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Analysis of Two-Level Complex Shifted Laplace Preconditioner and Deflation-Based Preconditioner for Helmholtz Equation

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Abstract A Long time deflation preconditioner is used to speed up the convergence of the Krylov subspace method. The discretization of Helmholtz equation with Dirichlet boundary condition by finite difference method obtained any linear system. Resolving a large wavenumber requires a larger number of Grid points, i.e. large linear systems. Thus due to the large linear system, many (sparse) direct methods have taken more memory, So we have used the (iterative technique) Krylov subspace method. One of the problems of the Krylov subspace method is the required preconditioner for better convergence. We use (CSLP) as a preconditioner and drive eigenvalues of (CSLP). However, with increasing wavenumber CSLP shows slow convergence behavior. Then we use another projection-type preconditioner as a deflation preconditioner. A rigorous Fourier analysis (RFA) is a separate research idea to examine the con- vergence of the iterative method included in this article. We analyze the deflation preconditioner with a complex shifted Laplace preconditioner (CSLP) which exhibition spectral behavior of the preconditioner, which is favorable to the Krylov method.

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1 Introduction

Ordinary and partial differential equation arise from the mathematical modelling of the dynamical system in engineering and science. There are many classical methods to solve ODE's and PDE's, but some time these methods become fail to obtain exact solution. Therefore, researchers try to find approximate solution from numerical methods such as Finite Element Methods, Finite Difference Methods and Finite Volume Method [9, 14].

There are many methods available in the literature [2, 12, 17, 22] to solve linear system but due large number of unknowns , these methods like Cholesky factorization, QR factorization, LU factorization become fail [3, 17–19]. // Therefore, numerical solution technique are often used like iterative methods like the Gauss-Jacobi or Gauss-Seidel can be found [12] have been mostly applicable from the starting point of the numerical treatment of Partial Differential equations.

1.1 Model Problem

To study we assume 1D Helmholtz Equation with Dirichlet Boundary Condition is

$$-\Delta u(x) - k^2 u(x) = f(x), \quad \text{on} \quad \Omega = (0,1), \tag{1}$$

with Dirichlet's boundary conditions

$$u(x) = Oatx \in 0, 1.$$

Many discretization techniques have been used considering geometry of the domain. Finite difference, Finite element and Finite volume are amongst them. A detail of discretization can be found in [8, 13, 20]. Discretizing of the Helmholtz problem Eq. (01) using finite difference method, the following linear system is obtained.

$$A_h u_h = f_h, \tag{2}$$

where A_h is the triadiogonal matrix

$$A_h = \frac{1}{h^2}\begin{bmatrix} 2-k^2h^2 & -1 & & & & \\ -1 & 2-k^2h^2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2-k^2h^2 & -1 \\ & & & & -1 & 2-k^2h^2 \end{bmatrix}$$

The boundary conditions are included into the system. Specifically, Helmholtz equation with a large wave number**k**. Then, at that point, there are iterative solvers like Jacobi and Gauss Seidel, who do not perform very well when applied to indefinite problems.

2 Solvers and Preconditioners

For better convergence, we apply Krylov subspace method[7]. That method is attractive for sparse linear system. There are several kinds of Krylov subspace method[7]and different formed are sorted for different frameworks with specific properties. The Krylov methods always require preconditioners for better

convergence[8].
The linear system $Au = f$ into another linear system (with good converging properties). Preconditioner is a matrix, very close to original matrix A as well as less expensive to invert as compared to A. There are two classes of preconditioners.

- Matrix Based Preconditoiner
- Operator Based Precondtioner

The diagonal preconditioner is one of the simplest of the preconditioning, i.e.

$$P = diag(A). \tag{3}$$

Applying the Preconditioner P on the linear system (02) yields the following preconditioned linear system,

$$P_h^-1 A_h u_h = P_h^-1 f_h \tag{4}$$

with boundary conditions

$$u(x) = O \quad \text{at} \quad x \in 0, 1 \; .$$

The matrix of Incomplete LU factorization is a meagre approximation of the LU factorization which serve as a preconditioner [1, 10]. Another example as simple operator-based Preconditioner is Discrete Poisson Operator Δ [1]. Later, an additional term $-k^2$ is added to the Laplace operator than the Real Shift Laplace preconditioner describe in [6] creating this preconditioner added to real shift term α in Laplace operator.

2.1 Complex Shifted Laplace Preconditioner (CSLP)

For Helmholtz problem, The complex shifted Laplace preconditioner(CSLP), is defined as [5, 16]

$$M_h g(\alpha, \beta) = -\Delta_h - (\alpha - \iota\beta)k^2 \tag{5}$$

After applying this preconditioner, linear system given in Eq. (02) will be

$$M_h^- 1(\alpha, \beta) A_h u_h = M_h^- 1(\alpha, \beta) f_h. \tag{6}$$

Note that eigenvalues of coefficient matrix $M_h^{-1}(\alpha, \beta) A_h$ are complex and can be plotted in complex plane.

2.2 Deflation Preconditioner

When wave number k is made large, it is generally observed eigenvalues become near to zero[8]. Because of these small eigenvalues, Krylov solver takes too much time (Slow iterative process) towards the convergence [4, 15].To speed up the process of convergence eigenvalues are required to be zero or one[8].Therefore, to convert these eigenvalues to zeros, the Deflation Preconditioner is best option to be used as a preconditioner [15]. The basic idea of the deflation is small eigenvalues move to zero. Also, Deflation Preconditioner can be defined as [17]

$$D_h = I_h - A_h Q_h \tag{7}$$

The next step is to apply Deflation preconditioner in Eq. (06) to obtain the following linear system,

$$D_h M_h^- 1(\alpha, \beta) A_h u_h = D_h M_h^- 1(\alpha, \beta) f_h. \tag{8}$$

The linear system given in Eq. (08) is possible now to be solved with any Krylov method. Krylov method works on principle of minimizing residual r_h^i at at $i^t h$ iteration. Therefore, following the same framework of analysis of two-grid method of Multigrid method, as given in [7, 11, 15, 21, 23, 24], analysis will be conducted on the performance of the Krylov solver with the choice above preconditioner for the obtained linear system as given in Eq. (11). When comparing to two grid analyses, in our setup the Jacobi smoother $I_h - D_h^- 1A_h$ will be replaced by CSLP i.e., first we shall analyze operator

$$M_h^- 1(\alpha, \beta)A_h. \tag{9}$$

Next step will be analysing our deflation preconditioner. It is important to note that deflation is very similar to coarse grid correction (CGC) operator as defined in [15, 24], therefore, the operator will be focused for the analysis will be

$$(I_h \breve{} A_h Q_h)M_h^- 1A_h \tag{10}$$

where

$$Q_h = I_{2h}^h A_{2h}^{-1} I_h^{2h}. \tag{11}$$

3 Methodology

In this section we discuss convergence analysis of Helmholtz Operator and we used rigorous Fourier analysis to drive convergence factor of Complex Shifted Laplace Preconditioner (CSLP) with combination of Deflation Preconditioner defined in section one. First, we wish to find spectrum of Helmholtz Operator (or simply $\lambda_h(A_h)$) respectively CSLP than explain the structures of rigorous Fourier analysis. Secondly, we work out on spectrum of deflated-preconditioned operator for One dimensional Helmholtz problem by rigorous Fourier analysis.

3.1 Asymptotic Analysis of CSLP

In this portion, we shall find the eigenvalues of Helmholtz operator as well as complex shifted Laplace preconditioner.Suppose in Eq. (1) $f(x) = 0$

$$-\Delta u(x) - k^2 u(x) = f(x), \quad \text{on} \quad \Omega = (0, 1),$$

We apply method of separation of variable of above equation to obtain Eigen function of Helmholtz operator is define in equation (12)

$$\Psi(ih)_h^j = sin(j\pi ih)_{i=1}^{n-1}. \tag{12}$$

The objective of our research is to find spectrum of Deflated Preconditioner operator, so we utilize the stencil of Helmholtz operator with Eigen function of Helmholtz operator

$$\frac{1}{h^2}\begin{bmatrix}-1 & 2 - l^2 & -1\end{bmatrix} Bsin\left(j\pi ih\right) = \lambda_h \psi_h(x)$$

To obtain, the eigen value of Helmholtz operator A_h

$$\lambda_h(A_h) = \frac{1}{h^2}\left[4sin^2\frac{(j\pi h)}{2} - l^2\right]. \tag{13}$$

We know that complex shifted Laplace preconditioner (CSLP) in Eq. (05) We find above Eigen values of Helmholtz operator similarly corresponding Eigen value of complex shifted Laplace preconditioner $\lambda_h\left(M_h\right)$ and $\lambda_h(M_h^{-1}A_h)$ is

$$\lambda_h\left(M_h(\alpha,\beta)\right) = \frac{1}{h^2}\left[4sin^2\frac{(j\pi h)}{2} - l^2(\alpha - i\beta)\right] \tag{14}$$

$$\lambda_h\left(M_h^{-1}(\alpha,\beta)A_h\right) = \frac{\frac{1}{h^2}\left[4sin^2\frac{(j\pi h)}{2} - l^2\right]}{\frac{1}{h^2}\left[4sin^2\frac{(j\pi h)}{2} - l^2(\alpha - i\beta)\right]} \tag{15}$$

$$\lambda_h\left(M_h^{-1}(\alpha,\beta)A_h\right) = \frac{\left[4sin^2\frac{(j\pi h)}{2} - l^2\right]}{\left[4sin^2\frac{(j\pi h)}{2} - l^2(\alpha - i\beta)\right]} \tag{16}$$

So, the next section we suppose Deflation preconditioner act as a coarse grid correction of two grid method

3.2 Deflation Preconditioner and Deflated Helmholtz Operator

We shall make use of analogy from Coarse Grid Correction (CGC) operator of multigrid, which eventually resembles with our deflation operator. As we know that the given system is

$$A_h u_h = f_h$$

we apply Complex Shifted Laplace Preconditioner (CSLP) in Eq. (06) replace by (smoother) we obtained approximate value w_h.Also we know that the error is simply given by $v_h = u_h - w_h$. The residual equation obtained from

$$r_h = f_h - A_h w_h. \tag{17}$$

Substituting $f_h = A_h u_h$in Eq. (17),

$$r_h = A_h u_h - A_h w_h = A_h u_h - w_h = A_h v_h r_h = A_h v_h \tag{18}$$

The residual Eq. (18) is similar to original Eq. (02). Solution of the residual equation is difficult as compared to original equation. For coarse grid equation of Eq. (18) is

$$A_{2h}\nu_{2h} = r_{2h}, \tag{19}$$

where R_{2h} is define as restriction from fine grid residual to the coarse grid

$$r_{2h} = I_h^{2h} r_h \tag{20}$$

where I_h^{2h} denote the restriction. So, the weighted restriction of model problem (01) is given by

$$I_h^{2h} r_h = \frac{1}{4} r_h\left(x - h\right) + \frac{1}{2} r_h\left(x\right) + \frac{1}{4} r_h\left(x + h\right) \quad \text{for } x \in \Omega_{2h}. \tag{21}$$

The equivalent matrix is

$$I_h^{2h} = \frac{1}{4} \times \begin{bmatrix} 1 & 2 & 1 & & & & & & \\ & & 1 & 2 & 1 & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & 2 & 1 & \\ & & & & & & 1 & 2 & 1 \end{bmatrix}_{(\frac{n}{2}-1)\times(n-1)} \tag{22}$$

We have defined above r_{2h} in Eq. (20) from the residual r_h.we get exact solution v_{2h} of the coarse grid Eq. (19)

$$\widetilde{\nu_h} = I_{2h}^{h}\nu_{2h}, \tag{23}$$

where I_{2h}^{h} is prolongation and it transfers from coarse to fine grid. Linear interpolation is the simplest of prolongation operators,

$$I_{2h}^{h}\nu_{2h}(x) = \begin{cases} v_{2h} & \text{if} x \in \Omega_2 h \\ \frac{1}{2}[_{2h}(x-h) +_{2h}(x+h)] & \text{otherwise} \end{cases} \tag{24}$$

The equivalent matrix defines by

$$I_{2h}^{h} = \frac{1}{2}\begin{bmatrix} 1 & & & & \\ 2 & & & & \\ 1 & 1 & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & 2 & \\ & & & 1 & 1 \\ & & & & 2 \\ & & & & 1 \end{bmatrix}_{(n-1)\times(\frac{n}{2}-1)} \tag{25}$$

Therefore $u_h = w_h + v_h$ is an exact solution and let $\widetilde{v_h}$ is defined in Eq. (23) is approximate to v_h .we refresh the value w_h by

$$u_h^{new} = w_h + \widetilde{v_h}. \tag{26}$$

All procedure from w_h to u_h^{new} by Eq. (18) to (26) is called coarse grid correction operator or Deflation preconditioner.

$$u_h^{new} = w_h + D_h(M_h^{-1}A_h) \tag{27}$$

$$D_h = \left(I - A_hQ_h\right)(M_h^{-1}A_h)Q_h = I_{2h}^{h}A_{2h}^{-1}I_h^{2h} \tag{28}$$

The corresponding deflation operator is represented by

$$D_h = I_h - A_hI_{2h}^{h}A_{2h}^{-1}I_h^{2h} \tag{29}$$

3.3 Analysis of Deflation Preconditioner

From Eq. (16), we obtained continuous spectrum of Complex shifted Laplace Preconditioner (CSLP) and in Eq. (12) we used $\psi_h^j = (j = 1, 2, \ldots, n-1)$, a discreet Eigen function of A_h. These Eigen function $\psi_h^j = (j = 1, 2, \ldots, n-1)$ form an orthonormal basis of $\mathbb{R}^{n-1}$, therefore

$$(\psi_h^j, \psi_h^l) = \sum_{i=1}^{n-1} \sin\left(j\pi ih\right)\sin\left(l\pi ih\right) = \delta_{j.l} \qquad 1 \le j, l \le n-1 \tag{30}$$

So, the matrix V_h constructed by orthonormal vectors as unitary column $V_h = V_h^{-1}$

$$V_h = [\psi_h^1, \psi_h^{n-1}, \ \psi_h^2, \psi_h^{n-2}, \ \ldots, \ \psi_h^j, \psi_h^{n-j}, \ \ldots, \ \psi_h^{(\frac{n}{2}-1)}, \psi_h^{(\frac{n}{2}+1)}, \psi_h^{\frac{n}{2}}] \tag{31}$$

Therefore, we operate unitary column on deflated-preconditioned operator $_{h,2h}(\alpha, \beta)$ and it does not change the spectral radius

$$\sigma\left(\breve{E}_{h,2h}\,(\alpha, \beta)\right) = \sigma_{h,2h}(\alpha, \beta), \tag{32}$$

where $\breve{E}_{h,2h} = V_h^{-1}{}_{h,2h}V_h$.

The Analysis of operator $_{h,2h}$ can be seen as a two-grid analysis so we transform into block diagonal matrix as given that

$$\breve{E}_{h,2h} = h, 2h1h, 2h2h, 2h(n2-1)^{\frac{n}{2}}_{h,2h} \tag{33}$$

With (2×2) of matrices $E^j_{h,2h} for 1 \leq j \leq \left(\frac{n}{2} - 1\right)$ (1×1)(34) matrix of $^{\frac{n}{2}}_{h,2h}$

The resolution of spectral radius $\sigma(^j_{h,2h})$ can be decreased the by computation of the spectral radii of (at most) $2 \times 2)$ matrices

$$\sigma\left(_{h,2h}\right) = \sigma\left(E_{h,2h}\right) = max^j_{h,2h} \ \text{ for } \ 1 \leq j \leq \frac{n}{2} \tag{35}$$

for the purpose of verifing Eq. (33) and (34), transform to the iteration matrix

$$_{h,2h} = D_h M_h^{-1} A_h \tag{36}$$

with

$$D_h = I_h - I^h_{2h} A^{-1}_{2h} I^{2h}_h A_h$$

into

$$\breve{E}_{h,2h} = \left(I_h - \breve{I}^h_{2h} \breve{A}^{-1}_{2h} \breve{I}^{2h}_h \breve{A}_h\right) \tag{37}$$

and

$$\left(M_h^{-1} A_h\right)$$

with

$$\breve{A}_h = V_h^{-1} A_h V_h,$$

and

$$\left(M_h^{-1} A_h\right) = V_h^{-1}\left(M_h^{-1} A_h\right) V_h($$

Then explain the quantities

$$\begin{aligned} \breve{I}^h_{2h} &= V_h^{-1} I^h_{2h} V_{2h}, \\ \breve{A}_{2h} &= V^{-1}_{2h} A_{2h} V_{2h}, \\ \breve{I}^{2h}_h = V^{-1}_{2h} I^{2h}_h V_h \end{aligned} \tag{38}$$

Again, establish Fourier transformation at level 2h. So the Eq. (12) will be reproduced as

$$\psi\,(ih)^j_{2h} = sin\,(2j\pi ih)^{\frac{n}{2}-1}_{i=1} \quad j = 1, 2, \ldots, \frac{n}{2} - 1. \tag{39}$$

Equivalent to (31), we obtained matrix V_{2h},

$$V_{2h} = [\psi^1_{2h},\ \psi^2_{2h}, \psi^3_{2h}\ , \ \ldots\ ,\ \psi^{\frac{n}{2}-1}_{2h}]. \tag{40}$$

We know that continuous spectrum of Helmholtz operator A_h given that (12) and (13) with $\lambda_h(A_h) = \frac{1}{h^2}\left[4sin^2\frac{(j\pi h)}{2} - l^2\right]$.For convenience, we introduce

$$\begin{aligned} s_j &= sin^2\frac{(j\pi h)}{2}, \\ c_j &= cos^2\frac{(j\pi h)}{2} \end{aligned} \tag{41}$$

Notice that $\lambda_h^{n-j} = \frac{1}{h^2}\left[4s_{n-j} - l^2\right] = \frac{1}{h^2}\left[4c_j - l^2\right]$.
Thus, we obtained block diagonal form of A_h is

$$\breve{A}_h = V_h^{-1}A_hV_h = \begin{bmatrix} A_h^1 & & & & \\ & A_h^2 & & & \\ & & \ddots & & \\ & & & A_h^{(\frac{n}{2}-1)} & \\ & & & & A_h^{\frac{n}{2}} \end{bmatrix} \tag{42}$$

Then diagonal block (2×2) of A_h^j is

$$A_h^j = \frac{1}{h^2}\begin{bmatrix} 4s_j - l^2 & \\ & 4c_j - l^2 \end{bmatrix} for 1 \leq j \leq \frac{n}{2} - 1 \tag{43}$$

and

$$A_h^{\frac{n}{2}} = \frac{2}{h^2} - l^2.$$

The Eq. (14) can be rewrite $\lambda_h\left(M_h\right) = \frac{1}{h^2}\left[4s_j - l^2(\alpha - i\beta)\right]$ for block diagonal form of $M_h(\alpha, \beta)$ is

$$\breve{M}_h(\alpha, \beta) = V_h^{-1}M_h(\alpha, \beta)V_h = \begin{bmatrix} M_h^1(\alpha, \beta) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & M_h^{\frac{n}{2}}(\alpha, \beta) \end{bmatrix}, \tag{44}$$

with the blocks

$$M_h^j(\alpha, \beta) = \frac{1}{h^2}\begin{bmatrix} 4s_j - l^2(\alpha - i\beta) & \\ & 4c_j - l^2(\alpha - i\beta) \end{bmatrix} for 1 \leq j \leq \frac{n}{2} - 1 \tag{45}$$

and

$$M_h^{\frac{n}{2}}(\alpha, \beta) = \frac{2}{h^2} - l^2(\alpha - i\beta)$$

Also, the Eq. (16) can be rewritten as

$$\lambda_h\left(M_h^{-1}(\alpha, \beta)A_h\right) = \frac{4s_j - l^2}{4s_j - l^2(\alpha - i\beta)}.$$

The block diagonal form of $M_h^{-1}(\alpha, \beta)A_h$ is

$$M_h^{-1}(\alpha, \beta)A_h = V_h^{-1}M_h^{-1}(\alpha, \beta)A_hV_h = \begin{bmatrix} (M_h^{-1}(\alpha, \beta)A_h)^1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & (M_h^{-1}(\alpha, \beta)A_h)^{\frac{n}{2}} \end{bmatrix}, \tag{46}$$

with the blocks,

$$M_h^{-1}(\alpha,\beta)A_h=\frac{1}{h^2}\begin{bmatrix}\frac{4s_j-l^2}{4s_j-l^2(\alpha-i\beta)} & \\ & \frac{4c_j-l^2}{4c_j-l^2(\alpha-i\beta)}\end{bmatrix}\quad \text{for } 1\le j\le \frac{n}{2}-1 \tag{47}$$

and

$$M_h^{-1}(\alpha,\beta)A_h=\frac{2-l^2}{2-l^2(\alpha-i\beta)}.($$

For$A_{2h}\psi_{2h}(x)=\lambda_{2h}\psi_{2h}(x)$ with $\lambda_{2h}=\frac{1}{4h^2}\left[4sin^2(j\pi h)-l^2\right]$ and $sin^2(j\pi h)=4s_jc_j$, so we get diagonal matrix, which is

$$\breve{A}_{2h}=V_{2h}^{-1}A_{2h}V_{2h}=\begin{bmatrix}A_{2h}^1 & & & \\ & A_{2h}^2 & & \\ & & \ddots & \\ & & & A_{2h}^{(\frac{n}{2}-1)}\end{bmatrix} \tag{48}$$

with $A_{2h}^j=\frac{1}{4h^2}\left[4d_j-l^2\right]$.

Then we shall study restriction operator I_h^{2h} and prolongation operator I_{2h}^h are defined in Eq. (21) and (24) respectively. So, the restriction operator I_h^{2h} is defined as follows

$$I_h^{2h}r_h=\frac{1}{4}r_h(x-h)+\frac{1}{2}r_h(x)+\frac{1}{4}r_h(x+h)\,for x\epsilon\Omega_{2h}.$$

We obtain

$$\begin{aligned}I_h^{2h}(sinj\pi x)&=\left[\sin(j\pi(x-h)))+2sin(j\pi x)+\sin(j\pi(x+h))\right]/4I_h^{2h}(sinj\pi x)\\&=[1+cos(jh)]sin(jx)2I_h^{2h}(sinj\pi x)=[2cos^2(jh)2]sin(\frac{j\pi x}{2})I_h^{2h}(sinj\pi x)\\&=c_j\sin(j\pi x),\end{aligned}$$

hence

$$I_h^{2h}\psi_h^j=\frac{1}{2}c_j\psi_{2h}^j \tag{49}$$

This recognition applies to all j, so, substitute $n-j$ for j,

$$I_h^{2h}\psi_h^{n-j}=\frac{1}{2}c_{n-j}\psi_{2h}^{n-j}=\frac{1}{2}s_j\psi_{2h}^{n-j}. \tag{50}$$

As $0\le j\le\frac{n}{2} for the equality\left(sin2j\pi ih\right)=-\sin(2(n-j)\pi ih)$ leads to

$$\psi_{2h}^{n-j}=-\psi_{2h}^j. \tag{51}$$

So, we obtain

$$\psi_{2h}^{n-j}=-\frac{1}{2}s_j\psi_{2h}^j \tag{52}$$

Then following expression obtained from Eq. (51) to (52)

$$I_h^{2h}=V_{2h}^{-1}I_h^{2h}V_h=\begin{bmatrix}q_h^1 & & & & 0\\ & q_h^2 & & & 0\\ & & \ddots & & \vdots\\ & & & q_h^{(\frac{n}{2}-1)} & 0\end{bmatrix} \tag{53}$$

with $q^j = \frac{1}{2}\begin{bmatrix} c_j & -s_j \end{bmatrix}$ for $1 \leq j \leq \frac{n}{2} - 1$(54). And as we see that, in Eq. (53), the last column is zero, so this pattern proceeds from $I_h^{2h}\psi_h^{n2} = 0$. And prolongation operator I_{2h}^h defined as in Eq. (24) associated with restriction operator in this way $I_{2h}^h = \left(I_h^{2h}\right)^T$

$$
\begin{aligned}
I_{2h}^h &= V_h^{-1} I_{2h}^h V_{2h} && (54)\\
&= V_h^{-1}\left(I_h^{2h}\right)^T V_{2h} && (55)\\
&= V_h^T\left(I_h^{2h}\right)^T V_{2h} && (56)\\
&= \left(V_{2h}^T I_h^{2h} V_h\right)^T && (57)\\
&= \left(I_h^{2h}\right)^T && (58)
\end{aligned}
$$

Then transposing Eq. (53), we obtain I_{2h}^h

$$
I_{2h}^h = V_h^{-1} I_{2h}^h V_{2h} = \begin{bmatrix} p_h^1 & & & \\ & p_h^2 & & \\ & & \ddots & \\ & & & p_h^{(\frac{n}{2}-1)} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad (59)
$$

with

$$
p^j = \frac{1}{2}\begin{bmatrix} c_j \\ -s_j \end{bmatrix} \quad \text{for} \quad 1 \leq j \leq \frac{n}{2} - 1 \qquad (60)
$$

Therefore, all parts of (36) have a block diagonal structure this follows that verify the block diagonal structure of deflated-preconditioned operator $_{h,2h}(\alpha, \beta) inEq.(33)and(34).So, theblocksof^j_{h,2h}(2 \times 2)$ for $1 \leq j \leq n2 - 1\$and^{\frac{n}{2}(11)}_{h,2h}$, and Eq. (43), (47), (48), (54) and (57) give the following expression

$$
{}^j_{h,2h} = I - p^j\left(A_{2h}^j\right)^{-1} q^j A_h^j M_h^{-1}(\alpha, \beta) A_h \qquad (61)
$$

for $1 \leq j \leq \frac{n}{2} - 1$ This can be written in simplified form as

$$
{}^{\frac{n}{2}}_{h,2h} = \frac{2 - l^2}{2 - l^2(\alpha - \iota\beta)}.
$$

Substituting the expression of p^j, A_{2h}^j, q^j ,A_h^j and b_h^j in Eq. (58) then we obtain

$$
{}^j_{h,2h} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} c_j \\ -s_j \end{bmatrix} \frac{4h^2}{16 s_j c_j - l^2} \frac{1}{2}\begin{bmatrix} c_j & -s_j \end{bmatrix} \frac{1}{h^2}\begin{bmatrix} 4s_j - l^2 & \\ & 4c_j - l^2 \end{bmatrix}\right) M_h^{-1}(\alpha, \beta) A_h
$$

Simplying above equation, by hand or by mapple, we get the following equation

$$
{}^j_{h,2h} = \begin{bmatrix} \frac{16 s_j c_j - l^2 - 4 s_j c_j c_j - l^2 c_j c_j}{16 s_j c_j - l^2} & \frac{4 c_j s_j c_j - c_j s_j l^2}{16 s_j c_j - l^2} \\ \frac{4 s_j c_j s_j - l^2 s_j c_j}{16 s_j c_j - l^2} & \frac{16 s_j c_j - l^2 - 4 s_j c_j s_j - s_j s_j l^2}{16 s_j c_j - l^2} \end{bmatrix} \begin{bmatrix} \frac{4 s_j - l^2}{h^2(4 s_j - l^2(\alpha - i\beta))} & \\ & \frac{4 c_j - l^2}{h^2(4 c_j - l^2(\alpha - i\beta))} \end{bmatrix}
$$

and for convenience, we denote last equation as

$$ {}^{j}_{h,2h} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{62} $$

where

$$ a = \frac{(4s_j - l^2)16s_jc_j - (4s_j - l^2)l^2}{h^2(16s_jc_j - l^2)(4s_j - l^2(\alpha - i\beta))} - \frac{(4s_j - l^2)4s_jc_jc_j + (4s_j - l^2)l^2c_jc_j)}{h^2(16s_jc_j - l^2)(4s_j - l^2(\alpha - i\beta))} $$

$$ b = \frac{(4c_j - l^2)4c_js_jc_j - (4c_j - l^2)c_js_jl^2}{h^2(16s_jc_j - l^2)(4c_j - l^2(\alpha - i\beta))} $$

$$ c = \frac{(4s_j - l^2)4s_jc_js_j - (4s_j - l^2)l^2s_jc_j}{h^2(16s_jc_j - l^2)(4s_j - l^2(\alpha - i\beta))} $$

$$ d = \frac{(4c_j - l^2)16s_jc_j - (4c_j - l^2)l^2}{h^2(16s_jc_j - l^2)(4c_j - l^2(\alpha - i\beta))} - \frac{((4c_j - l^2)4s_jc_js_j + (4c_j - l^2)s_js_jl^2)}{h^2(16s_jc_j - l^2)(4c_j - l^2(\alpha - i\beta))} $$

4 Results

In this section we will manipulate the above mentioned eigenvalues of Helmholtz Equation and spectrum of CSLP over Helmholtz operator. Figures 1 and 2 we study the eigenvalues of Helmholtz equation with changing the wave number $k = 10$ and $k = 100$ and we see in both plots increasing the number of negative eigenvalues against increasing the wave number.

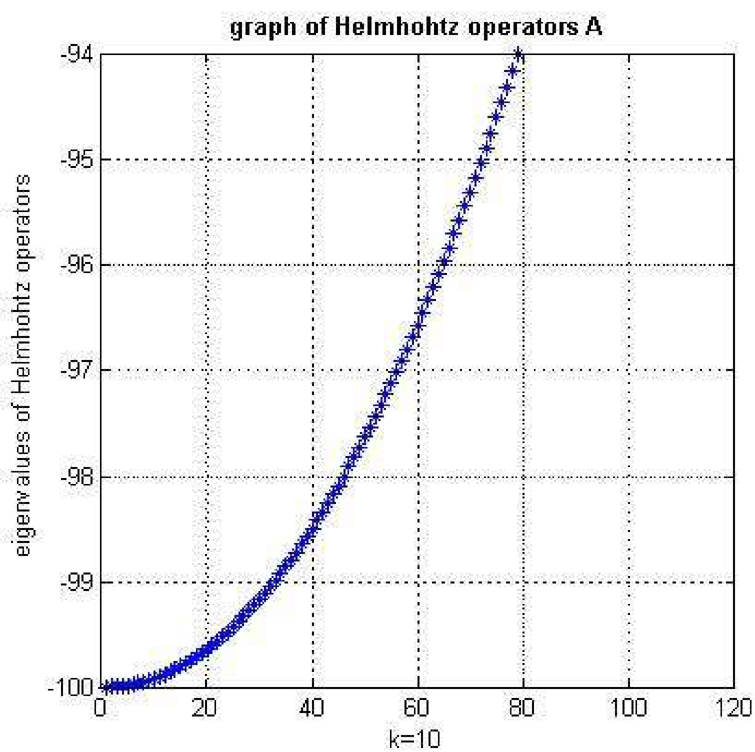

Figure 1. Eigen values of the 1D Helmholtz equation with wave number $k = 10$

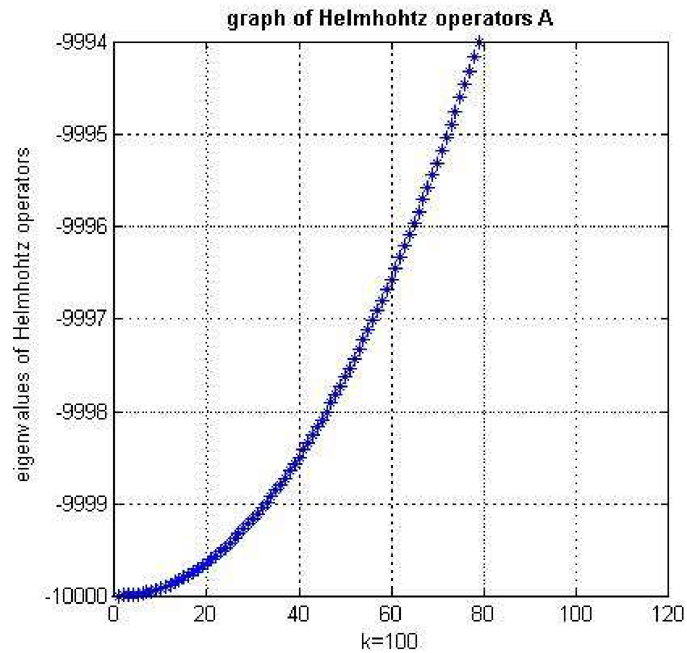

Figure 2. Eigen values of the 1D Helmholtz equation with wave number $k = 100$

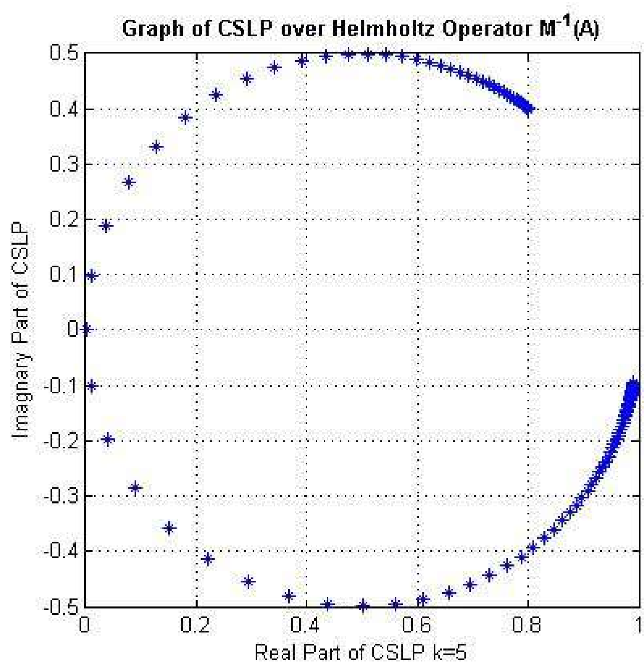

Figure 3. Spectrum of CSLP with shifts $(\alpha, \beta) = (1, 0.5)$ for wave number $k = 5$

At the beginning of the development of CSLP, only an imaginary shift was included. With the Laplace operator described in[22]. It has been observed later that CSLP with real and imaginary shifts reduced the number of iterations of the global method. If we say in simple words that increasing the imaginary part, the CSLP more attractive for multigrid approximation but this can be Spoiling the global convergence of krylov iterations so the real shift still preferred which causes outstanding limiting in global number of krylov iteration.The spectrum of CSLP is usually clustered in a complex plane close to one.Other spectral properties of CSLP also described in detail in [8], [23]

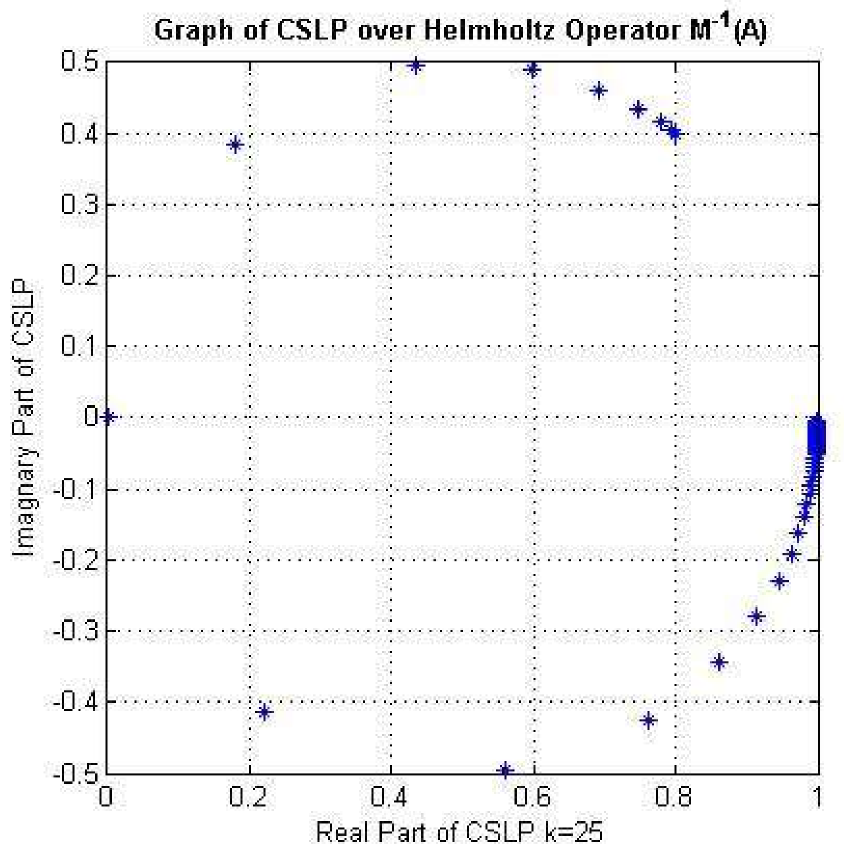

Figure 4. Spectrum of CSLP with shifts $(\alpha, \beta) = (1, 0.5)$ for wave number $k = 25$

We see that in figure 3-4 when wave number increases k=5 to k=25 the eigenvalues of CSLP near to zero these eigenvalues deteriorate the convergence of Krylov subspace method. so, deflation preconditioner is best option to rid the bad eigenvalues. In the rest of this section the expression (59) utilizes to study the behavior of spectrum of deflated preconditioner operator changing in wave number.

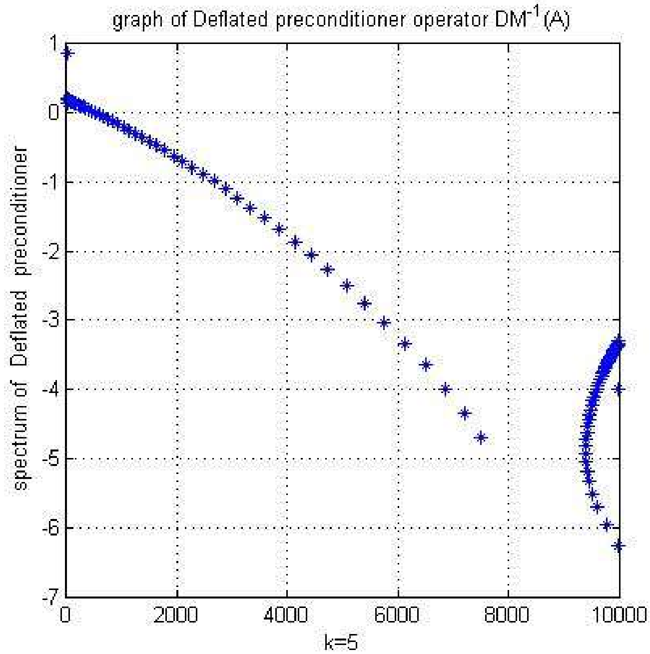

Figure 5. Spectrum of deflated preconditioner operator $^{j}_{h,2h}(\alpha = 1, \beta = 0.5)$ for wave number $k = 5$

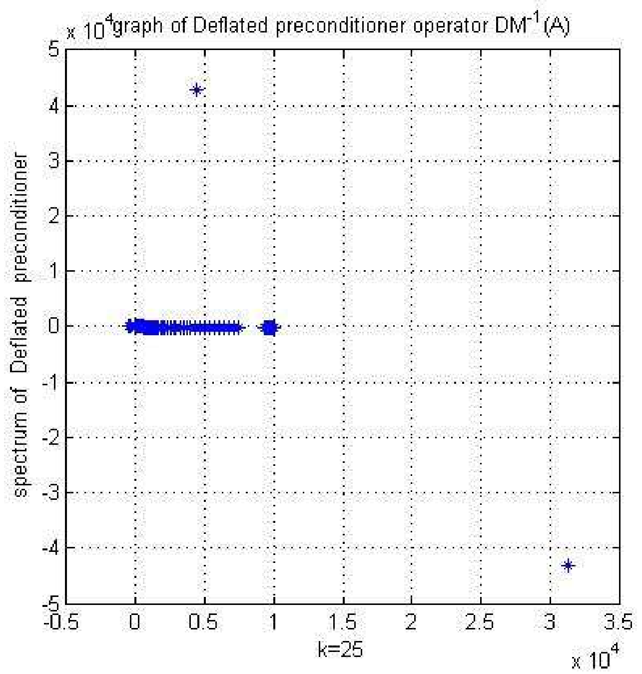

Figure 6. Spectrum of deflated preconditioner operator $^{j}_{h,2h}(\alpha = 1, \beta = 0.5)$ for wave number $k = 25$

5 Conclusions

In this article we have studied the one-dimensional Helmholtz equation with Dirichlet boundary condition. The preconditioner is required to apply Krylov subspace method on Helmholtz equation. We analyze the spectrum of CSLP preconditioner and this preconditioner found to be effective in improving the convergence of Krylov subspace method. The real and imaginary shifts have been settled as $(1, 0.5)$ and also perform rigorous Fourier analysis of two level preconditioner (CSLP with Deflation) of the Krylov subspace method for the Helmholtz equation.

Author Contributions

Rao Faisal Rajput: Methodology, Software, Writing- Original draft preparation. **A. H. Sheikh**: Conceptualization, Writing- Reviewing and Editing, Validation.**K. B. Amur**: Conceptualization, Writing-Reviewing and Editing, Supervision.

Compliance with Ethical Standards

It is declare that all authors don't have any conflict of interest. It is also declare that this article does not contain any studies with human participants or animals performed by any of the authors. Furthermore, informed consent was obtained from all individual participants included in the study.

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