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Solution of Time-Fractional Third-Order Partial Differential Equations of One and Higher Dimensions

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Abstract The purpose of this study is to develop the third order time fractional partial differential equations (PDEs) in one and higher dimensions, by taking Laplace Adomian decomposition method (LADM) and q-homotopy analysis transform method (q-HATM). To define fractional derivative, the Caputo operator is used for both fractional and integer orders. The solutions are obtained in the form of series. To understand the procedure of the suggested procedure, three numerical examples are taken. The graphs are plotted for the proposed solution at different values of fractional order γ which is 0<γ1. Both proposed methods are implemented by using (LADM) and (q-HATM) showing that the proposed technique is found to be better and accurate instrument for solving linear and non-linear time fractional PDEs. The Novelty of the proposed study is that the provided solution for fractional order partial differential equations has never been attempted for third order, this means that the provided solution can solve the third order and could be generalized for the higher order also.

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1. Introduction

In recent years, the time fractional calculus has played an important role in the areas of applied mathematics. Many scientists played an important role in fractional calculus such as L' Hospital, Wallis, Bernoulli, Leibniz, Riemann Liouville and Euler and Wallis. There are different applications of Fractional calculus in the field of Science Engineering, such as, robotic technology, viscoelasticity, signal processing, damping, fluid mechanics, electro-chemistry, telecommunication and biomedical, electro-magnetism, etc. The fractional derivative as seen as Caputo sense. Many authors are used different technique in the field of fractional partial differential equations, such as, Laplace transform method, homotopy analysis method, Adomian decomposition method, fractional reduced differential transform method, Burger equation, Schörodinger linear and non-linear equations, Variational iteration method, Lagrange polynomials, Volterra integro-differential equations, Telegraph equation etc.

There are different methods to solve the fractional differential equations [1-5], analytically and numerically. To solve the linear and nonlinear integral and integro-differential equations Adomian has developed the Adomian decomposition method (ADM). That use the series form of solution. The Laplace Adomian decomposition method (LADM) is a combine form of Adomian decomposition method and Laplace transform, considered as a powerful method that have been developed so far. In [6] an application of the Laplace Adomian decomposition method is given as a solution of system of delay differential equations with initial value problem. Some recent results for nonlinear equations are provided by [7] by using the mathematical and computer modeling. Analytical solution of Third Order Dispersive Fractional Partial Differential Equations is shown by [8] as an application of LADM. The solution of time space nonlinear fractional differential equations was presented by [9] that have shown the comparison between Laplace decomposition method and Adomian decomposition method. An analytic study on the third order dispersive partial differential equations was presented by [11]. In [13] an analytical and approximated solutions of space-time fractional telegraph equations Via Laplace transform are given. Lio in [14] has first introduced the Homotopy analysis method (HAM) and applied homotopy-analysis based technique for the solution of nonlinear problems also, the introduction to the Homotopy Analysis Method was explained in [15]. Based on the homotopy analysis method [17] introduced the technique for nonlinear problems. An application of q- Homotopy Analysis Method as Fingero-Imbibition phenomena in double phase flow through porous media was shown by [18]. The q-homotopy analysis transform method (q-HATM) along with the Laplace transform and q-homotopy analysis method (q-HAM) were combined in [19]. This method has been a very useful method for solving analytical solutions of one and higher dimensions, as fractional telegraph equations and fractional physical problems. q-homotopy analysis transform method for space and time-fractional Kdv-Burgers equation.

One of the applications of fractional order partial differential equation is telegraphic equations. Different researchers have put their efforts in order to give numerical and analytical methods such as [20] gives the homotopy analysis method (HAM), also [21] presents the solution by using Legendre polynomial and block-pulse function, [22] highlighted this issue with the help of radial basis function (RBF), in [23] the Chebyshev tau methods has been used to address the above mentioned issue, [24] formulates the solution by Haar-inc collocation method. Adomian decomposition method (ADM) is being used by [25] to solve fractional order partial differential equations which was further improvised by [26] by the name of Laplace decomposition method, fictitious time decomposition method (FTIM) developed by [27] to solve the telegraphic equations. In comparison with all above mentioned techniques natural transformation decomposition method (NTDM) with is the combination of Adomian decomposition methods and natural transformation methods provides best results to solve nonlinear fractional order partial differential equation in a very simple way [28, 29]. With the help of (NTDM) several number of physical problems has been solve so far such as [30] solve fractional telegraph equation, [31] effectively solve the fractional-order PDE's with proportional delay, [32] solve the non-linear PDEs, also system of polytropic gas is solved for the fractional uncertain flow is given by [33], the solution of physical models and fractional-order heat and wave equation is given by [34] and [35] respectively, diffusion equations are solved by [36]. Based on the above discussion the aim of the proposed study is to develop a non-complex solution for solving time fractional third order equation of higher dimensions which is more accurate compared with other existing techniques.

The main contribution of this study is to apply LADM and q-HATM for solving time fractional third order nonlinear PDE's. This method shows an analytical series solution of PDE's in the form of polynomial to obtain the near accurate result or near exact solutions for PDE's [37-38]. The uniqueness of the suggested technique is, that it can solve third order time fractional PDE's for n dimensions, which has never been highlighted before and by using this technique, more than third order can also be solved. Proposed methodology is easy way to apply in various fields of engineering and science such as telegraphic equation can easily be solved by using this approach. Following are the equations with the consideration of fractional nonlinear nonhomogeneous PDE's of one and higher dimensions.

One-dimension time fractional third order PDE

$$\frac{\partial^{3\gamma}\varphi_{(x,t)}}{\partial t^{3\gamma}} + a\frac{\partial^{2\gamma}\varphi_{(x,t)}}{\partial t^{2\gamma}} + b\frac{\partial^{\gamma}\varphi_{(x,t)}}{\partial t^{\gamma}} + c\varphi_{(x,t)} = d\frac{\partial^{3}\varphi_{(x,t)}}{\partial x^{3}}\ 0 < \gamma \leq 1, \quad (1)$$

with initial and boundary conditions

$$\varphi_{(x,0)} = \zeta_1(x), \quad \varphi_{t(x,0)} = \zeta_2(x), \quad \varphi_{tt(x,0)} = \zeta_3(x).$$

Two-dimensions time fractional third order PDE

$$\frac{\partial^{3\gamma}\varphi_{(x,y,t)}}{\partial t^{3\gamma}} + a\frac{\partial^{2\gamma}\varphi_{(x,y,t)}}{\partial t^{2\gamma}} + b\frac{\partial^{\gamma}\varphi_{(x,y,t)}}{\partial t^{\gamma}} + c\varphi_{(x,t)} = d\frac{\partial^{3}\varphi_{(x,y,t)}}{\partial x^{3}} + e\frac{\partial^{3}\varphi_{(x,y,t)}}{\partial y^{3}} \qquad 0 < \gamma \leq 1, \quad (2)$$

with initial and boundary conditions

$$\varphi_{(x,y,0)} = \zeta_1(x,y), \qquad \varphi_{t(x,y,0)} = \zeta_2(x,y), \qquad \varphi_{tt(x,y,0)} = \zeta_3(x,y).$$

Three-dimensions time fractional third order PDE

$$\frac{\partial^{3\gamma}\varphi_{(x,y,z,t)}}{\partial t^{3\gamma}} + a\frac{\partial^{2\gamma}\varphi_{(x,y,z,t)}}{\partial t^{2\gamma}} + b\frac{\partial^{\gamma}\varphi_{(x,y,z,t)}}{\partial t^{\gamma}} + c\varphi_{(x,t)} = d\frac{\partial^{3}\varphi_{(x,y,z,t)}}{\partial x^{3}} + e\frac{\partial^{3}\varphi_{(x,y,z,t)}}{\partial y^{3}} + f\frac{\partial^{3}\varphi_{(x,y,z,t)}}{\partial z^{3}}, \qquad 0 < \gamma \leq 1, \quad (3)$$

with initial and boundary conditions

$$\varphi_{(x,y,z,0)} = \zeta_1(x,y,z), \quad \varphi_{t(x,y,z,0)} = \zeta_2(x,y,z), \quad \varphi_{tt(x,y,z,0)} = \zeta_3(x,y,z)$$

where a, b, c, d, e, f are constants.

2. Definitions and Preliminary Concepts

Basic definitions and preliminaries that will be used in this study is going to be recall in this section along with some properties of fractional calculus,

Definition 1. The left sided Riemann-Liouville fractional integral of order $\gamma > 0$ of a function $f \epsilon C_{\mu,}\ \mu \geq -1$ is define as [1-2]:

$$J^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{1-\gamma}},$$

$$J^0 f(t) = f(t).$$

Where Γ denotes the gamma function define as

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

Definition 2. The left sided Caputo fractional derivative of f,

$f \epsilon C_{-1}^{m}, m \in N\ U\{0\}$

$$D_t^{\gamma} f(t) = \begin{cases} \dfrac{d^n f(t)}{dt^n}, & \gamma = n\epsilon N, \\ \dfrac{1}{\Gamma(n-\gamma)} \displaystyle\int_0^t (t-\tau)^{n-\gamma-1} f^n(\tau) d\tau, & n-1 < \gamma < n, \ n\epsilon N. \end{cases}$$

if $(t) = t^{\alpha}$, we have

$$D_t^{\gamma} t^{\alpha} = \begin{cases} \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma+1)} t^{\alpha-\gamma}, & n-1 < \gamma \leq n, \ \alpha > n-1, \ \alpha\epsilon\mathbb{R}, \\ 0, & n-1 < \gamma \leq n, \ \alpha \leq n-1, \end{cases}$$

Definition 3. The Laplace transform of the continuous function $\varphi(t)$ is define as [2-3],

$\varphi(s) = L[\varphi(t)] = \int_0^{\infty} e^{-st} \varphi(t) dt$, where s is real or complex number.

Definition 4. The Laplace transform $L[\varphi(x,t)]$ of the Caputo fractional derivative is define as [4-5],

$L\left[D_t^{n\gamma} \varphi(x,t)\right] = s^{n\gamma} L[\varphi(x,t)] - \sum_{k=0}^{n-1} s^{n\gamma-k-1} \varphi^k(x,0), \ \ n-1 < n\gamma \leq n.$

where L is Laplace Operator.

3. Idea of Fractional Laplace Adomian Decomposition Method (Fladm)

In this discussed the solution of third order fractional partial differential equation by FLADM

$$D^{3\gamma} \varphi_{(x,t)} + p\varphi_{(x,t)} + q\varphi_{(x,t)} = r_{(x,t)} \tag{4}$$

$x, t \geq 0, \qquad\qquad m-1 < \gamma < m$

where,

$D^{3\gamma} = \frac{\partial^{3\gamma}}{\partial t^{3\gamma}}$ is the Caputo operator,

$p \ and \ q$ are linear and non-linear functions, r is the source function.

The initial condition is $\varphi_{(x,0)} = k_{(x)}$ (5)

Take equation (4), apply Laplace transformation, we get

$$\mathcal{L}\left[D^{3\gamma} \varphi_{(x,t)}\right] + \mathcal{L}[p\varphi_{(x,t)} + q\varphi_{(x,t)}] = \mathcal{L}[r_{(x,t)}] \tag{6}$$

according to definition (4), the fractional derivative in terms of Laplace transformation

$$\mathcal{L}\left[D^{3\gamma} \varphi_{(x,t)}\right] = S^{3\gamma} \mathcal{L}\left[\varphi_{(x,t)}\right] - \sum_{k=0}^{n-1} S^{3\gamma-1-k} \varphi_{(0)}^{(k)} \tag{7}$$

substituting equation (7) in equation (6), after simplifying, we get

$$S^{3\gamma} \mathcal{L}\left[\varphi_{(x,t)}\right] = \sum_{k=0}^{n-1} S^{3\gamma-1-k} \varphi_{(0)}^{(k)} + \mathcal{L}[r_{(x,t)}] - \mathcal{L}[p\varphi_{(x,t)} + q\varphi_{(x,t)}] \tag{8}$$

divide by $S^{3\gamma}$, we get

$$\mathcal{L}[\varphi_{(x,t)}] = \frac{1}{S^{3\gamma}}\sum_{k=0}^{n-1} S^{3\gamma-1-k}\varphi_{(0)}^{(k)} + \frac{1}{S^{3\gamma}}\mathcal{L}[r_{(x,t)}] - \frac{1}{S^{3\gamma}}\mathcal{L}[p\varphi_{(x,t)} + q\varphi_{(x,t)}] \quad (9)$$

The ADM solution $\varphi_{(x,t)}$ is represented by the following infinite series

$$\varphi_{(x,t)} = \sum_{i=0}^{\infty} \varphi_{i(x,t)} \quad (10)$$

the non-linear function (if any) in the problem is define by the infinite series of Adomian polynomial,

$$q\varphi_{(x,t)} = \sum_{i=0}^{\infty} B_i \quad (11)$$

$$B_i = \frac{1}{i!}\left[\frac{\partial^i}{\partial\lambda^i}\left[q\sum_{i=0}^{\infty}\mu^i\varphi_i\right]\right]_{\mu=0} \quad (12)$$

substituting equations (9) and (10) in equation (8), we get

$$\mathcal{L}\left[\sum_{i=0}^{\infty}\varphi_{i(x,t)}\right] = \frac{1}{S^{3\gamma}}\sum_{k=0}^{n-1} S^{3\gamma-1-k}\varphi_{(0)}^{(k)} + \frac{1}{S^{3\gamma}}\mathcal{L}[r_{(x,t)}] - \frac{1}{S^{3\gamma}}\mathcal{L}\left[p\sum_{i=0}^{\infty}\varphi_{i(x,t)} + \sum_{i=0}^{\infty}B_i\right] \quad (13)$$

using LADM, we have

$$\mathcal{L}[\varphi_{0(x,t)}] = \frac{1}{S^{3\gamma}}\sum_{k=0}^{n-1} S^{3\gamma-1-k}\varphi_{(0)}^{(k)} + \frac{1}{S^{3\gamma}}\mathcal{L}[r_{(x,t)}] \quad (14)$$

substituting equation (13) in equation (12), after simplification we get

$$\mathcal{L}\left[\sum_{i=0}^{\infty}\varphi_{i(x,t)}\right] - \mathcal{L}[\varphi_{0(x,t)}] = -\frac{1}{S^{3\gamma}}\mathcal{L}\left[p\sum_{i=0}^{\infty}\varphi_{i(x,t)} + \sum_{i=0}^{\infty}B_i\right] \quad (15)$$

$$\mathcal{L}[\varphi_{i+1(x,t)}] = -\frac{1}{S^{3\gamma}}\mathcal{L}\left[p\sum_{i=0}^{\infty}\varphi_{i(x,t)} + \sum_{i=0}^{\infty}B_i\right], \qquad i \geq 1 \quad (16)$$

applying inverse Laplace transformation in equations (13) and (15), we get

$$\varphi_{0(x,t)} = \mathcal{L}^{-1}\left[\frac{1}{S^{3\gamma}}\left[\sum_{k=0}^{n-1} S^{3\gamma-1-k}\varphi_{(0)}^{(k)} + \mathcal{L}[r_{(x,t)}]\right]\right] \quad (17)$$

$$\varphi_{i+1(x,t)} = -\mathcal{L}^{-1}\left[\frac{1}{S^{3\gamma}}\mathcal{L}\sum_{i=0}^{\infty}[p\varphi_{i(x,t)} + B_i]\right] \quad (18)$$

4. Idea of q-Homotopy Analysis Transform Method

This section will present the basic theory and solution procedure of q-HATM for non-linear partial differential equations. General form of fractional non-linear non-homogeneous third order differential equation is given as,

$$D_t^{3\gamma}\varphi_{(x,t)} + u\varphi_{(x,t)} + v\varphi_{(x,t)} = w_{(x,t)} \qquad n-1 < \gamma \leq n \quad (19)$$

where,

$D_t^{3\gamma}$ = third order fractional derivative due to Caputo

u = linear differential operator

v = non-linear differential operator

$w_{(x,t)}$ = Source term

Take equation (19), applying Laplace transform

$$\mathcal{L}[D_t^{3\gamma}\varphi_{(x,t)}]+\mathcal{L}[u\varphi_{(x,t)}]+\mathcal{L}[v\varphi_{(x,t)}]=\mathcal{L}[w_{(x,t)}] \qquad (20)$$

Substituting equation (7) in equation (20), after divide by $S^{3\gamma}$, we get

$$\mathcal{L}\left[\varphi_{(x,t)}\right]-\frac{1}{S^{3\gamma}}\sum_{k=0}^{n-1}S^{3\gamma-1-k}\varphi_{(x,0)}^{(k)}+\frac{1}{S^{3\gamma}}\{\mathcal{L}\left[u\varphi_{(x,t)}\right]+\mathcal{L}\left[v\varphi_{(x,t)}\right]\}-\mathcal{L}[w_{(x,t)}]\}=0 \qquad (21)$$

Non-linear operator is defined as,

$$v\left[\psi_{(x,t;q)}\right]=\mathcal{L}\left[\psi_{(x,t;q)}\right]-\frac{1}{S^{3\gamma}}\sum_{k=0}^{n-1}S^{3\gamma-1-k}\psi_{(x,t;q)}^{(k)}(0^{+})+\frac{1}{S^{3\gamma}}\{\mathcal{L}\left[u\psi_{(x,t;q)}\right]+\mathcal{L}\left[v\psi_{(x,t;q)}\right]-\mathcal{L}\left[w_{(x,t)}\right]\} \qquad (22)$$

where $q\epsilon[0,\frac{1}{n}]$ and $\psi_{(x,t;q)}$ is real function of x, t and q.

now construct a homotopy

$$(1-nq)\,\mathcal{L}\left[\psi_{(x,t;q)}-\varphi_{0(x,t)}\right]=hqH_{(x,t)}v[\psi_{(x,t;q)}] \qquad (23)$$

$n\geq 1$, is embedding parameter

$H_{(x,t)}=$ non-zero auxiliary function

$h=$ auxiliary parameter

$\varphi_{0(x,t)}=$ initial approximation of $\varphi_{(x,t)}$

$\psi_{(x,t;q)}=$ unknown function

for the embedding parameter $q=0$ and $q=\frac{1}{n}$

for the following result holds

$$\psi_{(x,t;0)}=\varphi_{0(x,t)}$$

$$\psi_{\left(x,t;\frac{1}{n}\right)}=\varphi_{(x,t)} \qquad (24)$$

consequently, as q increase from 0 to $\frac{1}{n}$ the solution $\psi_{(x,t;q)}$ transform from initial guess $\varphi_{0(x,t)}$ to the solution $\varphi_{(x,t)}$.

now expending the function $\psi_{(x,t;q)}$ in series form by applying Taylor's theorem about q, we have

$$\psi_{(x,t;q)}=\psi_{(x,t;0)}+\frac{q}{1!}\frac{\partial\psi_{(x,t;0)}}{\partial q}+\frac{q^2}{2!}\frac{\partial^2\psi_{(x,t;0)}}{\partial q^2}+\frac{q^3}{3!}\frac{\partial^3\psi_{(x,t;0)}}{\partial q^3}+\cdots+\frac{q^m}{m!}\frac{\partial^m\psi_{(x,t;0)}}{\partial q^m} \qquad (25)$$

where $m=m^{th}$ derivative

$$\varphi_{m(x,t)}=\frac{1}{m!}\frac{\partial^m\psi_{(x,t;q)}}{\partial q^m}\Big|_{q=0} \qquad (26)$$

substituting equations (24) and (26) in equation (25), we get

$$\psi_{(x,t;q)}=\varphi_{0(x,t)}+\sum_{m=1}^{\infty}\varphi_{m(x,t)}\,q^m \qquad (27)$$

series (27) converges at $q=\frac{1}{n}$

$$\psi_{(x,t;q)} = \varphi_{0(x,t)} + \sum_{m=1}^{\infty} \varphi_{m(x,t)} \left(\frac{1}{n}\right)^m \quad (28)$$

where h is the auxiliary parameter, with the initial guess $\varphi_{0(x,t)}$ and the properly chosen asymptotic parameter n. m-th order deformation equation is obtained by differentiating equation (23) m-times with respect to q, and after simplification, we get

$$\frac{\partial^m}{\partial q^m} \mathcal{L}\left[\psi_{(x,t;q)} - \varphi_{0(x,t)}\right] - n\frac{\partial^m}{\partial q^m} q\mathcal{L}\left[\psi_{(x,t;q)} - \varphi_{0(x,t)}\right] = h\frac{\partial^m}{\partial q^m} qv\left[\psi_{(x,t;q)}\right] \quad (29)$$

$$K_m = \begin{cases} 0 & m \leq 1 \\ n & m > 1 \end{cases} \quad (30)$$

substituting equations (26) and (30) in equation (29), we get

$$\mathcal{L}\left[\varphi_{m(x,t)} - K_m \varphi_{m-1(x,t)}\right] = h\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}} v\left[\psi_{(x,t;q)}\right] \quad (31)$$

$$\Re_m(\bar{\varphi}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}} v\left[\psi_{(x,t;q)}\right]|_{q=0} \quad (32)$$

substituting equation (32) in equation (31), we get

$$\varphi_{m(x,t)} = K_m \varphi_{m-1(x,t)} = hH_{(x,t)}\mathcal{L}^{-1}[\,\Re_m(\bar{\varphi}_{m-1})] \quad (33)$$

5. Numerical Examples

To show the performance of the proposed techniques three examples of one, two and three dimensions is taken and their solution by LADM and q-HATM are as follows,

Example 1. Consider the following Third order Time FPDEQ in one dimension by Solving LADM and q-HATM.

$$\frac{\partial^{3\gamma}\varphi_{(x,t)}}{\partial t^{3\gamma}} + 3\frac{\partial^{2\gamma}\varphi_{(x,t)}}{\partial t^{2\gamma}} + 3\frac{\partial^{\gamma}\varphi_{(x,t)}}{\partial t^{\gamma}} + \varphi_{(x,t)} = \frac{\partial^3\varphi_{(x,t)}}{\partial x^3}, \quad 0 < \gamma \leq 1, \qquad t \geq 0 \quad (34)$$

with initial condition

$$\varphi_{(x,0)} = e^x, \qquad \varphi_{t(x,0)} = -2e^x, \qquad \varphi_{tt(x,0)} = e^x \quad (35)$$

Solution:

taking equation (34), apply Laplace transformation

$$\mathcal{L}\left[\frac{\partial^{3\gamma}\varphi_{(x,t)}}{\partial t^{3\gamma}}\right] = -3\mathcal{L}\left[\frac{\partial^{2\gamma}\varphi_{(x,t)}}{\partial t^{2\gamma}}\right] - 3\mathcal{L}\left[\frac{\partial^{\gamma}\varphi_{(x,t)}}{\partial t^{\gamma}}\right] - \mathcal{L}\left[\varphi_{(x,t)}\right] + \mathcal{L}\left[\frac{\partial^3\varphi_{(x,t)}}{\partial x^3}\right] \quad (36)$$

substituting equation (7) in Equation (36), we get

$$S^{3\gamma}\mathcal{L}\left[\varphi_{(x,t)}\right] - S^{3\gamma-1}\varphi_{(x,0)} - S^{3\gamma-2}\varphi_{t(x,0)} - S^{3\gamma-3}\varphi_{tt(x,0)} =$$

$$-3\mathcal{L}\left[\frac{\partial^{2\gamma}\varphi_{(x,t)}}{\partial t^{2\gamma}}\right] - 3\mathcal{L}\left[\frac{\partial^{\gamma}\varphi_{(x,t)}}{\partial t^{\gamma}}\right] - \mathcal{L}\left[\varphi_{(x,t)}\right] + \mathcal{L}\left[\frac{\partial^3\varphi_{(x,t)}}{\partial x^3}\right] \quad (37)$$

after simplifying and applying inverse Laplace transform, we get

$$\varphi_{(x,t)} = \mathcal{L}^{-1}\left[\frac{1}{S}\varphi_{(x,0)} + \frac{1}{S^2}\varphi_{t(x,0)} + \frac{1}{S^3}\varphi_{tt(x,0)}\right] + \mathcal{L}^{-1}[\frac{1}{S^{3\gamma}}\mathcal{L}\{-3\frac{\partial^{2\gamma}\varphi_{(x,t)}}{\partial t^{2\gamma}}$$

$$-3\frac{\partial^{\gamma}\varphi_{(x,t)}}{\partial t^{\gamma}} - \varphi_{(x,t)} + \frac{\partial^3\varphi_{(x,t)}}{\partial x^3}\}] \quad (38)$$

using ADM, we get

$$\varphi_{0(x,t)} = \mathcal{L}^{-1}\left[\frac{1}{S}\varphi_{(x,0)} + \frac{1}{S^2}\varphi_{t(x,0)} + \frac{1}{S^3}\varphi_{tt(x,0)}\right] \quad (39)$$

substituting equation (35) in equation (39), we get

$$\varphi_{\circ(x,t)} = e^x\mathcal{L}^{-1}\left[\frac{1}{S} + \frac{-2}{S^2} + \frac{1}{S^3}\right]$$

$$= e^x(1 - 2t + \tfrac{1}{2}t^2) \quad (40)$$

$$\varphi_{j+1(x,t)} = \mathcal{L}^{-1}[\tfrac{1}{S^{3\gamma}}\mathcal{L}\{-3\frac{\partial^{2\gamma}\varphi_{j(x,t)}}{\partial t^{2\gamma}} - 3\frac{\partial^{\gamma}\varphi_{j(x,t)}}{\partial t^{\gamma}} - \varphi_{j(x,t)} + \frac{\partial^3\varphi_{j(x,t)}}{\partial x^3}\}] \quad (41)$$

for j=0,1,2, we get

$$\varphi_{1(x,t)} = e^x(\tfrac{-3}{\Gamma(\gamma+3)}t^{\gamma+2} + \tfrac{6}{\Gamma(2\gamma+2)}t^{2\gamma+1} - \tfrac{3}{\Gamma(2\gamma+3)}t^{2\gamma+2}) \quad (42)$$

$$\varphi_{2(x,t)} = e^x(\tfrac{9}{\Gamma(2\gamma+3)}t^{2\gamma+2} - \tfrac{18}{\Gamma(3\gamma+2)}t^{3\gamma+1} + \tfrac{18}{\Gamma(3\gamma+3)}t^{3\gamma+2}$$

$$-\tfrac{18}{\Gamma(4\gamma+2)}t^{4\gamma+1} + \tfrac{9}{\Gamma(4\gamma+3)}t^{4\gamma+2}) \quad (43)$$

$$\varphi_{3(x,t)} = e^x(-\tfrac{27}{\Gamma(3\gamma+3)}t^{3\gamma+2} + \tfrac{54}{\Gamma(4\gamma+2)}t^{4\gamma+1} - \tfrac{81}{\Gamma(4\gamma+3)}t^{4\gamma+2}$$

$$+\tfrac{108}{\Gamma(5\gamma+2)}t^{5\gamma+1} - \tfrac{81}{\Gamma(5\gamma+3)}t^{5\gamma+2} + \tfrac{54}{\Gamma(6\gamma+2)}t^{6\gamma+1} - \tfrac{27}{\Gamma(6\gamma+3)}t^{6\gamma+2}). \quad (44)$$

substituting equations (40), (42), (43), (44) in equation (10), after simplify, we get

$$\varphi_{(x,t)} = e^x[\,1 - 2t + \tfrac{1}{2}t^2 - \tfrac{3}{\Gamma(\gamma+3)}t^{\gamma+2} + \tfrac{6}{\Gamma(2\gamma+2)}t^{2\gamma+1} + \tfrac{6}{\Gamma(2\gamma+3)}t^{2\gamma+2}$$

$$-\tfrac{18}{\Gamma(3\gamma+2)}t^{3\gamma+1} - \tfrac{9}{\Gamma(3\gamma+3)}t^{3\gamma+2} + \tfrac{36}{\Gamma(4\gamma+2)}t^{4\gamma+1} - \tfrac{72}{\Gamma(4\gamma+3)}t^{4\gamma+2}$$

$$+\tfrac{108}{\Gamma(5\gamma+2)}t^{5\gamma+1} - \tfrac{81}{\Gamma(5\gamma+3)}t^{5\gamma+2} + \tfrac{54}{\Gamma(6\gamma+2)}t^{6\gamma+1} - \tfrac{27}{\Gamma(6\gamma+3)}t^{6\gamma+2} \ldots]. \quad (45)$$

now we are solving equations (34) and (35) of example 1 by q-HATM

taking equation (36), substituting equations (7) and (35), after simplification, we get

$$\mathcal{L}[\varphi_{(x,t)}] - e^x\left(\tfrac{1}{S} - \tfrac{2}{S^2} + \tfrac{1}{S^3}\right) + \tfrac{1}{S^{3\gamma}}\mathcal{L}\left[3\frac{\partial^{2\gamma}\varphi_{(x,t)}}{\partial t^{2\gamma}} + 3\frac{\partial^{\gamma}\varphi_{(x,t)}}{\partial t^{\gamma}} + \varphi_{(x,t)} - \frac{\partial^3\varphi_{(x,t)}}{\partial x^3}\right] = 0 \quad (46)$$

using equation (22), define the non-linear operator, we get

$$N[\psi_{(x,t;q)}] = \mathcal{L}[\psi_{(x,t;q)}] - e^x\left(\frac{1}{S} - \frac{2}{S^2} + \frac{1}{S^3}\right)$$

$$+\tfrac{1}{S^{3\gamma}}\mathcal{L}\left[3\frac{\partial^{2\gamma}\psi_{(x,t;q)}}{\partial t^{2\gamma}} + 3\frac{\partial^{\gamma}\psi_{(x,t;q)}}{\partial t^{\gamma}} + \psi_{(x,t;q)} - \frac{\partial^3\psi_{(x,t;q)}}{\partial x^3}\right] \quad (47)$$

using equation (32) and equation (30), we get

$$\mathbb{R}_m(\bar{\varphi}_{m-1}) = \mathcal{L}[\varphi_{m-1(x,t)}] - (1-\frac{k_m}{n})e^x\left(\frac{1}{S}-\frac{2}{S^2}+\frac{1}{S^3}\right)$$

$$+\frac{1}{S^{3\gamma}}\mathcal{L}\left[3\frac{\partial^{2\gamma}\varphi_{m-1}}{\partial t^{2\gamma}}+3\frac{\partial^{\gamma}\varphi_{m-1}}{\partial t^{\gamma}}+\varphi_{m-1}-\frac{\partial^3\varphi_{m-1}}{\partial x^3}\right]=0 \quad (48)$$

for $m = 0$, equation (48) after solving, we get

$$\mathbb{R}_m(\bar{\varphi}_{m-1}) = -e^x\left(\frac{1}{S}-\frac{2}{S^2}+\frac{1}{S^3}\right) \quad (49)$$

substituting equation (49) in equation (32), after apply inverse Laplace transform, we get

$$\varphi_{0(x,t)} = -he^x(1-2t+\frac{1}{2}t^2) \quad (50)$$

for $m = 1$, equation (48) after solving, we get

$$\mathbb{R}_m(\bar{\varphi}_{m-1}) = -(h+1)e^x\left(\frac{1}{S}-\frac{2}{S^2}+\frac{1}{S^3}\right)-3he^x\frac{1}{S^{3\gamma}}[S^{2\gamma-3} \quad -2S^{\gamma-2}+S^{\gamma-3}]. \quad (51)$$

substituting equation (51) in equation (32), after apply inverse Laplace transform, we get

$$\varphi_{1(x,t)} = -h(h+1)e^x\left(1-2t+\frac{1}{2}t^2\right)\ 3h^2e^x\left(\frac{1}{\Gamma(\gamma+3)}t^{\gamma+2}+\frac{2}{\Gamma(2\gamma+2)}t^{2\gamma+1}-\frac{1}{\Gamma(2\gamma+3)}t^{2\gamma+2}\right). \quad (52)$$

for $m = 2$, equation (48) after solving, we get

$$\mathbb{R}_m(\bar{\varphi}_{m-1}) = -h(h+1)e^x\left(\frac{1}{S}-\frac{2}{S^2}+\frac{1}{S^3}\right)+3e^x[\frac{-h^2}{S^{\gamma+3}}+\frac{2h^2}{S^{2\gamma+2}}-\frac{h^2}{S^{2\gamma+3}}$$

$$-\frac{h(h+1)}{S^{\gamma+3}}-\frac{3h^2}{S^{2\gamma+3}}+\frac{6h^2}{S^{3\gamma+2}}-\frac{3h^2}{S^{3\gamma+3}}+\frac{2h(h+1)}{S^{2\gamma+2}}-\frac{h(h+1)}{S^{2\gamma+3}}$$

$$-\frac{3h^2}{S^{3\gamma+3}}+\frac{6h^2}{S^{4\gamma+2}}-\frac{3h^2}{S^{4\gamma+3}}]. \quad (53)$$

substituting equation (53) in equation (32), after applying inverse Laplace transform, we get

$$\varphi_{2(x,t)} = -h(n+h)(h+1)e^x\left(1-2t+\frac{1}{2}t^2\right)+3e^x[\frac{-h^2(n+2h+1)}{\Gamma(\gamma+3)}t^{\gamma+2}$$

$$+\frac{2h^2(n+2h+1)}{\Gamma(2\gamma+2)}t^{2\gamma+1}-\frac{h^2(n+5h+1)}{\Gamma(2\gamma+3)}t^{2\gamma+2}+\frac{6h^3}{\Gamma(3\gamma+2)}t^{3\gamma+1}$$

$$-\frac{6h^3}{\Gamma(3\gamma+3)}t^{3\gamma+2}+\frac{6h^3}{\Gamma(4\gamma+2)}t^{4\gamma+1}-\frac{3h^3}{\Gamma(4\gamma+3)}t^{4\gamma+2}]. \quad (54)$$

for $m = 3$, equation (48) after solving, we get

$$\mathbb{R}_m(\bar{\varphi}_{m-1}) = -h(n+h)(h+1)e^x\left(\frac{1}{S}-\frac{2}{S^2}+\frac{1}{S^3}\right)+3e^x[\frac{-h^2(n+2h+1)}{S^{\gamma+3}}$$

$$+\frac{2h^2(n+2h+1)}{S^{2\gamma+2}}-\frac{h^2(n+5h+1)}{S^{2\gamma+3}}+\frac{6h^3}{S^{3\gamma+2}}-\frac{6h^3}{S^{3\gamma+3}}+\frac{6h^3}{S^{4\gamma+2}}$$

$$-\frac{3h^3}{s^{4\gamma+3}}-\frac{h(n+h)(h+1)}{s^{\gamma+3}}-\frac{3h^2(n+2h+1)}{s^{2\gamma+3}}+\frac{6h^2(n+2h+1)}{s^{3\gamma+2}}\quad-\frac{3h^2(n+5h+1)}{s^{3\gamma+3}}+\frac{18h^3}{s^{4\gamma+2}}-\frac{18h^3}{s^{4\gamma+3}}+\frac{18h^3}{s^{5\gamma+2}}-\frac{9h^3}{s^{5\gamma+3}}+\frac{2h(n+h)(h+1)}{s^{2\gamma+2}}-$$
$$\frac{h(n+h)(h+1)}{s^{2\gamma+3}}-\frac{3h^2(n+2h+1)}{s^{3\gamma+3}}\quad+\frac{6h^2(n+2h+1)}{s^{4\gamma+2}}-\frac{3h^2(n+5h+1)}{s^{4\gamma+3}}+\frac{18h^3}{s^{5\gamma+2}}-\frac{18h^3}{s^{5\gamma+3}}+\frac{18h^3}{s^{6\gamma+2}}\quad-\frac{9h^3}{s^{6\gamma+3}}].$$

(55)

substituting equation (55) in equation (32), after apply inverse Laplace transform, we get

$$\varphi_{3(x,t)}=-h(n+h)^2(h+1)e^x\left(1-2t+\frac{1}{2}t^2\right)+3e^x[\frac{-h^2(n+h)(n+3h+2)}{\Gamma(\gamma+3)}t^{\gamma+2}$$
$$+\frac{2h^2(n+h)(n+3h+2)}{\Gamma(2\gamma+2)}t^{2\gamma+1}-\frac{h^2\{(n+h)(n+6h+2)+3h(n+2h+1)\}}{\Gamma(2\gamma+3)}t^{2\gamma+2}$$
$$+\frac{6h^3(2n+3h+1)}{\Gamma(3\gamma+2)}t^{3\gamma+1}-\frac{3h^3(4n+9h+2)}{\Gamma(3\gamma+3)}t^{3\gamma+2}+\frac{6h^3(2n+6h+1)}{\Gamma(4\gamma+2)}t^{4\gamma+1}$$
$$-\frac{3h^3(2n+12h+1)}{\Gamma(4\gamma+3)}t^{4\gamma+2}+\frac{36h^4}{\Gamma(5\gamma+2)}t^{5\gamma+1}-\frac{27h^4}{\Gamma(5\gamma+3)}t^{5\gamma+2}$$
$$+\frac{18h^4}{\Gamma(6\gamma+2)}t^{6\gamma+1}-\frac{9h^4}{\Gamma(6\gamma+3)}t^{6\gamma+2}]. \tag{56}$$

substituting equations (50), (52), (54) and (56) in equation (10), we get

$$\varphi_{(x,t)}=-h\{1+(h+1)+(n+h)(h+1)(n+h+1)\}e^x(1-2t+\frac{1}{2}t^2)$$
$$-\frac{3h^2\{1+(h+1)+(n+h)+(n+h)(n+3h+2)\}e^x}{\Gamma(\gamma+3)}t^{\gamma+2}$$
$$+\frac{6h^2\{1+(h+1)+(n+h)+(n+h)(n+3h+2)\}e^x}{\Gamma(2\gamma+2)}t^{2\gamma+1}$$
$$-\frac{3h^2\{2h+2(h+1)+(n+h)+(n+h)(n+6h+2)+3h(n+2h+1)\}e^x}{\Gamma(2\gamma+3)}t^{2\gamma+2}$$
$$+\frac{18h^3\{1+2(n+h)+(h+1)\}e^x}{\Gamma(3\gamma+2)}t^{3\gamma+1}-\frac{9h^3\{h+4(n+h)+4(h+1)\}e^x}{\Gamma(3\gamma+3)}t^{3\gamma+2}$$
$$+\frac{36h^3\{h+(n+h)+(h+1)\}e^x}{\Gamma(4\gamma+2)}t^{4\gamma+1}-\frac{18h^3\{4h+(n+h)+(h+1)\}e^x}{\Gamma(4\gamma+3)}t^{4\gamma+2}$$
$$+\frac{108h^4e^x}{\Gamma(5\gamma+2)}t^{5\gamma+1}-\frac{81h^4e^x}{\Gamma(5\gamma+3)}t^{5\gamma+2}+\frac{54h^4e^x}{\Gamma(6\gamma+2)}t^{6\gamma+1}-\frac{27h^4e^x}{\Gamma(6\gamma+3)}t^{6\gamma+2}]. \tag{57}$$

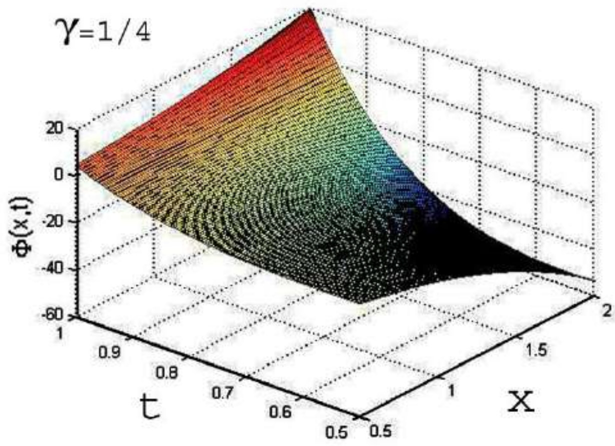

Figure 1(a). $\gamma = 0.25$ of example 1

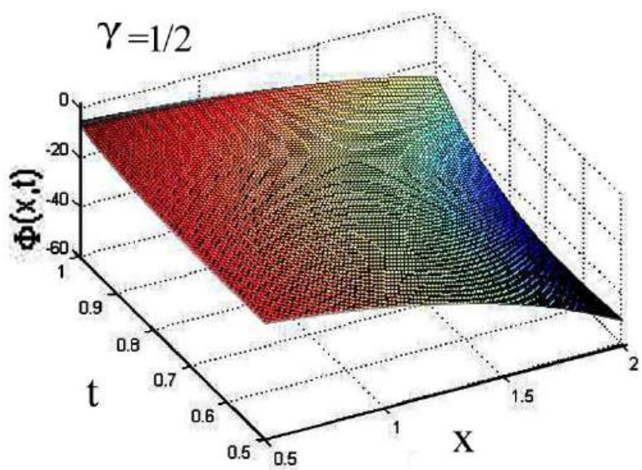

Figure 1(b). $\gamma = 0.50$ of example 1

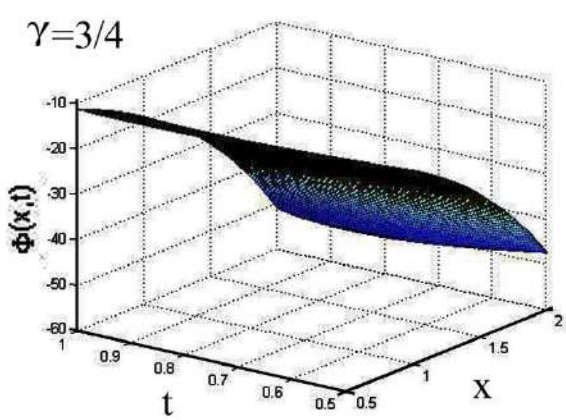

Figure1(c). $\gamma = 0.75$ of example 1

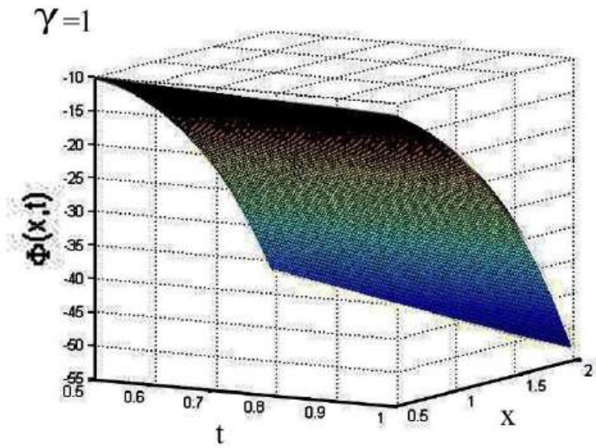

Figure1(d). $\gamma = 1.0$ of example 1

Example 2: Consider the following Third order time FPDE in two dimensions by solving LADM and q-HATM.

$$\frac{\partial^{3\gamma}\varphi_{(x,y,t)}}{\partial t^{3\gamma}} + 3\frac{\partial^{2\gamma}\varphi_{(x,y,t)}}{\partial t^{2\gamma}} + 2\frac{\partial^{\gamma}\varphi_{(x,y,t)}}{\partial t^{\gamma}} + \varphi_{(x,y,t)} = \frac{1}{2}\frac{\partial^3\varphi_{(x,y,t)}}{\partial x^3} + \frac{1}{2}\frac{\partial^3\varphi_{(x,y,t)}}{\partial y^3} \tag{58}$$

$$0 < \gamma \leq 1, \qquad t \geq 0$$

With initial conditions

$$\varphi_{(x,y,0)} = e^{x+y}, \quad \varphi_{t(x,y,0)} = -3e^{x+y}, \qquad \varphi_{tt(x,y,0)} = 2e^{x+y} \tag{59}$$

Solution:

Taking equation (58), apply Laplace transformation

$$\mathcal{L}\left[\frac{\partial^{3\gamma}\varphi_{(x,y,t)}}{\partial t^{3\gamma}}\right] = \mathcal{L}[-3\frac{\partial^{2\gamma}\varphi_{(x,y,t)}}{\partial t^{2\gamma}} - 2\frac{\partial^{\gamma}\varphi_{(x,y,t)}}{\partial t^{\gamma}} - \varphi_{(x,y,t)} + \frac{1}{2}\frac{\partial^3\varphi_{(x,y,t)}}{\partial x^3} + \frac{1}{2}\frac{\partial^3\varphi_{(x,y,t)}}{\partial y^3}] \tag{60}$$

Substituting equation (7) in equation (60), we get

$$S^{3\gamma}\mathcal{L}\left[\varphi_{(x,y,t)}\right] - S^{3\gamma-1}\varphi_{(x,y,0)} - S^{3\gamma-2}\varphi_{t(x,y,0)} - S^{3\gamma-3}\varphi_{tt(x,y,0)}\}$$

$$= \mathcal{L}[-3\frac{\partial^{2\gamma}\varphi_{(x,y,t)}}{\partial t^{2\gamma}} - 2\frac{\partial^{\gamma}\varphi_{(x,y,t)}}{\partial t^{\gamma}} - \varphi_{(x,y,t)} + \frac{1}{2}\frac{\partial^3\varphi_{(x,y,t)}}{\partial x^3} + \frac{1}{2}\frac{\partial^3\varphi_{(x,y,t)}}{\partial y^3}] \tag{61}$$

After simplification and applying inverse Laplace transformation, we get

$$\varphi_{(x,y,t)} = \mathcal{L}^{-1}[\frac{1}{S}\varphi_{(x,y,0)} + \frac{1}{S^2}\varphi_{t(x,y,0)} + \frac{1}{S^3}\varphi_{tt(x,y,0)}]$$

$$+\mathcal{L}^{-1}[\frac{1}{S^{3\gamma}}\mathcal{L}\left\{-3\frac{\partial^{2\gamma}\varphi_{(x,y,t)}}{\partial t^{2\gamma}} - 2\frac{\partial^{\gamma}\varphi_{(x,y,t)}}{\partial t^{\gamma}} - \varphi_{(x,y,t)} + \frac{1}{2}\frac{\partial^3\varphi_{(x,y,t)}}{\partial x^3} + \frac{1}{2}\frac{\partial^3\varphi_{(x,y,t)}}{\partial y^3}\right\}] \tag{62}$$

using the ADM formula, we get

$$\varphi_{\circ(x,y,t)} = \mathcal{L}^{-1}\left[\frac{1}{S}\varphi_{(x,y,0)} + \frac{1}{S^2}\varphi_{t(x,y,0)} + \frac{1}{S^3}\varphi_{tt(x,y,0)}\right] \tag{63}$$

substituting equation (59) in equation (63), we get

$$= e^{x+y}\mathcal{L}^{-1}\left[\frac{1}{s} - 3\frac{1!}{s^2} + \frac{2!}{s^3}\right]$$

$$= e^{x+y}(1 - 3t + t^2) \tag{64}$$

$$\varphi_{j+1(x,y,t)} = \mathcal{L}^{-1}[\frac{1}{s^{3\gamma}}\mathcal{L}\{-3\frac{\partial^{2\gamma}\varphi_{j(x,y,t)}}{\partial t^{2\gamma}} - 2\frac{\partial^{\gamma}\varphi_{j(x,y,t)}}{\partial t^{\gamma}} - \varphi_{j(x,y,t)} + \frac{1}{2}\frac{\partial^3\varphi_{j(x,y,t)}}{\partial x^3} + \frac{1}{2}\frac{\partial^3\varphi_{j(x,y,t)}}{\partial y^3}\}] \tag{65}$$

for j=0,1,2, we get

$$\varphi_{1(x,y,t)} = e^{x+y}(\frac{-6}{\Gamma(\gamma+3)}t^{\gamma+2} + \frac{6}{\Gamma(2\gamma+2)}t^{2\gamma+1} - \frac{4}{\Gamma(2\gamma+3)}t^{2\gamma+2}). \tag{66}$$

$$\varphi_{2(x,y,t)} = e^{x+y}(\frac{18}{\Gamma(2\gamma+3)}t^{2\gamma+2} - \frac{18}{\Gamma(3\gamma+2)}t^{3\gamma+1} + \frac{24}{\Gamma(3\gamma+3)}t^{3\gamma+2} - \frac{12}{\Gamma(4\gamma+2)}t^{4\gamma+1}$$

$$+\frac{8}{\Gamma(4\gamma+3)}t^{4\gamma+2}). \tag{67}$$

$$\varphi_{3(x,y,t)} = e^{x+y}(-\frac{54}{\Gamma(3\gamma+3)}t^{3\gamma+2} + \frac{54}{\Gamma(4\gamma+2)}t^{4\gamma+1} - \frac{108}{\Gamma(4\gamma+3)}t^{4\gamma+2}$$

$$+\frac{72}{\Gamma(5\gamma+2)}t^{5\gamma+1} - \frac{72}{\Gamma(5\gamma+3)}t^{5\gamma+2} + \frac{24}{\Gamma(6\gamma+2)}t^{6\gamma+1} - \frac{16}{\Gamma(6\gamma+3)}t^{6\gamma+2}). \tag{68}$$

Substituting equations (64), (66), (67), and (68) in equation (10), after simplification, we get

$$\varphi_{(x,y,t)} = e^{x+y}[\ 1 - 2t + \frac{1}{2}t^2 - \frac{3}{\Gamma(\gamma+3)}t^{\gamma+2} + \frac{6}{\Gamma(2\gamma+2)}t^{2\gamma+1} + \frac{6}{\Gamma(2\gamma+3)}t^{2\gamma+2}$$

$$-\frac{18}{\Gamma(3\gamma+2)}t^{3\gamma+1} - \frac{9}{\Gamma(3\gamma+3)}t^{3\gamma+2} + \frac{36}{\Gamma(4\gamma+2)}t^{4\gamma+1} - \frac{72}{\Gamma(4\gamma+3)}t^{4\gamma+2}$$

$$+\frac{108}{\Gamma(5\gamma+2)}t^{5\gamma+1} - \frac{81}{\Gamma(5\gamma+3)}t^{5\gamma+2} + \frac{54}{\Gamma(6\gamma+2)}t^{6\gamma+1}$$

$$-\frac{27}{\Gamma(6\gamma+3)}t^{6\gamma+2} \ldots] \tag{69}$$

now we solving equations (58) and (59) of example 2 by q-HATM.

Taking equation (60), substituting equations (7) and (59), after simplification, we get

$$\mathcal{L}\left[\varphi_{(x,y,t)}\right] - e^{x+y}\{\frac{1}{s} - \frac{3}{s^2} + \frac{2}{s^3}\}$$

$$+\frac{1}{s^{3\gamma}}\mathcal{L}\left[3\frac{\partial^{2\gamma}v_{(x,y,t)}}{\partial t^{2\gamma}} + 2\frac{\partial^{\gamma}v_{(x,y,t)}}{\partial t^{\gamma}} + \varphi_{(x,y,t)} - \frac{1}{2}\frac{\partial^3\varphi_{(x,y,t)}}{\partial x^3} - \frac{1}{2}\frac{\partial^3\varphi_{(x,y,t)}}{\partial y^3}\right] = 0 \tag{70}$$

using equation (22), define the non-linear operator, we get

$$N\left[\psi_{(x,y,t;q)}\right] = \mathcal{L}\left[\psi_{(x,y,t;q)}\right] - e^{x+y}\left(\frac{1}{S} - \frac{3}{S^2} + \frac{2}{S^3}\right)$$

$$+\frac{1}{s^{3\gamma}}\mathcal{L}[3\frac{\partial^{2\gamma}\psi_{(x,y,t;q)}}{\partial t^{2\gamma}} + 2\frac{\partial^{\gamma}\psi_{(x,y,t;q)}}{\partial t^{\gamma}} + \psi_{(x,y,t;q)} - \frac{1}{2}\frac{\partial^3\psi_{(x,y,t;q)}}{\partial x^3} - \frac{1}{2}\frac{\partial^3\psi_{(x,y,t;q)}}{\partial y^3}] = 0 \tag{71}$$

using equation (32) and equation (30), after simplification, we get

$$\mathbb{R}_m(\bar{\varphi}_{m-1}) = \mathcal{L}[\varphi_{m-1(x,y,t)}] - (1-\frac{k_m}{n})e^{x+y}\left(\frac{1}{S}-\frac{3}{S^2}+\frac{2}{S^3}\right)+\frac{1}{S^{3\gamma}}\mathcal{L}\left[3\frac{\partial^{2\gamma}\varphi_{m-1}}{\partial t^{2\gamma}}+2\frac{\partial^{\gamma}\varphi_{m-1}}{\partial t^{\gamma}}+\varphi_{m-1}-\frac{1}{2}\frac{\partial^3\varphi_{m-1}}{\partial x^3}-\frac{1}{2}\frac{\partial^3\varphi_{m-1}}{\partial y^3}\right]=0 \quad (72)$$

For $m = 0$, equation (72) after simplification, we get

$$\mathbb{R}_m(\bar{\varphi}_{m-1}) = -e^{x+y}\left(\frac{1}{S}-\frac{3}{S^2}+\frac{2}{S^3}\right) \quad (73)$$

substituting equation (73) in equation (32) after applying inverse Laplace transformation, we get

$$\varphi_{0(x,y,t)} = -he^{x+y}(1-3t+t^2) \quad (74)$$

for $m = 1$, equation (72) after simplification, we get

$$\mathbb{R}_m(\bar{\varphi}_{m-1}) = -(h+1)e^{x+y}\left(\frac{1}{S}-\frac{3}{S^2}+\frac{2}{S^3}\right)-2he^{x+y}\left(\frac{3}{S^{\gamma+3}}-\frac{3}{S^{2\gamma+2}}+\frac{2}{S^{2\gamma+3}}\right). \quad (75)$$

substituting equation (75) in equation (32) after apply inverse Laplace transformation, we get

$$\varphi_{1(x,y,t)} = -h(h+1)e^{x+y}(1-3t+t^2)+e^{x+y}\left[\frac{-6h^2}{\Gamma(\gamma+3)}t^{\gamma+2}+\frac{6h^2}{\Gamma(2\gamma+2)}t^{2\gamma+1}-\frac{4h^2}{\Gamma(2\gamma+3)}t^{2\gamma+2}\right]. \quad (76)$$

for $m = 2$, equation (72) after simplification, we get

$$\mathbb{R}_m(\bar{\varphi}_{m-1}) = -h(h+1)e^{x+y}\left(\frac{1}{S}-\frac{3}{S^2}+\frac{2}{S^3}\right)+e^{x+y}\left[\frac{-6h^2}{S^{\gamma+3}}+\frac{6h^2}{S^{2\gamma+2}}-\frac{4h^2}{S^{2\gamma+3}}-\frac{6h(h+1)}{S^{\gamma+3}}-\frac{18h^2}{S^{2\gamma+3}}+\frac{18h^2}{S^{3\gamma+2}}-\frac{12h^2}{S^{3\gamma+3}}+\frac{6h(h+1)}{S^{2\gamma+2}}-\frac{4h(h+1)}{S^{2\gamma+3}}-\frac{12h^2}{S^{3\gamma+3}}+\frac{12h^2}{S^{4\gamma+2}}-\frac{8h^2}{S^{4\gamma+3}}\right]. \quad (77)$$

substituting equation (77) in equation (32) after apply inverse Laplace transformation, we get

$$\varphi_{2(x,y,t)} = -h(n+h)(h+1)e^{x+y}(1-3t+t^2)+e^{x+y}\left[\frac{-6h^2\{(n+h)+(h+1)\}}{\Gamma(\gamma+3)}t^{\gamma+2}+\frac{6h^2\{(n+h)+(h+1)\}}{\Gamma(2\gamma+2)}t^{2\gamma+1}-\frac{2h^2\{9h+2(n+h)+2(h+1)\}}{\Gamma(2\gamma+3)}t^{2\gamma+2}+\frac{18h^3}{\Gamma(3\gamma+2)}t^{3\gamma+1}-\frac{24h^3}{\Gamma(3\gamma+3)}t^{3\gamma+2}+\frac{12h^3}{\Gamma(4\gamma+2)}t^{4\gamma+1}-\frac{8h^3}{\Gamma(4\gamma+3)}t^{4\gamma+2}\right]. \quad (78)$$

for $m = 3$, equation (72) after simplification, we get

$$\mathbb{R}_m(\bar{\varphi}_{m-1}) = -h(n+h)(h+1)e^{x+y}\left(\frac{1}{S}-\frac{3}{S^2}+\frac{2}{S^3}\right)+e^{x+y}\left[\frac{-6h^2\{(n+h)+(h+1)\}}{S^{\gamma+3}}+\frac{6h^2\{(n+h)+(h+1)\}}{S^{2\gamma+2}}-\frac{2h^2\{9h+2(n+h)+2(h+1)\}}{S^{2\gamma+3}}+\frac{18h^3}{S^{3\gamma+2}}-\frac{24h^3}{S^{3\gamma+3}}+\frac{12h^3}{S^{4\gamma+2}}-\frac{8h^3}{S^{4\gamma+3}}\right.$$

$$-\frac{6h(n+h)(h+1)}{S^{\gamma+3}}-\frac{18h^2\{(n+h)+(h+1)\}}{S^{2\gamma+3}}+\frac{18h^2\{(n+h)+(h+1)\}}{S^{3\gamma+2}}$$

$$-\frac{6h^2\{9h+2(n+h)+2(h+1)\}}{S^{3\gamma+3}}+\frac{54h^3}{S^{4\gamma+2}}-\frac{72h^3}{S^{4\gamma+3}}+\frac{36h^3}{S^{5\gamma+2}}-\frac{24h^3}{S^{5\gamma+3}}$$

$$+\frac{6h(n+h)(h+1)}{S^{2\gamma+2}}-\frac{4h(n+h)(h+1)}{S^{2\gamma+3}}-\frac{12h^2\{(n+h)+(h+1)\}}{S^{3\gamma+3}}+\frac{12h^2\{(n+h)+(h+1)\}}{S^{4\gamma+2}}$$

$$-\frac{4h^2\{9h+2(n+h)+2(h+1)\}}{S^{4\gamma+3}}+\frac{36h^3}{S^{5\gamma+2}}-\frac{48h^3}{S^{5\gamma+3}}+\frac{24h^3}{S^{6\gamma+2}}-\frac{16h^3}{S^{6\gamma+3}}]. \quad (79)$$

substituting equation (79) in equation (32) after applying inverse Laplace transformation, we get,

$$\varphi_{3(x,y,t)}=-h(n+h)^2(h+1)e^{x+y}(1-3t+t^2)$$

$$+e^{x+y}[\frac{-6h^2(n+h)\{(n+h)+2(h+1)\}}{\Gamma(\gamma+3)}t^{\gamma+2}+\frac{6h^2(n+h)\{(n+h)+2(h+1)\}}{\Gamma(2\gamma+2)}t^{2\gamma+1}$$

$$-\frac{2h^2\{2(n+h)^2+18h(n+h)+4(n+h)(h+1)+9h(h+1)\}}{\Gamma(2\gamma+3)}t^{2\gamma+2}$$

$$+\frac{18h^3\{2(n+h)+(h+1)\}}{\Gamma(3\gamma+2)}t^{3\gamma+1}-\frac{6h^3\{9h+8(n+h)+4(h+1)\}}{\Gamma(3\gamma+3)}t^{3\gamma+2}+\frac{6h^3\{9h+4(n+h)+2(h+1)\}}{\Gamma(4\gamma+2)}t^{4\gamma+1}-\frac{4h^3\{27h+4(n+h)+2(h+1)\}}{\Gamma(4\gamma+3)}t^{4\gamma+2}$$

$$+\frac{72h^4}{\Gamma(5\gamma+2)}t^{5\gamma+1}-\frac{72h^4}{\Gamma(5\gamma+3)}t^{5\gamma+2}+\frac{24h^4}{\Gamma(6\gamma+2)}t^{6\gamma+1}-\frac{16h^4}{\Gamma(6\gamma+3)}t^{6\gamma+2}]. \quad (80)$$

substituting equations (74), (76), (78) and (80) in equation (10), we get,

$$\varphi_{(x,y,t)}=-h\{1+(h+1)+(n+h)(h+1)(n+h+1)\}e^{x+y}(1-3t+t^2)$$

$$+e^{x+y}[\frac{-6h^2\{1+(n+h)^2+(n+h)+2(n+h)(h+1)+(h+1)\}}{\Gamma(\gamma+3)}t^{\gamma+2}$$

$$+\frac{6h^2\{1+(n+h)^2+2(n+h)(h+1)\}}{\Gamma(2\gamma+2)}t^{2\gamma+1}$$

$$-\frac{2h^2\{7h+2(n+h)^2+4(n+h)(h+1)+2(n+h)+18h(n+h)+4(h+1)+9h(h+1)\}}{\Gamma(2\gamma+3)}t^{2\gamma+2}$$

$$+\frac{18h^3\{1+2(n+h)+(h+1)\}}{\Gamma(3\gamma+2)}t^{3\gamma+1}-\frac{6h^3\{5h+8(n+h)+8(h+1)\}}{\Gamma(3\gamma+3)}t^{3\gamma+2}$$

$$+\frac{6h^3\{7h+4(n+h)+4(h+1)\}}{\Gamma(4\gamma+2)}t^{4\gamma+1}-\frac{4h^3\{25h+4(n+h)+4(h+1)\}}{\Gamma(4\gamma+3)}t^{4\gamma+2}$$

$$+\frac{72h^4}{\Gamma(5\gamma+2)}t^{5\gamma+1}-\frac{72h^4}{\Gamma(5\gamma+3)}t^{5\gamma+2}+\frac{24h^4}{\Gamma(6\gamma+2)}t^{6\gamma+1}-\frac{16h^4}{\Gamma(6\gamma+3)}t^{6\gamma+2}].$$

(81)

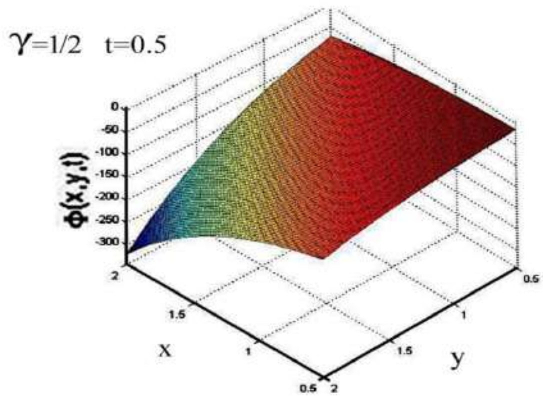

Figure 2(a). $\gamma = 0.5, t = 0.5$ of example 2

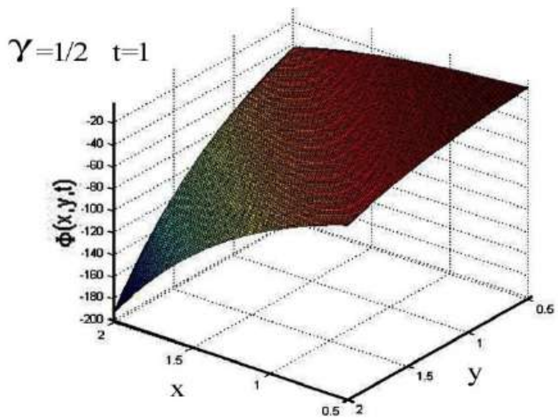

Figure 2(b). $\gamma = 0.5, t = 1.0$ of example 2

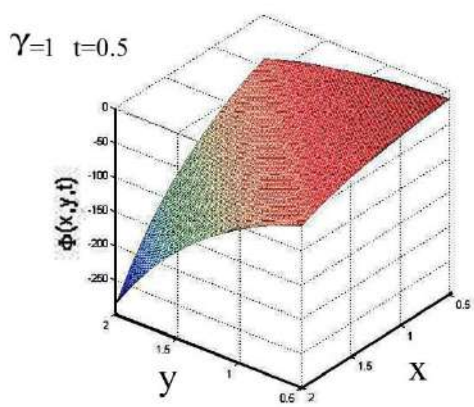

Figure 2(c). $\gamma = 1.0, t = 0.5$ of example 2

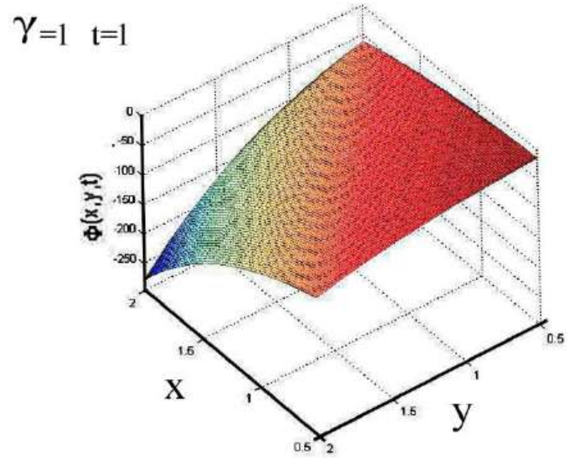

Figure 2(d). $\gamma = 1.0, t = 1.0$ of example 2

Example 3: Consider the following Third order Time FPDE in three dimensions by solving LADM and q-HATM

$$\frac{\partial^{3\gamma}\varphi_{(x,y,z,t)}}{\partial t^{3\gamma}} + 3\frac{\partial^{2\gamma}\varphi_{(x,y,z,t)}}{\partial t^{2\gamma}} + 6\frac{\partial^{\gamma}\varphi_{(x,y,z,t)}}{\partial t^{\gamma}} + 9\varphi_{(x,y,z,t)} = \frac{1}{2}\frac{\partial^3\varphi_{(x,y,z,t)}}{\partial x^3} + \frac{\partial^3\varphi_{(x,y,z,t)}}{\partial y^3} + \frac{1}{2}\frac{\partial^3\varphi_{(x,y,z,t)}}{\partial Z^3} \quad (82)$$

$$0 < \gamma \leq 1, \qquad t \geq 0$$

with initial conditions

$$\varphi_{(x,y,z,0)} = e^{x+2y+z}, \varphi_{t(x,y,z,0)} = -2e^{x+2y+z}, \ \varphi_{tt(x,y,z,0)} = e^{x+2y+z} \quad (83)$$

Solution:

Taking equation (82), apply Laplace transformation

$$\mathcal{L}\left[\frac{\partial^{3\gamma}\varphi_{(x,y,z,t)}}{\partial t^{3\gamma}}\right] = \mathcal{L}[-3\frac{\partial^{2\gamma}\varphi_{(x,y,z,t)}}{\partial t^{2\gamma}} - 6\frac{\partial^{\gamma}\varphi_{(x,y,z,t)}}{\partial t^{\gamma}} - 9\varphi_{(x,y,z,t)} + \frac{1}{2}\frac{\partial^3\varphi_{(x,y,z,t)}}{\partial x^3}$$

$$+\frac{\partial^3\varphi_{(x,y,z,t)}}{\partial y^3} + \frac{1}{2}\frac{\partial^3\varphi_{(x,y,z,t)}}{\partial z^3}] \quad (84)$$

substituting equation (7) in equation (84), we get

$$S^{3\gamma}\mathcal{L}[\varphi_{(x,y,z,t)}] - S^{3\gamma-1}\varphi_{(x,y,z,0)} - S^{3\gamma-2}\varphi_{t(x,y,z,0)} - S^{3\gamma-3}\varphi_{tt(x,y,z,0)}$$

$$= \mathcal{L}[-3\frac{\partial^{2\gamma}\varphi_{(x,y,z,t)}}{\partial t^{2\gamma}} - 6\frac{\partial^{\gamma}\varphi_{(x,y,z,t)}}{\partial t^{\gamma}} - 9\varphi_{(x,y,z,t)} + \frac{1}{2}\frac{\partial^3\varphi_{(x,y,z,t)}}{\partial x^3} + \frac{\partial^3\varphi_{(x,y,z,t)}}{\partial y^3} + \frac{1}{2}\frac{\partial^3\varphi_{(x,y,z,t)}}{\partial z^3}]$$

(85)

after Simplify and using inverse Laplace transformation, we get,

$$\varphi_{(x,y,z,t)} = \mathcal{L}^{-1}[\frac{1}{S}\varphi_{(x,y,z,0)} + \frac{1}{S^2}\varphi_{t(x,y,z,0)} + \frac{1}{S^3}\varphi_{tt(x,y,z,0)}]$$

$$+\mathcal{L}^{-1}[\frac{1}{S^{3\gamma}}\mathcal{L}\{-3\frac{\partial^{2\gamma}\varphi_{(x,y,z,t)}}{\partial t^{2\gamma}} - 6\frac{\partial^{\gamma}\varphi_{(x,y,z,t)}}{\partial t^{\gamma}} - 9\varphi_{(x,y,z,t)} + \frac{1}{2}\frac{\partial^3\varphi_{(x,y,z,t)}}{\partial x^3} \qquad \frac{\partial^3\varphi_{(x,y,z,t)}}{\partial y^3} + \frac{1}{2}\frac{\partial^3\varphi_{(x,y,z,t)}}{\partial z^3}\}]$$

(86)

using ADM formula, we get

$$\varphi_{\circ(x,y,z,t)} = \mathcal{L}^{-1}\left[\frac{1}{S}\varphi_{(x,y,z,0)} + \frac{1}{S^2}\varphi_{t(x,y,z,0)} + \frac{1}{S^3}\varphi_{tt(x,y,z,0)}\right] \tag{87}$$

substituting equation (83) in equation (87), we get

$$\varphi_{\circ(x,y,z,t)} = e^{x+2y+z}\mathcal{L}^{-1}[\frac{1}{S} - \frac{2}{S^2} + \frac{1}{S^3}]$$

$$= e^{x+2y+z}(1 - 2t + \frac{1}{2}t^2) \tag{88}$$

$$\varphi_{j+1(x,y,z,t)} = \mathcal{L}^{-1}[\frac{1}{S^{3\gamma}}\mathcal{L}\{-3\frac{\partial^{2\gamma}\varphi_{j(x,y,z,t)}}{\partial t^{2\gamma}} - 6\frac{\partial^{\gamma}\varphi_{j(x,y,z,t)}}{\partial t^{\gamma}} - 9\varphi_{j(x,y,z,t)}$$

$$+\frac{1}{2}\frac{\partial^3\varphi_{j(x,y,z,t)}}{\partial x^3} + \frac{\partial^3\varphi_{j(x,y,z,t)}}{\partial y^3} + \frac{1}{2}\frac{\partial^3\varphi_{j(x,y,z,t)}}{\partial z^3}\}] \tag{89}$$

for j=0,1,2, after solving, we get

$$\varphi_{1(x,y,z,t)} = e^{x+2y+z}(-\frac{3}{\Gamma(\gamma+3)}t^{\gamma+2} + \frac{12}{\Gamma(2\gamma+2)}t^{2\gamma+1} - \frac{6}{\Gamma(2\gamma+3)}t^{2\gamma+2}) \tag{90}$$

$$\varphi_{2(x,y,z,t)} = e^{x+2y+z}(\frac{9}{\Gamma(2\gamma+3)}t^{2\gamma+2} - \frac{36}{\Gamma(3\gamma+2)}t^{3\gamma+1} + \frac{36}{\Gamma(3\gamma+3)}t^{3\gamma+2}$$

$$-\frac{72}{\Gamma(4\gamma+2)}t^{4\gamma+1} + \frac{36}{\Gamma(4\gamma+3)}t^{4\gamma+2}). \tag{91}$$

$$\varphi_{3(x,y,z,t)} = e^{x+2y+z}(-\frac{27}{\Gamma(3\gamma+3)}t^{3\gamma+2} + \frac{108}{\Gamma(4\gamma+2)}t^{4\gamma+1} - \frac{162}{\Gamma(4\gamma+3)}t^{4\gamma+2}$$

$$+\frac{432}{\Gamma(5\gamma+2)}t^{5\gamma+1} - \frac{324}{\Gamma(5\gamma+3)}t^{5\gamma+2} + \frac{432}{\Gamma(6\gamma+2)}t^{6\gamma+1} - \frac{216}{\Gamma(6\gamma+3)}t^{6\gamma+2}). \tag{92}$$

substituting equations (88), (90), (91), (92) in equation (10), after simplify, we get

$$\varphi_{(x,y,z,t)} = e^{x+2y+z}[1 - 2t + \frac{1}{2}t^2 - \frac{3}{\Gamma(\gamma+3)}t^{\gamma+2} + \frac{12}{\Gamma(2\gamma+2)}t^{2\gamma+1} + \frac{3}{\Gamma(2\gamma+3)}t^{2\gamma+2} - \frac{36}{\Gamma(3\gamma+2)}t^{3\gamma+1} + \frac{9}{\Gamma(3\gamma+3)}t^{3\gamma+2} + \frac{36}{\Gamma(4\gamma+2)}t^{4\gamma+1} - \frac{126}{\Gamma(4\gamma+3)}t^{4\gamma+2} + \frac{432}{\Gamma(5\gamma+2)}t^{5\gamma+1} - \frac{324}{\Gamma(5\gamma+3)}t^{5\gamma+2} + \frac{432}{\Gamma(6\gamma+2)}t^{6\gamma+1} - \frac{216}{\Gamma(6\gamma+3)}t^{6\gamma+2} + \cdots \ldots \ldots \ldots]. \tag{93}$$

now solving equations (82) and (83) of example 3 by q-HATM

taking equation (84), Substituting equations (7) and (83), after simplification, we get

$$\mathcal{L}\left[\varphi_{(x,y,z,t)}\right] - e^{x+2y+z}(\frac{1}{S} - \frac{2}{S^2} + \frac{1}{S^3}\}$$

$$+\frac{1}{S^{3\gamma}}\mathcal{L}[3\frac{\partial^{2\gamma}\varphi_{(x,y,z,t)}}{\partial t^{2\gamma}} + 6\frac{\partial^{\gamma}\varphi_{(x,y,z,t)}}{\partial t^{\gamma}} + 9\varphi_{(x,y,z,t)} - \frac{1}{2}\frac{\partial^3\varphi_{(x,y,z,t)}}{\partial x^3}$$

$$-\frac{\partial^3\varphi_{(x,y,z,t)}}{\partial y^3} - \frac{1}{2}\frac{\partial^3\varphi_{(x,y,z,t)}}{\partial z^3}] = 0 \tag{94}$$

using equation (22), define the non-linear operator, we get

$$N\left[\psi_{(x,y,z,t;q)}\right] = \mathcal{L}\left[\psi_{(x,y,z,t;q)}\right] - e^{x+2y+z}\left(\frac{1}{S} - \frac{2}{S^2} + \frac{1}{S^3}\right)$$

$$+\frac{1}{S^{3\gamma}}\mathcal{L}[3\frac{\partial^{2\gamma}\psi_{(x,y,z,t;q)}}{\partial t^{2\gamma}} + 6\frac{\partial^{\gamma}\psi_{(x,y,z,t;q)}}{\partial t^{\gamma}} + 9\psi_{(x,y,z,t;q)}$$

$$-\frac{1}{2}\frac{\partial^3\psi_{(x,y,z,t;q)}}{\partial x^3} - \frac{\partial^3\psi_{(x,y,z,t;q)}}{\partial y^3} - \frac{1}{2}\frac{\partial^3\psi_{(x,y,z,t;q)}}{\partial z^3}] = 0 \quad (95)$$

taking equations (32) and (30), after simplification, we get

$$\mathbb{R}_m(\vec{\varphi}_{m-1}) = \mathcal{L}\left[\varphi_{m-1(x,y,z,t)}\right] - (1 - \frac{k_m}{n})e^{x+2y+z}\left(\frac{1}{S} - \frac{2}{S^2} + \frac{1}{S^3}\right)$$

$$+\frac{1}{S^{3\gamma}}\mathcal{L}[3\frac{\partial^{2\gamma}\varphi_{m-1}}{\partial t^{2\gamma}} + 6\frac{\partial^{\gamma}\varphi_{m-1}}{\partial t^{\gamma}} + 9\varphi_{m-1} - \frac{1}{2}\frac{\partial^3\varphi_{m-1}}{\partial x^3} - \frac{\partial^3\varphi_{m-1}}{\partial y^3} - \frac{1}{2}\frac{\partial^3\varphi_{m-1}}{\partial z^3}]. \quad (96)$$

for $m = 0$, equation (96) after simplification, we get

$$\mathbb{R}_m(\vec{\varphi}_{m-1}) = -e^{x+2y+z}\left(\frac{1}{S} - \frac{2}{S^2} + \frac{1}{S^3}\right) \quad (97)$$

substitution equation (97) in equation (32), after simplify and apply inverse Laplace transformation, we get

$$\varphi_{0(x,y,z,t)} = -he^{x+2y+z}(1 - 2t + \frac{1}{2}t^2) \quad (98)$$

for $m = 1$, equation (96) after simplification, we get

$$\mathbb{R}_m(\vec{\varphi}_{m-1}) = -(h+1)e^{x+2y+z}\left(\frac{1}{S} - \frac{2}{S^2} + \frac{1}{S^3}\right)$$

$$+3e^{x+2y+z}[-h\frac{1}{S^{\gamma+3}} + 4h\frac{1}{S^{2\gamma+2}} - 2h\frac{1}{S^{2\gamma+3}}]. \quad (99)$$

substituting equation (99) in equation (32), after simplify and apply inverse Laplace transformation, we get

$$\varphi_{1(x,y,z,t)} = -h(h+1)e^{x+2y+z}\left(1 - 2t + \frac{1}{2}t^2\right) + 3e^{x+2y+z}[\frac{-h^2}{\Gamma(\gamma+3)}t^{\gamma+2}$$

$$+\frac{4h^2}{\Gamma(2\gamma+2)}t^{2\gamma+1} - \frac{2h^2}{\Gamma(2\gamma+3)}t^{2\gamma+2}].$$

(100)

for $m = 2$, equation (96) after simplification, we get

$$\mathbb{R}_m(\vec{\varphi}_{m-1}) = -h(h+1)e^{x+2y+z}\left(\frac{1}{S} - \frac{2}{S^2} + \frac{1}{S^3}\right) + 3e^{x+2y+z}[\frac{-h^2}{S^{\gamma+3}} + \frac{4h^2}{S^{2\gamma+2}}$$

$$-\frac{2h^2}{S^{2\gamma+3}} - \frac{h(h+1)}{S^{\gamma+3}} - \frac{3h^2}{S^{2\gamma+3}} + \frac{12h^2}{S^{3\gamma+2}} - \frac{6h^2}{S^{3\gamma+3}} + \frac{4h(h+1)}{S^{2\gamma+2}} - \frac{2h(h+1)}{S^{2\gamma+3}}$$

$$-\frac{6h^2}{S^{3\gamma+3}} + \frac{24h^2}{S^{4\gamma+2}} - \frac{12h^2}{S^{4\gamma+3}}].$$

(101)

substituting equation (101) in equation (32), after simplify and apply inverse Laplace transformation, we get

$$\varphi_{2(x,y,z,t)} = -h(n+h)(h+1)e^{x+2y+z}\left(1-2t+\frac{1}{2}t^2\right)$$

$$+3e^{x+2y+z}[\frac{-h^2\{(n+h)+(h+1)\}}{\Gamma(\gamma+3)}t^{\gamma+2}+\frac{4h^2\{(n+h)+(h+1)\}}{\Gamma(2\gamma+2)}t^{2\gamma+1}-\frac{h^2\{3h+2(n+h)+2(h+1)\}}{\Gamma(2\gamma+3)}t^{2\gamma+2}$$
$$+\frac{12h^3}{\Gamma(3\gamma+2)}t^{3\gamma+1}-\frac{12h^3}{\Gamma(3\gamma+3)}t^{3\gamma+2}$$
$$+\frac{24h^3}{\Gamma(4\gamma+2)}t^{4\gamma+1}-\frac{12h^3}{\Gamma(4\gamma+3)}t^{4\gamma+2}].$$

(102)

for $m=3$, equation (96), after simplification, we get,

$$\mathbb{R}_m(\bar{\varphi}_{m-1}) = -h(n+h)(h+1)e^{x+2y+z}\left(\frac{1}{S}-\frac{2}{S^2}+\frac{1}{S^3}\right)$$

$$+3e^{x+2y+z}[\frac{-h^2\{(n+h)+(h+1)\}}{S^{\gamma+3}}+\frac{4h^2\{(n+h)+(h+1)\}}{S^{2\gamma+2}}$$

$$-\frac{h^2\{3h+2(n+h)+2(h+1)\}}{S^{2\gamma+3}}+\frac{12h^3}{S^{3\gamma+2}}-\frac{12h^3}{S^{3\gamma+3}}+\frac{24h^3}{S^{4\gamma+2}}-\frac{12h^3}{S^{4\gamma+3}}$$

$$-\frac{h(n+h)(h+1)}{S^{\gamma+3}}-\frac{3h^2\{(n+h)+(h+1)\}}{S^{2\gamma+3}}+\frac{12h^2\{(n+h)+(h+1)\}}{S^{3\gamma+2}}$$

$$\frac{3h^2\{3h+2(n+h)+2(h+1)\}}{S^{3\gamma+3}}+\frac{36h^3}{S^{4\gamma+2}}-\frac{36h^3}{S^{4\gamma+3}}+\frac{72h^3}{S^{5\gamma+2}}-\frac{36h^3}{S^{5\gamma+3}}$$

$$+\frac{4h(n+h)(h+1)}{S^{2\gamma+2}}-\frac{2h(n+h)(h+1)}{S^{2\gamma+3}}-\frac{6h^2\{(n+h)+(h+1)\}}{S^{3\gamma+3}}$$

$$+\frac{24h^2\{(n+h)+(h+1)\}}{S^{4\gamma+2}}-\frac{6h^2\{3h+2(n+h)+2(h+1)\}}{S^{4\gamma+3}}+\frac{72h^3}{S^{5\gamma+2}}-\frac{72h^3}{S^{5\gamma+3}}+\frac{144h^3}{S^{6\gamma+2}}-\frac{72h^3}{S^{6\gamma+3}}].$$

(103)

substituting equation (103) in equation (32), after simplify and apply inverse Laplace transformation, we get

$$\varphi_{3(x,y,z,t)} = -h(n+h)^2(h+1)e^{x+2y+z}\left(1-2t+\frac{1}{2}t^2\right)$$

$$+3e^{x+2y+z}[\frac{-h^2(n+h)\{(n+h)+2(h+1)\}}{\Gamma(\gamma+3)}t^{\gamma+2}$$

$$+\frac{4h^2(n+h)\{(n+h)+2(h+1)\}}{\Gamma(2\gamma+2)}t^{2\gamma+1}-\frac{h^2\{2(n+h)^2+6h(n+h)+4(n+h)(h+1)+3h(h+1)\}}{\Gamma(2\gamma+3)}t^{2\gamma+2}$$

$$+\frac{12h^3\{2(n+h)+(h+1)\}}{\Gamma(3\gamma+2)}t^{3\gamma+1}-\frac{3h^3\{3h+8(n+h)+4(h+1)\}}{\Gamma(3\gamma+3)}t^{3\gamma+2}$$

$$+\frac{12h^3\{3h+4(n+h)+2(h+1)\}}{\Gamma(4\gamma+2)}t^{4\gamma+1}-\frac{6h^3\{9h+4(n+h)+2(h+1)\}}{\Gamma(4\gamma+3)}t^{4\gamma+2}$$

$$+\frac{144h^4}{\Gamma(5\gamma+2)}t^{5\gamma+1}-\frac{108h^4}{\Gamma(5\gamma+3)}t^{5\gamma+2}+\frac{144h^4}{\Gamma(6\gamma+2)}t^{6\gamma+1}-\frac{72h^4}{\Gamma(6\gamma+3)}t^{6\gamma+2}].$$

(104)

substituting equations (98), (100), (102), and (104) in equation (10), we get

$$\varphi_{(x,y,z,t)} = e^{x+2y+z}[-h\{1+(n+h)^2(h+1)+(n+h)(h+1)+(h+1)\}$$

$$(1-2t+\frac{1}{2}t^2)$$

$$-\frac{3h^2\{1+(n+h)^2+2(n+h)(h+1)+(n+h)+(h+1)\}}{\Gamma(\gamma+3)}t^{\gamma+2}$$

$$+\frac{12h^2\{1+(n+h)^2+2(n+h)(h+1)+(n+h)+(h+1)\}}{\Gamma(2\gamma+2)}t^{2\gamma+1}$$

$$-\frac{3h^2\{h+2(n+h)^2+6(n+h)(h+1)+4h(n+h)+3h(h+1)+4(h+1)\}}{\Gamma(2\gamma+3)}t^{2\gamma+2}$$

$$+\frac{36h^3\{1+2(n+h)+(h+1)\}}{\Gamma(3\gamma+2)}t^{3\gamma+1}-\frac{9h^3\{1+8(n+h)+7(h+1)\}}{\Gamma(3\gamma+3)}t^{3\gamma+2}$$

$$+\frac{36h^3\{h+4(n+h)+4(h+1)\}}{\Gamma(4\gamma+2)}t^{4\gamma+1}$$

$$-\frac{18h^3\{7h+4(n+h)+4(h+1)\}}{\Gamma(4\gamma+3)}t^{4\gamma+2}+\frac{432h^4}{\Gamma(5\gamma+2)}t^{5\gamma+1}$$

$$-\frac{324h^4}{\Gamma(5\gamma+3)}t^{5\gamma+2}+\frac{432h^4}{\Gamma(6\gamma+2)}t^{6\gamma+1}-\frac{216h^4}{\Gamma(6\gamma+3)}t^{6\gamma+2}+---].$$

(105)

6 Results and Discussion

With the help of graph 1 and 2 the behavior of the proposed approximate series solution has been shown by taking $\gamma = 0.25, 0.5, 0.75$ *and* 1.0. In the proposed work the value of γ could be $0 < \gamma \leq 1$ for space time fractional order partial differential equations. Both designed techniques have applied to get the numerical results in series form at different values of x, y, z, and t when h= -1, $n = 1$ which is the standard q-HATM. The solution of the problem 1 at fractional order taking $\gamma = 0.25, 0.5, 075$ *and* 1 are plotted in figure 1(a), 1(b) 1(c) and 1(d). Similar representation of example 2 has been expressed with the help of figure 2(a), 2(b) 2(c) and 2(d). All the figures are representing the different dynamical behavior by the solutions of different non linear fractional order problems. This is stated earlier the comparison of the proposed technique is not compared by exact solution since the exact solution for the third order has never been solve earlier. Over we can say that the graphical representation gives the confirmation and reliability of the proposed work. Moreover, If the value of fraction order γ varied from one to two, the representation of graphs for the solution, shows the different dynamical behavior of each example, which helps in choosing the optimum fractional order of the model, and by the help of this, the physical scenario of each example can be describe very easily.

7 Conclusion and Future Recommendations

In this paper, an efficient analytical series solution of third order time fractional PDEs are given. To show are proposed method we have considered three examples of one, two and three dimensions by using two methods that is LADM and q-HATM that showed that the suggested technique is simpler accurate and straight forward. To describe fractional derivative, Caputo operator for both fractional and integers order is calculated. The q-HATM, auxiliary parameter h and embedding parameter n (n≥ 1) are control and adjust the convergence of the series solution by selecting a suitable value. It can be concluded that in standard q-HATM that is h=1, n=1, for third order time fractional PDEs both give same results. This shows that the proposed techniques have a very broad capacity of utilization towards the analytical solution to many fractional physical problems emerging in various fields of science and engineering. For the future work, the proposed method can be used for solving nonlinear variable-order time fractional diffusion-wave equation, fractional variational problems and fractional optimal control problems. The proposed methods can also be used for the approximation method of higher order fractional integrals and fractional derivatives. The limitation of the proposed study is that we have considered the value of γ between 0 and 1. The proposed approach will not give any solution beyond this domain.

Main Contributions

Mr. Safdar contributed in developing the proposed problem by applying the direct methods. **Dr. Fozia Hanif** contributed the main idea for the mathematical formulation of the proposed model. **Mr. Ilyas** as third author contributed to carrying out the simulations. **Dr. Rehan Shams** as fourth author has done the grammatical corrections **Dr. Muhammad Rehan** as a fifth author contributed his expertise in technical writing and **Dr. Syed Inayatullah** as a sixth author done the overall supervision.

Compliance with Ethical Standards

It is declared that all authors don't have any conflict of interest.

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