(R^2-r^2\right)\right] \tag{32}$$

First-order solution, Replacing the zeroth order solution from equation Eq.(32), into Eq.(26) and subject to conditions Eq.(27) and Eq.(28) is given by

$$v_1 = \frac{\rho^3 g^3}{16\mu^3}\left[\frac{H(t)}{L}+1\right]^3(r^4-R^4) \tag{33}$$

Replacing the zeroth and first order solution into Eq. (33) and subject to conditions from equations Eq. (30) and Eq. (31) the Second-order solution is given as

$$v_2 = \frac{9\rho^5 g^5}{144\mu^5}\left[\frac{H(t)}{L}+1\right]^5(R^6-r^6) \tag{34}$$

In resultant the perturbation method is utilised, Solution is corrected up to second order.

$$v_z = \frac{\rho g\,(H(t)+L)}{2L}\left[\beta \mathrm{R}+\frac{1}{2\mu}\left(R^2-r^2\right)\right]+\frac{\epsilon\rho^3 g^3}{16\mu^3}\left[\frac{H(t)}{L}+1\right]^3(r^4-R^4)+\frac{9\epsilon^2\rho^5 g^5}{144\mu^5}\left[\frac{H(t)}{L}+1\right]^5(R^6-r^6) \tag{35}$$

4.4 Flow Rate

The flow rate per unit width is obtained by the formula;

$$Q = \int_0^{2\pi} \int_0^R r v_z(r,t) \mathrm{d}r \mathrm{d}\theta = \int_0^R 2\pi r v_z(r,t) \mathrm{d}r \tag{36}$$

It is worthwhile to Calculate the flow rate, by using the equation of velocity profile Eq.(35) into Eq. (36).

$$Q = \frac{\rho g \pi\,(H(t)+L)}{8\mu L}\left(4\beta\mu R^3 + R^4\right) - \frac{\epsilon\rho^3 g^3 \pi R^6}{24\mu^3}\left(\frac{H(t)}{L}+1\right)^3 + \frac{3\epsilon^2\rho^5 g^5 \pi R^8}{64\mu^5}\left(\frac{H(t)}{L}+1\right)^5 \tag{37}$$

4.5 The Average Velocity

The average velocity's Formula is defined as.

$$\vec{V} = \frac{Q}{\pi R^2} \tag{38}$$

in the consequence of applying Eq. (37) to the fluid going down the pipe, the average velocity is

$$\vec{V} = \frac{\rho g \pi\,(H(t)+L)}{8\mu L}\left(4\beta\mu R + R^2\right) - \frac{\epsilon\rho^3 g^3 \pi R^4}{24\mu^3}\left(\frac{H(t)}{L}+1\right)^3 + \frac{3\epsilon^2\rho^5 g^5 \pi R^6}{64\mu^5}\left(\frac{H(t)}{L}+1\right)^5 \tag{39}$$

4.6 Mass Equilibrium

The mass balance of the Complete tank is[13]

$$\frac{d}{dt}\left(\pi R_T^2 H(t)\right) = -Q(t) \tag{40}$$

Putting flow rate ignoring second and higher order from Eq.(37) into Eq.(40) letter on separating variables on both sides of the equation and the resulting mathematical form of the balance of the tank is

$$H(t) = \left[\left((H_o+L)^{-}2 - \frac{\epsilon\rho^2 g^2 R^3}{3\mu^2 L^2\,(4\beta\mu + R)}\right) e^{\frac{\rho g\left(4\beta\mu R^3 + R^4\right)t}{4\mu L R_T^2}} + \frac{\epsilon\rho^2 g^2 R^3}{3\mu^2 L^2\,(4\beta\mu + R)}\right]^{\frac{-1}{2}} - L \tag{41}$$

4.7 The Time Depth Of The Mass Relation

Independent mathematical relation For the depth of the tank and time is given as follows[7]:

$$t = \left(\frac{-4\mu L R_T^2}{\rho g\left(4\beta\mu R^3 + R^4\right)}\right) \ln\left(\frac{(H_O+L)^{-2} - \frac{\epsilon\rho^2 g^2 R^3}{3\mu^2 L^2 (4\beta\mu + R)}}{(H(t)+L)^{-2} - \frac{\epsilon\rho^2 g^2 R^3}{3\mu^2 L^2 (4\beta\mu + R)}}\right) \tag{42}$$

The time required to complete drainage (Time of efflux) has been gotten while considering $H(t) = 0$ in Eq.(42)

$$t_{eff} = \left(\frac{4\mu L R_T^2}{\rho g\left(4\beta\mu R^3 + R^4\right)}\right) \ln\left(\frac{3\mu^2 L^2\,(4\beta\mu + R)\,(H_o+L)^{-2} - \epsilon\rho^2 g^2 R^3}{3\mu^2\,(4\beta\mu + R) - \epsilon\rho^2 g^2 R^3}\right) \tag{43}$$

Remark: Taking $\epsilon = 0$ and $\beta = 0$ in Eq.(43), the Solution of the Newtonian fluid was obtained by Bernoulli's equation defined in [3].

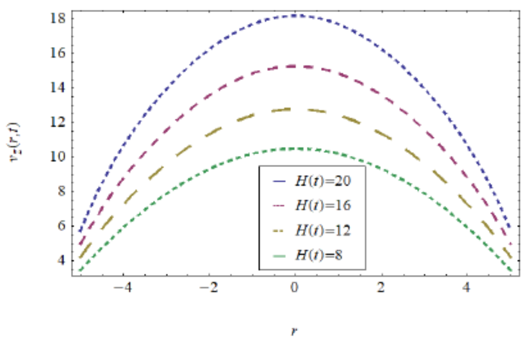

Figure 2. Result of H(t) on velocity field, when ρ = 0.78g/cm^3, μ = 11.5cP, R = 5cm, L = 10cm, ϵ = 0.01β = 0.1

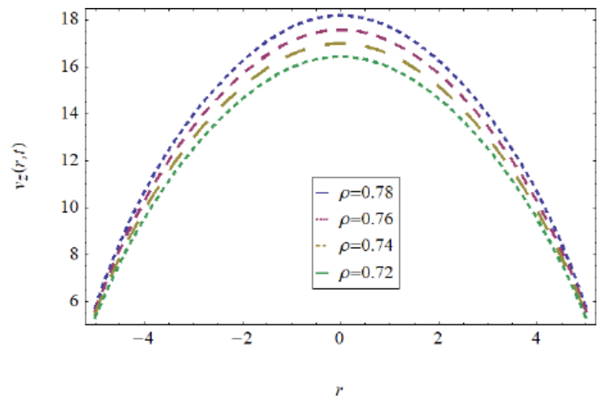

Figure 3. Influence of ρ on velocity distribution, when R = 5cm; μ = 11.5cP; L = 10cm; H = 20cm; ϵ = 0.01β = 0.1

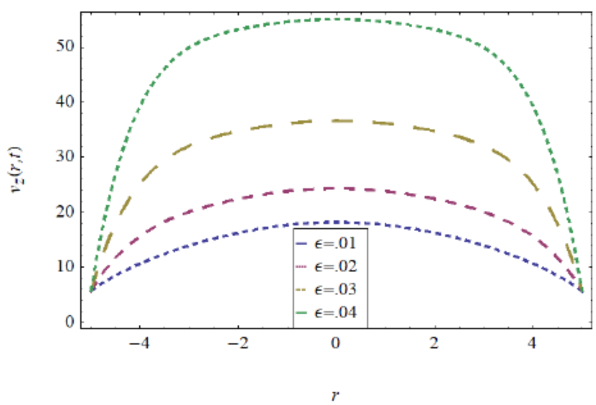

Figure 4. Influence of ϵ on velocity Field, whenρ = 0.78g/cm^3; R = 5cm; μ = 11.5cP; L = 10cm; H(t) = 20cmβ = 0.1

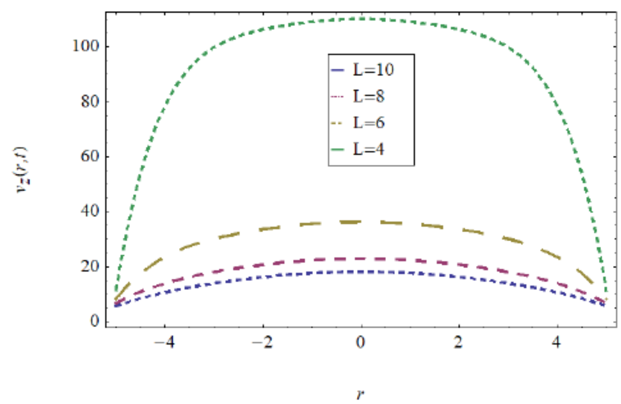

Figure 5. effect of L on velocity profile,ρ = 0.78g/cm^3, μ = 11.5cP, R = 5cm, H = 20cm, ϵ = 0.01β = 0.1

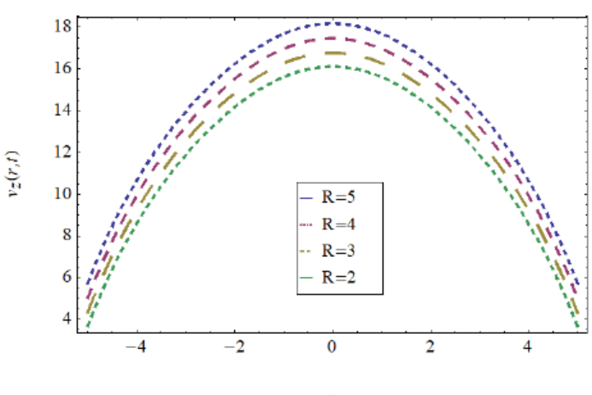

Figure 6. result of R on velocity field, when ρ = 0.78g/cm^3, μ = 0.78g/cm^3, L = 10cm, H = 20cm, ϵ = 0.01β = 0.1

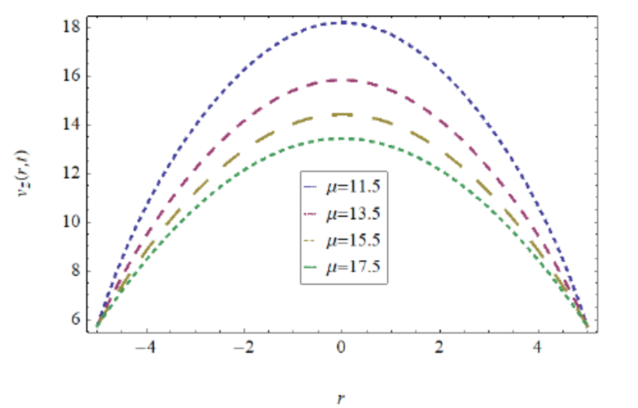

Figure 7. Consequence of μ on velocity distribution, when R = 5cmc$P$, ρ = 0.78g/cm^3, H = 20cm, L = 10cm, ϵ = 0.01β = 0.1

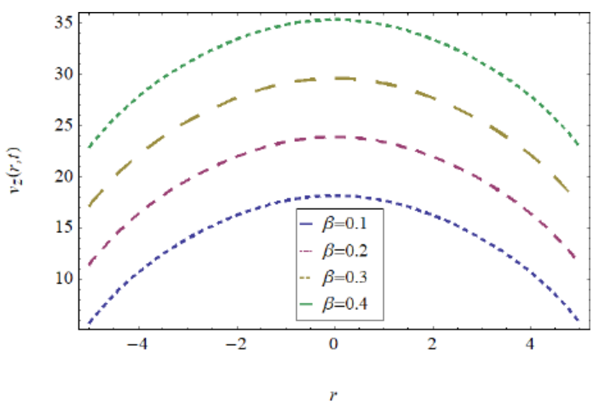

Figure 8. influence of β on velocity field when, R = 5cm, ρ = 0.78g/cm^3, L = 10cm, H = 20cm, ϵ = 0.01

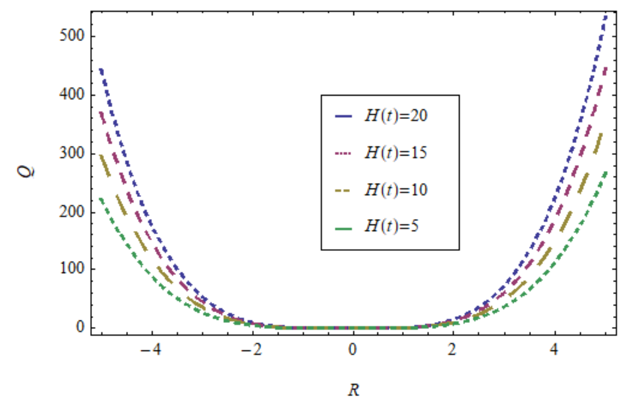

Figure 9. Consequence of H(t) on flow rate while μ = 11.5cP, R = 5cm, ρ = 0.78g/cm^3, H = 20cm, L = 10cm, ϵ = 0.01β = 0.1

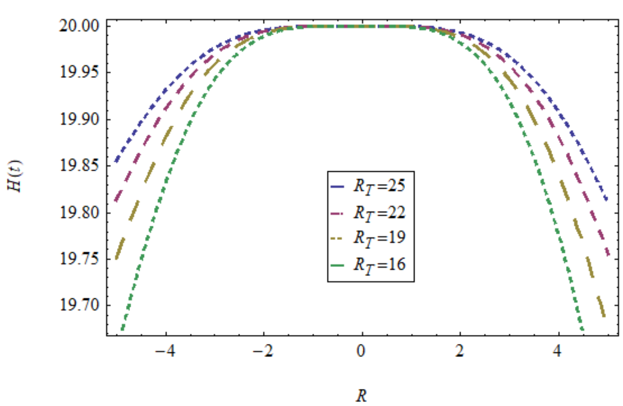

Figure 10. result of R_T on depth of W.R.t R, when R = 5cm, μ = 0.78g/cm^3, L = 10cm, H = 20cm, ϵ = 0.01β = 0.1

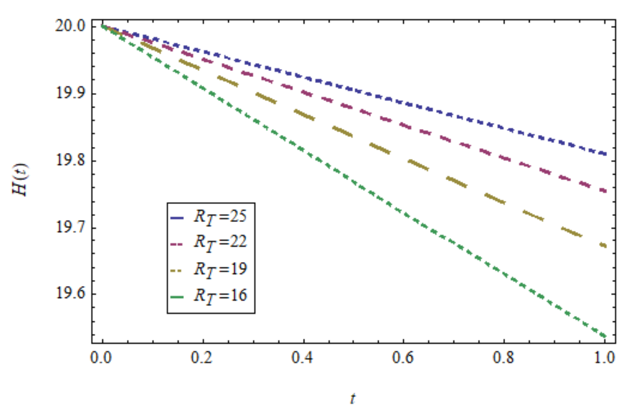

Figure 11. Consequence of R_T on depth W.r.t to t while μ = 11.5cP, ρ = 0.78g/cm^3, H = 20cm, L = 10cm, ϵ = 0.01β = 0.1

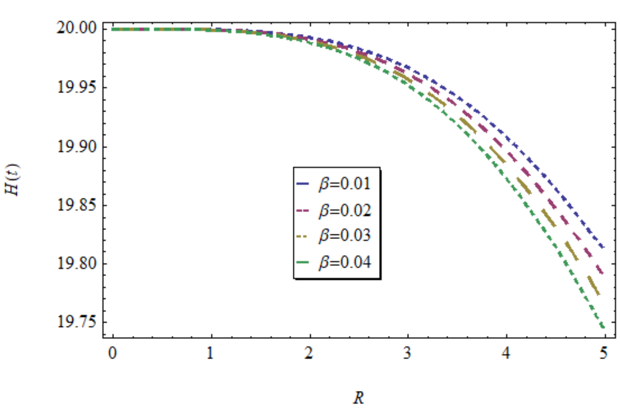

Figure 12. influence of β on depth W.r.t to text R when R = 5cm, μ = 11.5$cp\rho$ = 0, 78g/cm ^{3}, L = 10cm, H = 20cm, ϵ = 0.01

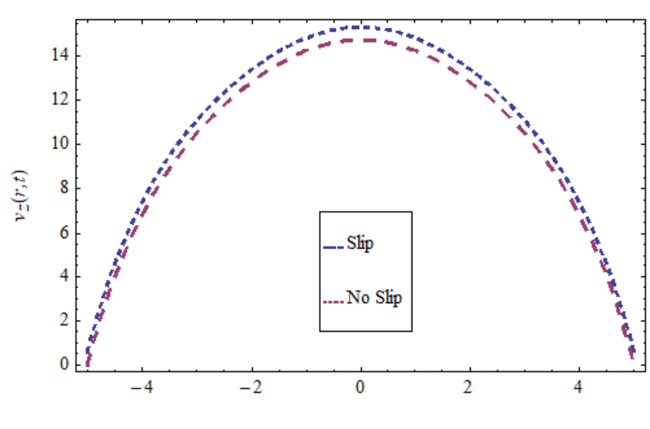

Figure 13. Comparison of Slip and No-Slip when μ = 11.5cP, ρ = 0.78g/cm^3, H = 20cm, L = 10cm, ϵ = 0.01R = 5$cm\beta$ = 0.1

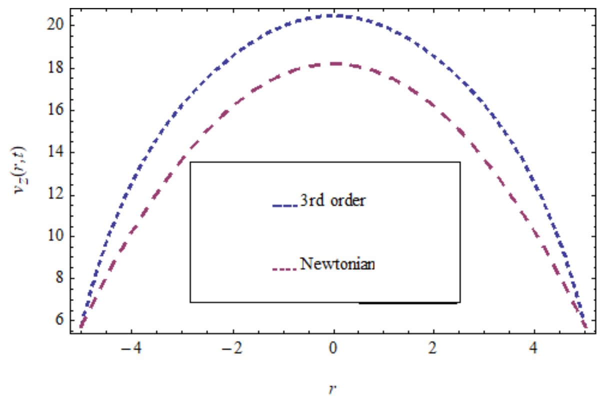

Figure 14. Comparison of Third-order fluid and Newtonian fluid for velocity field, when μ = 11.5cP, ρ = 0.78g/cm^3, H = 20cm, L = 10cm, ϵ = 0.01R = 5c$m\beta$ = 0.1

R	Third Order Fluid	Newtonian Fluid
0	20.4779	18.186
1	19.9864	17.6975
2	18.5744	16.202
3	16.2701	13.7093
4	12.5885	10.2197
5	5.733	5.733

Table 1. comparing third-order and Newtonian fluid velocity profile

5 Results and discussion

This section is all about for results of tank drainage containing in-compressible third-order fluid; perturbation solution is obtained for the non-linear Partial differential equation with Slip boundary conditions. the nature of the solution is purely analytic which only possible because the resulting partial differential eqution is highely Non-Linear. The consequences of various components over the velocity distribution, depth, and flow rate also has been explored. The impacts of depth, pipe radius, fluid density, pipe length, dynamic viscosity, and slip parameter β on velocity distribution are shown in Fig. 2–8. Influence of depth on flow rate is exposed in Fig. 9 further effect of radius of the tank with respect to R and time , slip parameter β on depth is inspected in Fig.10 – 12.Comparison of slip with no-slip boundary condition and velocity profile of Newtonian fluid with third-order fluid's velocity profile is given graphically in Fig.13 and 14 respectively. It is observed from Fig 2 to 8 that including slip parameter if we're increasing all factors on which velocity of third-order fluid is dependent so that in result the magnitude of velocity behaves as directly proportion means it is increasing and while decreasing the factor on which velocity profile is dependent decreases. Flow rate of the visco-elastic Fluid increases while there is increase in depth in Fig 9.

r	Slip condition	N0-Slip condition
0	15.3182	14.7449
1	14.8267	14.2534
2	13.4147	12.8414
3	11.1104	10.5371
4	7.42878	6.85548
5	0.5733	0

Table 2. comparison of Slip condition and No-Slip condition

For both cases depth increases means with respect to radius and time which are being plotted in Fig 10 and 11. But in Fig 12 it is noticed that depth of the fluid decreases when slip parameter β increases. Comparative study of slip and no-slip are present in figure 13 as well as in table 2 from which it is concluded that velocity profile of fluid with slip condition is higher than velocity profile of fluid with no-slip condition. Comparison of third-order and Newtonian fluid's velocity profile is presented in Table 1 and Fig. 14 with fixed parameters shows that the velocity profile for third order fluid is higher than Newtonian fluid.

6 Conclusion

Perturbation technique is considered to find out the analytical Solution of isothermal, in-compressible and unsteady tank drainage flow for third-order fluid. Since the fluid seems to have some velocity at the pipe's walls, so that effect of Slip condition is analysed over velocity profile. From the analytic solution various findings are given such as average velocity, depth, flow rate of fluid, mathematical form of Mass equilibrium for complete tank and time of efflux". further mathematically relationship between time and depth of the tank have been obtained, which shows how the time varies with depth of the fluid. from the findings it is interesting that depth increases with respect radius and time. It's essential to reminder that as higher the viscosity for third-order fluid, higher the velocity of third order fluid. It drains faster as compared to Newtonian fluid means third order fluid posses higher velocity profile, also with slip condition it flows faster as compared to no-slip condition which concludes that slip is more efficient than No-Slip.

7 Author Contributions

Muneer Ahmed Mahar:Methodology, Writing- Original draft preparation **K. N. Memon**:Software, Conceptualization,.**A. M. Siddiqu**:suggestions and ideas. **Syed Feroz Shah**:Supervision. **Azam Ali**:Writing-Reviewing and Editing.

8 Compliance with Ethical Standards

It is declare that all authors don't have any conflict of interest.

9 Funding Information

No Funding

10 Author Information

ORCID:
K.N. Memon: 0000-0003-2163-2552.

References

[1] M Kamran Alam, KN Memon, AM Siddiqui, SF Shah, Muhammad Farooq, Muhammad Ayaz, Taher A Nofal, and Hijaz Ahmad. Modeling and analysis of high shear viscoelastic ellis thin liquid film phenomena. *Physica Scripta*, 96(5):055201, 2021.

[2] Lawrence K Forbes and Graeme C Hocking. Unsteady draining of a fluid from a circular tank. *Applied mathematical modelling*, 34(12):3958–3975, 2010.

[3] Vijay K Garg. *Applied computational fluid dynamics*. CRC Press, 1998.

[4] T Hayat, Naveed Ahmed, and M Sajid. Analytic solution for mhd flow of a third order fluid in a porous channel. *Journal of Computational and Applied Mathematics*, 214(2):572–582, 2008.

[5] Tasawar Hayat and AH Kara. Couette flow of a third-grade fluid with variable magnetic field. *Mathematical and Computer Modelling*, 43(1-2):132–137, 2006.

[6] Saeed Islam, Kamran Nazir Memon, Abdul Majeed Siddiqui, and Syed Feroz Shah. Analytical solution of tank drainage for electrically conducting power law fluid. 2018.

[7] Mansoor Ali Khaskheli, Kamran Nazir Memon, Abdul Hanan Sheikh, Abdul Majeed Siddiqui, and Syed Feroz Shah. Tank drainage for an electrically conducting newtonian fluid with the use of the bessel function. *Eng. Technol. Appl. Sci. Res*, 10(2), 2020.

[8] Mansoor Ali Khaskheli, Kamran Nazir Memon, Abdul Hanan Sheikh, Abdul Majeed Siddiqui, and Syed Feroz Shah. Tank drainage for an electrically conducting newtonian fluid with the use of the bessel function. *Eng. Technol. Appl. Sci. Res*, 10(2), 2020.

[9] Kamran Nazir Memon, Muhammad Kamran Alam, Jamel Baili, Zeeshan Nawaz, Abdul Hanan Shiekh, and Hijaz Ahmad. Analytical solution of tank drainage flow for electrically conducting newtonian fluid. *Thermal Science*, 25(Spec. issue 2):433–439, 2021.

[10] KN Memon, S Islam, AM Siddiqui, Sher Afzal Khan, Nazir Ahmad Zafar, and M Akram. Lift and drainage of electrically conducting power law fluid on a vertical cylinder. *Applied Mathematics & Information Sciences*, 8(1):45, 2014.

[11] KN Memon, Sher Afzal Khan, S Islam, Nazir Ahmad Zafar, Syed Feroz Shah, and AM Siddiqui. Unsteady drainage of electrically conducting power law fluid. *Applied Mathematics & Information Sciences*, 8(5):2287, 2014.

[12] KN Memon, SF Shah, and AM Siddiqui. Exact solution of unsteady tank drainage for ellis fluid. *Journal of Applied Fluid Mechanics*, 11(6):1629–1636, 2018.

[13] KN Memon, AM Siddiqui, and SF Shah. Exact solution of tank drainage for newtonian fluid with slip condition. *Sindh University Research Journal-SURJ (Science Series)*, 49(2), 2017.

[14] Melvin Mooney. Explicit formulas for slip and fluidity. *Journal of Rheology (1929-1932)*, 2(2):210–222, 1931.

[15] Syed Asad Raza Shah, KN Memon, SF Shah, AH Sheikh, and AM Siddiqui. Delta perturbation method for thin film flow of a third grade fluid on a vertical moving belt. *STATISTICS, COMPUTING AND INTER-DISCIPLINARY RESEARCH*, 4(1):61–73, 2022.

[16] Syed Mansoor Shah, KN Memon, Syed Feroz Shah, Abdul Hanan Sheikh, Abbas Ali Ghoto, and AM Siddiqui. Exact solution for ptt fluid on a vertical moving belt for lift with slip condition. *Indian Journal of Science and Technology*, 12:30, 2019.

[17] AM Siddiqui, M Akram, KN Memon, and S Islam. Khalid khan, withdrawal and drainage of thin film flow on a vertical cylinder. *Scientific Research and Essays*, 7(41):3554–3565, 2012.

[18] RH SOLANGI, SF SHAH, AM SIDDIQUI, and KN MEMON. Effect of slip condition on thin layer flow on an upright cylinder for drainage of electrically conducting power law fluid. *Sindh University Research Journal-SURJ (Science Series)*, 48(4), 2016.