VFAST Transactions on Mathematics
http://vfast.org/journals/index.php/VTM@ 2021, ISSN(e): 2309-0022, ISSN(p): 2411-6343
Volume 10, Number 2, July-December, 2022 pp: 164-174

A new three step derivative free method using weight function for numerical solution of non-linear equations arises in application problems

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Keywords: *Non-linear Equation, Convergence Analysis, Efficiency Index, Derivative free, weight function*
Subject Classification: *35J05, 35J10, 35K05, 35L05*

Journal Info:
Submitted: November 17, 2022
Accepted: December 27, 2022
Published: December 31, 2022

Abstract In this paper a three-step numerical method, using weight function, has been derived for finding the root of non-linear equations. The proposed method possesses the accuracy of order eight with four functional evaluations. The efficiency index of the derived scheme is 1.682. Numerical examples, application problems are used to demonstrate the performance of the presented schemes and compare them to other available methods in the literature of the same order. Matlab, Mathematica 2021 & Maple 2021 software were used for numerical results.

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1 Introduction

Iterative algorithms for computing approximate zeros of nonlinear equations are of significant importance in computational and applied mathematics due to their numerous applications in many fields of modern science, including engineering, mathematical chemistry, bio-mathematics, physics, and statistics, among others. Numerous engineering problems take the form of nonlinear functions. Analytical approaches generally do not tackle these types of engineering problems, hence we require an iterative technique. Newton [1] presented one of the most well-known and famous iterative algorithm around the end of the fifteenth century. This algorithm, commonly referred as Newton's technique, has been widely used for many years to solve non-linear equations.

Nowadays's scholars are intend to develop multipoint iterative, derivative free, optimal methods to solve nonlinear equations. A. Cordero et al. [2] proposed a general procedure to obtain optimal derivative free iterative methods by using polynomial interpolation for non-linear equations $f(x) = 0$. A new bracketing and derivative free method of quadratic convergence had been proposed by S. Jamali et al. [3] & derived from the Newton backward interpolation technique, numerical tests shows the stability of the method. S. Jamali et al. [4] presented a Stirling interpolation technique based second-order bracketing and derivative free method for the solution of non-linear equations. For solving non-linear equations, a procedure for design of Steffensen-type algorithms of various orders is proposed by A. Cordero & J. Torregrosa [5], many iterative techniques may be transformed into derivative-free iterative systems by using particular divided difference of first order has been suggested. For the solution of non-linear equation B. Neta [6] proposed a higher order derivative-free method and the novelty in the method is using Traub's method as first step. A novel high-order derivative-free approach for solving a non-linear equation is proposed. The utilization of Traub's approach as a first step is innovative, and introduce two new family of Chebyshev-Halley type algorithms without derivatives for solving non-linear equations numerically. I. K. Argyros et el. [7] suggested optimal fourth and eighth order of three and four point methods, require the three and four function evaluations only. A. Suhadolnik [8] presented several techniques that are based on combinations of bisection, regula falsi, and parabolic interpolation named combined bracketing methods for solution of the non-linear equations. [9–13] also proposed several modified methods using algebraic and interpolation technique. A new approach for solving non-linear equations based on Muller's algorithm has been developed by A. Suhadolnik [14]. F. Soleymani et al.[15] presented a Modified Jarratt Method with Twelfth-Order Without Memory using Hermite interpolation technique for numerical solution of nonlinear equations. S. Jamali et al.[16] recently proposed an optimal two point derivative free method for solution of nonlinear algebraic and transcendental equation in application problems. The Muller's approach is based on an interpolating polynomial constructed from the last three points of an iterative sequence. For solution of non-linear equations M. S. Petkovic et al. [17] Proposed two point derivative free family of methods. An arbitrary real parameter and suitable parameter function are used in these methods. A. Cordero et al. [18] Transform two renowned fourth-order Ostrowski's method and sixth-order improved Ostrowski's method in to derivative free methods by approximate derivatives by central-difference quotients.

2 Derivation of proposed method

It is proposed in [19], a non-optimal eighth order method with five function evaluation (three function and two first derivative)

i.e

$$\begin{aligned}
&\text{Step.1 } \psi_n = \xi_n - \frac{\phi(\xi_n)}{\phi'(\xi_n)} \\
&\text{Step.2 } \eta_n = \psi_n - \frac{\phi(\psi_n)}{\phi'(\xi_n)}\left(\frac{\phi^2(\xi_n)}{\phi^2(\xi_n) - 2\phi(\xi_n)\phi(\psi_n) + \phi^2(\psi_n)}\right) \\
&\text{Step.3 } \xi_{n+1} = \eta_n - \frac{\phi(\eta_n)}{\phi'(\eta_n)}
\end{aligned} \tag{1}$$

In three-point formula (1) requires five function evaluation per iteration, to reduce the number of function evaluation and make it derivative free, we approximate $\phi'(\xi_n) \approx \phi[\kappa_n, \xi_n]$ and $\phi'(\eta_n) \approx \phi[\psi_n, \eta_n]$using finite difference. And also introduce the weight function $(A(t_1) + B(t_2) + C(t_3))$ in third step of (1) to enhance the convergence.
Finally we got.

$$\begin{aligned} &\text{Step.1 } \psi_n = \xi_n - \frac{\phi(\xi_n)}{\phi[\kappa_n, \xi_n]} \\ &\text{Step.2 } \eta_n = \psi_n - \frac{\phi(\psi_n)}{\phi[\kappa_n, \xi_n]}\left(\frac{\phi^2(\xi_n)}{\phi^2(\xi_n) - 2\phi(\xi_n)\phi(\psi_n) + \phi^2(\psi_n)}\right) \\ &\text{Step.3 } \xi_{n+1} = \eta_n - (A(t_1) + B(t_2) + C(t_3))\frac{\phi(\eta_n)}{\phi[\psi_n, \eta_n]} \end{aligned} \tag{2}$$

we replace $\phi'(\xi_n) \approx \phi[\kappa_n, \xi_n]$, where [5] $\kappa_n = \xi_n + \lambda\phi^m(\xi_n)\ \forall\ \lambda \neq 0 \wedge m \geq 2$
taking$\lambda = 1 \wedge m = 3$.
Where $\phi[\kappa_n, \xi_n] = \frac{\phi(\kappa_n) - \phi(\xi_n)}{\kappa_n - \xi_n}$, $\phi[\psi_n, \eta_n] = \frac{\phi(\psi_n) - \phi(\eta_n)}{\psi_n - \eta_n}$ and for weight function $(A(t_1) + B(t_2) + C(t_3))$ we expand the taylor series about 1

$$\begin{aligned} A(t_3) &= A(1) + (t_3 - 1)A'(1) + \tfrac{1}{2}(t_3 - 1)^2 A''(1) + \tfrac{1}{6}(t_3 - 1)^3 A'''(1) + \ldots. \\ B(t_1) &= B(1) + (t_1 - 1)B'(1) + \tfrac{1}{2}(t_1 - 1)^2 B''(1) + \tfrac{1}{6}(t_1 - 1)^3 B'''(1) + \ldots. \\ C(t_2) &= C(1) + (t_2 - 1)C'(1) + \tfrac{1}{2}(t_2 - 1)^2 C''(1) + \tfrac{1}{6}(t_2 - 1)^3 C'''(1) + \ldots. \end{aligned} \tag{3}$$

By taking

$$\begin{aligned} &A(1) = 2,\ A'(1) = A''(1) = 6,\ A^n(1) = 0,\ \text{if } n > 2,\ B(1) = -1,\ B'(1) = B''(1) = 0, B^n(1) = 0, \\ &\text{if } n > 2,\ C(1) = 3,\ C'(1) = 2,\ C^n(1) = 0 \text{ if } n > 1 \end{aligned}$$

we get

$$\begin{aligned} A(t_1) &= 1 + t_1^2 \\ B(t_2) &= -1 \\ C(t_3) &= 1 + 2t_3 \end{aligned} \tag{4}$$

and $t_1 = \frac{\phi(\psi_n)}{\phi(\xi_n)}$, $t_2 = \frac{\phi(\eta_n)}{\phi(\psi_n)}$ & $t_3 = \frac{\phi(\eta_n)}{\phi(\xi_n)}$

3 Convergence Analysis

Theorem I: Let $a \in D$ be a simple zero of a sufficiently differentiable function $\phi : D \subset \Re \rightarrow \Re$ in an open interval D, which contains ξ_0 as an initial approximation of a. Then the method (2) is of order eighth and includes only four function evaluations per iteration, and no derivatives used.
Proof.
We write down the Taylor's series expansion of the function $\phi(\xi_n)$.

$$\phi(\xi_n) = \sum_{m=0}^{\infty} \frac{\phi^m(a)}{m!}(\xi_n - a)^m = \phi(a) + \phi'(a)(\xi_n - a) + \frac{\phi''(a)}{2!}(\xi_n - a)^2 + \frac{\phi'''(a)}{3!}(\xi_n - a)^3 + \ldots \tag{5}$$

For simplicity, we assume that $A_k = \left(\frac{1}{k!}\right)\frac{\phi^k(a)}{\phi'(a)}$, $k \geq 2$.
and assume that $\varepsilon_n = \xi_n - a$. thus, we have

$$\phi(\xi_n) = \phi'(a)\left[\varepsilon_n + A_2\varepsilon_n^2 + A_3\varepsilon_n^3 + A_4\varepsilon_n^4 + \ldots\right] \tag{6}$$

Furthermore, we have

$$\phi[\kappa_n, \xi_n] = \frac{\phi(\kappa_n) - \phi(\xi_n)}{\kappa_n - \xi_n} = \phi'(a)\begin{pmatrix} 1 + 2A_2\varepsilon_n + 3A_3\varepsilon_n^2 + \left(A_2\phi'^3(a) + 4A_4\right)\varepsilon_n^3 + \\ 3\phi'^3(a)\left(A_2^2 + A_3\right)\varepsilon_n^4 + \ldots + O\left(\varepsilon_n^9\right) \end{pmatrix} \tag{7}$$

Dividing (5) by (6) gives us

$$\frac{\phi(\xi_n)}{\phi[\kappa_n, \xi_n]} = \varepsilon_n - A_2\varepsilon_n^2 + 2\left(A_2^2 - A_3\right)\varepsilon_n^3 + \left(-4A_2^3 + 7A_2A_3 - A_2\phi'^3(a) - 3A_4\right)\varepsilon_n^4 + \ldots + O\left(\varepsilon_n^9\right) \tag{8}$$

and hence, we have

$$\text{Step.1 } \psi_n = \xi_n - \frac{\phi(\xi_n)}{\phi[\kappa_n, \xi_n]} = A_2\varepsilon_n^2 + (-2A_2^2 + A_2\phi'^2(a) + 2A_3)\varepsilon_n^3 + (4A_2^3 - A_2^2\phi'^2(a) - 7A_2A_3 + 3A_3\phi'^2(a) + 3A_4)\varepsilon_n^4 + \ldots + O(\varepsilon_n^9) \tag{9}$$

and

$$\phi(\psi_n) = \phi'(a)[A_2\varepsilon_n^2 + (-2A_2^2 + A_2\phi'^2 + 2A_3)\varepsilon_n^3 + (4A_2^3 - A_2^2\phi'^2(a) - 7A_2A_3 + 3A_3\phi'^2(a) + 3A_4)\varepsilon_n^4 + \ldots + O(\varepsilon_n^9)] \tag{10}$$

from equation (5) and (9) we got

$$\phi[\xi_n, \psi_n] = \phi'(a)(1 + A_2\varepsilon_n + (A_2^2 + A_3)\varepsilon_n^2 + (-2A_2^3 + 3A_2A_3 + A_4)\varepsilon_n^3 + \ldots + O(\varepsilon_n^9)) \tag{11}$$

$$\frac{\phi(\psi_n)}{\phi[\xi_n, \psi_n]} = A_2\varepsilon_n^2 + (2A_3 - 3A_2^2)\varepsilon_n^3 + (7A_2^3 - 10A_2A_3 + A_2\phi'^3(a) + 3A_4)\varepsilon_n^4 + \ldots + O(\varepsilon_n^9) \tag{12}$$

and

$$\frac{\phi^2(\xi_n)}{\phi^2(\xi_n) - 2\phi(\xi_n)\phi(\psi_n) + \phi^2(\psi_n)} = 1 + 2A_2\varepsilon_n + (4A_3 - 3A_2^2)\varepsilon_n^2 + 2(A_2^3 - 4A_2A_3 + A_2\phi'^3(a) + 3A_4)\varepsilon_n^3 + \ldots + O(\varepsilon_n^9) \tag{13}$$

$$\text{Step.2 } \eta_n = \psi_n - \frac{\phi(\psi_n)}{\phi[\kappa_n, \xi_n]}\left(\frac{\phi^2(\xi_n)}{\phi^2(\xi_n) - 2\phi(\xi_n)\phi(\psi_n) + \phi^2(\psi_n)}\right) = \left(2A_2^3 - A_2A_3\right)\varepsilon_n^4 + \left(-10A_2^4 + 14A_2^2A_3 - A_2^2\phi'^3(a) - 2A_2A_4 - 2A_3^2\right)\varepsilon_n^5 + \ldots + O\left(\varepsilon_n^9\right) \tag{14}$$

And

$$\phi(\eta_n) = \phi'(a)\left((2A_2^3 - A_2A_3)\varepsilon_n^4 + \begin{pmatrix} -10A_2^4 + 14A_2^2A_3 - \\ A_2^2\phi'^3(a) - 2A_2A_4 - 2A_3^2 \end{pmatrix}\varepsilon_n^5 + \ldots + O\left(\varepsilon_n^9\right)\right) \tag{15}$$

And

$$\phi[\psi_n, \eta_n] = \frac{\phi(\psi_n) - \phi(\eta_n)}{\psi_n - \eta_n} = \phi'(a)\begin{pmatrix} 1 + A_2^2\varepsilon_n^2 + 2A_2\left(A_3 - A_2^2\right)\varepsilon_n^3 + \\ A_2\left(6A_2^3 + A_2\left(\phi'^3(a) - 7A_3\right) + 3A_4\right)\varepsilon_n^4 + \\ \ldots + O\left(\varepsilon_n^9\right) \end{pmatrix} \tag{16}$$

And weight function

$$(A(t_1) + B(t_2) + C(t_3)) = 1 + A_2^2\varepsilon_n^2 + 2\left(A_2A_3 - A_2^3\right)\varepsilon_n^3 + A_2\left(A_2^3 - 2A_2A_3 + 2A_4\right)\varepsilon_n^4 + \ldots + O\left(\varepsilon_n^9\right) \tag{17}$$

Finally

$$\text{Step.3 } \xi_{n+1} = \eta_n - (A(t_1) + B(t_2) + C(t_3))\frac{\phi(\eta_n)}{\phi[\psi_n, \eta_n]} = A_2^2\left(2A_2^2 - A_3\right)\left(3A_2^3 + A_2\left(\phi'^3(a) - 4A_3\right) + A_4\right)\varepsilon_n^8 + O\left(\varepsilon_n^9\right) \tag{18}$$

Since the order of convergence of proposed method is four, and there are three function evaluation used in each iteration with no derivative. Its efficiency index is $4^{1/3} = 1.587401\ldots$

4 Application Problems

Maple 2021 and Mathematica 2021 were used for solution of following application problems, and Matlab was used to find the COC of application problems.

Problem 1. Planck's radiation law. See in [7, 20–22]

$$\phi_1 = \exp(-x) - 1 + \frac{x}{5} \tag{19}$$

Table 1. Numerical results for problem 1. for first four iterations and their absolute function values at $x_0 = 4$

Method	1^{st} iteration	2^{nd} iteration	3^{rd} iteration	4^{th} iteration
Proposed 8^{th}	4.96511428×10^{0} $9.67569418 \times 10^{-9}$	4.96511423×10^{0} $3.50682003 \times 10^{-68}$	4.96511423×10^{0} $1.04415339 \times 10^{-543}$	4.96511423×10^{0} $6.45016629 \times 10^{-4348}$
SM 8^{th}	4.96512223×10^{0} $1.54477412 \times 10^{-6}$	4.96511423×10^{0} $6.60093214 \times 10^{-38}$	4.96511423×10^{0} $4.01840927 \times 10^{-226}$	4.96511423×10^{0} $2.04525547 \times 10^{-1355}$
AKKB 8^{th}	4.96511467×10^{0} $8.42634074 \times 10^{-8}$	4.96511423×10^{0} $1.84817509 \times 10^{-40}$	4.96511423×10^{0} $9.38132174 \times 10^{-204}$	4.96511423×10^{0} $3.16132118 \times 10^{-1020}$
KBK 8^{th}	4.96511410×10^{0} $2.56680409 \times 10^{-8}$	4.96511423×10^{0} $3.29102226 \times 10^{-64}$	4.96511423×10^{0} $2.40345205 \times 10^{-511}$	4.96511423×10^{0} $1.94478951 \times 10^{-4088}$
JLM 8^{th}	4.96511426×10^{0} $6.35715655 \times 10^{-9}$	4.96511423×10^{0} $3.07879628 \times 10^{-54}$	4.96511423×10^{0} $3.97273188 \times 10^{-326}$	4.96511423×10^{0} $1.83374553 \times 10^{-1957}$

Table 2. Numerical results for problem 1., error fixed at *delta* = 10^{-3000}

Method	*IG*	*N*	*FE*
Proposed 8^{th}	4	4	16
SM 8^{th}	4	5	20
AKKB 8^{th}	4	5	20
KBK 8^{th}	4	4	16
JLM 8^{th}	4	5	20

Problem 2. (car stability). Application in mechanical engineering See in [23].

$$\phi_2(x) = x^4 - 1.9404x^2 + 0.75 \quad (20)$$

Table 3. Numerical results for problem 2. for first four iterations and their absolute function values at $x_0 = -0.5$.

Method	1^{st} iteration	2^{nd} iteration	3^{rd} iteration	4^{th} iteration
Proposed 8^{th}	-7.29955558×10^{-1} 9.28913981×10^{-10}	-7.29955559×10^{-1} 6.01242356×10^{-72}	-7.29955559×10^{-1} $1.85200488\times10^{-569}$	-7.29955559×10^{-1} $1.50100656\times10^{-4549}$
SM 8^{th}	-7.29686598×10^{-1} $3.43561979\times10^{--4}$	-7.29955559×10^{-1} 5.96113371×10^{-26}	-7.29955559×10^{-1} $4.96026069\times10^{-200}$	-7.29955559×10^{-1} $1.14001249\times10^{-1592}$
AKKB 8^{th}	-7.29809213×10^{-1} 1.86914376×10^{-4}	-7.29955559×10^{-1} 7.58360417×10^{-31}	-7.29955559×10^{-1} $5.75141912\times10^{-242}$	-7.29955559×10^{-1} $6.29462552\times10^{-1931}$
KBK 8^{th}	-7.29954445×10^{-1} 1.42205591×10^{-6}	-7.29955559×10^{-1} 2.25812858×10^{-46}	-7.29955559×10^{-1} $-9.12880222\times10^{-365}$	-7.29955559×10^{-1} $-6.51230669\times10^{-2912}$
JLM 8^{th}	-7.29955550×10^{-1} 1.20206197×10^{-8}	-7.29955559×10^{-1} 6.40069903×10^{-48}	-7.29955559×10^{-1} $1.45891770\times10^{-283}$	-7.29955559×10^{-1} $2.04574324\times10^{-1697}$

Table 4. Numerical results for problem 2., error fixed at $delta = 10^{-3000}$

Method	*IG*	*N*	*FE*
Proposed 8^{th}	−0.5	4	16
SM 8^{th}	−0.5	5	20
AKKB 8^{th}	−0.5	5	20
KBK 8^{th}	−0.5	5	20
JLM 8^{th}	−0.5	5	20

Problem 3. (The Shockley Diode Equation and Electric Circuit). See in [24].

$$\phi_3 = -0.5 + 0.1\,x + 1.4\ln(x + 1) \quad (21)$$

Table 5. Numerical results for problem 3. for first four iterations and their absolute function values at $x_0 = 0$.

Method	1^{st} iteration	2^{nd} iteration	3^{rd} iteration	4^{th} iteration
Proposed 8^{th}	3.89992153×10^{-1} 3.80387471×10^{-5}	3.89977198×10^{-1} 5.64267581×10^{-39}	3.89977198×10^{-1} $1.32263897\times10^{-309}$	3.89977198×10^{-1} $1.20528858\times10^{-2474}$
SM 8^{th}	3.90022793×10^{-1} 5.04816858×10^{-5}	3.89977198×10^{-1} 2.79924005×10^{-37}	3.89977198×10^{-1} $2.50206521\times10^{-295}$	3.89977198×10^{-1} $1.01945125\times10^{-2359}$
AKKB 8^{th}	3.90071054×10^{-1} 1.03914445×10^{-4}	3.89977198×10^{-1} 2.45440600×10^{-34}	3.89977198×10^{1} $2.37750739\times10^{-271}$	3.89977198×10^{-1} $1.84300132\times10^{-2167}$
KBK 8^{th}	3.89596121×10^{-1} 4.21985237×10^{-4}	3.89977198×10^{-1} 4.01193524×10^{-31}	3.89977198×10^{-1} $2.67875461\times10^{-247}$	3.89977198×10^{-1} $1.05819347\times10^{-1976}$
JLM 8^{th}	3.89992153×10^{-1} 1.65577677×10^{-5}	3.89977198×10^{-1} 1.03787154×10^{-32}	3.89977198×10^{-1} $6.29492624\times10^{-196}$	3.89977198×10^{-1} $3.13381179\times10^{-1175}$

Table 6. Numerical results for problem 3., error fixed at *delta* = 10^{-3000}

Method	*IG*	*N*	*FE*
Proposed 8^{th}	0	5	20
SM 8^{th}	0	5	20
AKKB 8^{th}	0	5	20
KBK 8^{th}	0	5	20
JLM 8^{th}	0	5	20

Problem 4. Beam Designing Model (An Engineering Problem) See in [25]**.**

$$\phi_4 = \frac{x^4 + 2.87x^2 - 10.28}{4.62} - x \tag{22}$$

Table 7. Numerical results for problem 4. for first four iterations and their absolute function values at $x_0 = -3.5$.

Method	1^{st} iteration	2^{nd} iteration	3^{rd} iteration	4^{th} iteration
Proposed 8^{th}	$-3.33038696 \times 10^{0}$ $3.52258020 \times 10^{-6}$	$-3.33038866 \times 10^{0}$ $6.26674991 \times 10^{-46}$	$-3.33038866 \times 10^{0}$ $6.28755917 \times 10^{-364}$	$-3.33038866 \times 10^{0}$ $6.45654017 \times 10^{-2908}$
SM 8^{th}	$-3.33039778 \times 10^{0}$ $1.88258338 \times 10^{-5}$	$-3.33038866 \times 10^{0}$ $7.72183249 \times 10^{-39}$	$-3.33038866 \times 10^{0}$ $6.18758077 \times 10^{-306}$	$-3.33038866 \times 10^{0}$ $1.05178003 \times 10^{-2442}$
AKKB 8^{th}	$-3.33044757 \times 10^{0}$ $1.21628040 \times 10^{-4}$	$-3.33038866 \times 10^{0}$ $6.36232200 \times 10^{-32}$	$-3.33038866 \times 10^{0}$ $3.57010797 \times 10^{-250}$	$-3.33038866 \times 10^{0}$ $3.50919087 \times 10^{-1996}$
KBK 8^{th}	$-3.33039293 \times 10^{0}$ $8.81113549 \times 10^{-6}$	$-3.33038866 \times 10^{0}$ $3.72411245 \times 10^{-43}$	$-3.33038866 \times 10^{0}$ $3.79307745 \times 10^{-342}$	$-3.33038866 \times 10^{0}$ $4.39281694 \times 10^{-2734}$
JLM 8^{th}	$-3.33038937 \times 10^{0}$ $1.45545266 \times 10^{-6}$	$-3.33038866 \times 10^{0}$ $6.70379768 \times 10^{-39}$	$-3.33038866 \times 10^{0}$ $6.40118774 \times 10^{-233}$	$-3.33038866 \times 10^{0}$ $4.85175162 \times 10^{-1397}$

Table 8. Numerical results for problem 4., error fixed at *delta* = 10^{-3000}

Method	*IG*	*N*	*FE*
Proposed 8^{th}	−3.5	5	20
SM 8^{th}	−3.5	5	20
AKKB 8^{th}	−3.5	5	20
KBK 8^{th}	−3.5	5	20
JLM 8^{th}	−3.5	5	20

Problem 5 Below algebraic and transcendental problems were taken from literature and tested in proposed method.

Functions
$\phi_5(x) = \sin(x) + \cos(x) + x$
$\phi_6(x) = e^{sin(x)} - x + 1$
$\phi_7(x) = x^4 + x^3 + 8x + 9$
$\phi_8(x) = x^6 + (1-2x)^5$
$\phi_9(x) = 1 + e^{x^2+x} - \cos\left(-x^2+1\right) + x^3$
$\phi_{10}(x) = \tanh(x) + 2x$

Table 9. Shows the value of $\lvert x_1 - x_0 \rvert$, $\lvert x_2 - x_1 \rvert$, $\lvert x_3 - x_2 \rvert$ & *COC* of different methods of order eight.

$\lvert x_n - x_{n-1} \rvert$	Proposed 8^{th}	SM 8^{th}	AKKB 8^{th}	KBK 8^{th}	JLM 8^{th}

Case 1. For $(\phi_5(x), -0.2)$

$\lvert x_1 - x_0 \rvert$	$2.5663e-1$	$2.5667e-1$	$2.5659e-1$	$2.5662e-1$	$2.5662e-1$
$\lvert x_2 - x_1 \rvert$	$3.1377e-6$	$4.1049e-5$	$3.2525e-5$	$4.0069e-6$	$7.4256-7$
$\lvert x_3 - x_2 \rvert$	$1.3122e-47$	$2.0867e-28$	$1.2030e-25$	$5.1459e-46$	$1.5995e-41$
COC	8.4228	6.1363	5.2430	8.3075	6.2592

Case 2. For $(\phi_6(x), 3)$

$\lvert x_1 - x_0 \rvert$	$3.6934e-1$	$3.6932e-1$	$3.6934e-1$	$3.6931e-1$	$3.6934e-1$
$\lvert x_2 - x_1 \rvert$	$2.3201e-6$	$1.1058e-5$	$1.6206e-6$	$2.7809e-5$	$1.4600e-7$
$\lvert x_3 - x_2 \rvert$	$1.7253e-48$	$1.5512e-32$	$4.1167e-33$	$1.5342e-39$	$2.3747e-45$
COC	8.2024	5.9360	4.9639	8.0087	5.9017

Case 3. For $(\phi_8(x), 1.01)$

$\lvert x_1 - x_0 \rvert$	$1.0000e-2$	$1.0000e-2$	$9.9997e-3$	$1.0000e-2$	$1.0000e-2$
$\lvert x_2 - x_1 \rvert$	$1.6348e-12$	$3.3753e-9$	$3.1459e-7$	$5.9699e-11$	$1.1601e-8$
$\lvert x_3 - x_2 \rvert$	$3.3124e-90$	$2.5462e-61$	$1.6935e-43$	$8.54943-77$	$3.4369e-44$
COC	8.2003	8.0539	8.0557	8.0063	5.9857

Case 4. For $(\phi_9(x), -0.8)$

$\lvert x_1 - x_0 \rvert$	$2.0000e-1$	$2.0000e-1$	$2.0000e-1$	$1.9999e-1$	$2.0000e-1$
$\lvert x_2 - x_1 \rvert$	$3.1850e-7$	$4.6487e-6$	$1.8970e-6$	$6.9811e-6$	$4.1646e-6$
$\lvert x_3 - x_2 \rvert$	$2.8535e-53$	$2.4506e-42$	$4.5696e-464$	$8.4973e-42$	$4.0472e-34$
COC	8.2021	7.8292	7.8874	8.0579	5.9837

Case 5. For $(\phi_{10}(x), 1)$

$\lvert x_1 - x_0 \rvert$	3.0001	3.0062	3.0075	Not Converge	3.0021
$\lvert x_2 - x_1 \rvert$	$7.5517e-5$	$6.1713e-3$	$7.4714e-3$	Not Converge	$2.1370e-3$
$\lvert x_3 - x_2 \rvert$	$2.2396e-47$	$3.8545e-25$	$7.8817e-24$	Not Converge	$2.6473e-25$
COC	8.5470	8.2617	8.0531	Not Converge	6.9598

5 Conclusion

In this paper, the main attention has been made to derive a three-step optimal derivative free method of eighth order for finding the root of a non-linear equation in application problems using weight function. Various application problems have been tested by proposed method and compared with other counterpart methods available in the literature. It's observed from the above tables that the proposed method is reducing the error faster than the other methods. It's accurate, consistent and its stability is much better as compared to counterpart methods available in the literature, above proposed method is good addition in literature. In the future we are interested in proposing a four-point derivative free optimal iterative method of order sixteen.

Credit Author Statement

Sanaullah Jamali: Solution of the problem, **Zubair Ahmed Kalhoro**: Conceptualization, Methodology, **Abdul Wasim Shaikh**: Visualization, Investigation. **Muhammad Saleem Chandio**: Supervision, Software support, **Sanaullah Dehraj**: Software, Validation, Writing- Reviewing and Editing

Compliance with Ethical Standards

It is declared that all authors don't have any conflict of interest. Furthermore, informed consent was obtained from all individual participants included in the study.

Funding Information

No funding

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References

[1] D. Kincaid and W. Cheney. *Numerical Analysis*. Brooks/Cole Publishing Company, 2009.

[2] A. Cordero, J. L. Hueso, E. Martínez, and J. R. Torregrosa. Generating optimal derivative free iterative methods for nonlinear equations by using polynomial interpolation. *Math. Comput. Model*, 57(7):1950–1956, 2013. doi: https://doi.org/10.1016/j.mcm.2012.01.012.

[3] S. Jamali, Z. A. Kalhoro, A. W. Shaikh, and M. S. Chandio. A New Second Order Derivative Free Method for Numerical Solution of Non-Linear Algebraic and Transcendental Equations using Interpolation Technique. *Journal of Mechanics of Continua Mathematical Sciences*, 16(4):75–84, apr 2021. ISSN 0973-8975. doi: https://doi.org/10.26782/jmcms.2021.04.00006.

[4] S. Jamali, Z. A. Kalhoro, A. W. Shaikh, and M. S. Chandio. An Iterative, Bracketing & Derivative-Free Method for Numerical Solution of Non-Linear Equations using Stirling Interpolation Technique. *Journal of Mechanics of Continua Mathematical Sciences*, 16(6):13–27, jun 2021. ISSN 0973-8975. doi: https://doi.org/10.26782/jmcms.2021.06.00002.

[5] A. Cordero and J. R. Torregrosa. Low-complexity root-finding iteration functions with no derivatives of any order of convergence. *J. Comput. Appl. Math*, 275:502–515, February 2015. doi: https://doi.org/10.1016/j.cam.2014.01.024.

[6] B. Neta. A new derivative-free method to solve nonlinear equations. *Mathematics*, 9(6):1–5, 2021. doi: https://doi.org/10.3390/math9060583.

[7] I. K. Argyros, M. Kansal, V. Kanwar, and S. Bajaj. Higher-order derivative-free families of chebyshev–halley type methods with or without memory for solving nonlinear equations. *Appl. Math*, 315:224–245, December 2017. doi: https://doi.org/10.1016/j.amc.2017.07.051.

[8] A. Suhadolnik. Combined bracketing methods for solving nonlinear equations. *Appl. Math*, 25(11):1755–1760, 2012. doi: https://doi.org/10.1016/j.aml.2012.02.006.

[9] U. K. Qureshi, Z. A. Kalhoro, A. A. Shaikh, and S. Jamali. Sixth Order Numerical Iterated Method of Open Methods for Solving Nonlinear Applications Problems. *Proceedings of the Pakistan Academy of Sciences: A. Physical and Computational Sciences*, 57(November):35–40, 2020.

[10] M. K. Mahmood and F. Ahmad. Recursive elucidation of polynomial congruences using root-finding numerical techniques. *Abstract and Applied Analysis*, 2014:9, 2014. doi: http://dx.doi.org/10.1155/2014/575064.

[11] S. Jamali, Z. A. Kalhoro, A. W. Shaikh, and M. S. Chandio. An Iterative, Bracketing & Derivative-Free Method for Numerical Solution of Non-Linear Equations using Stirling Interpolation Technique. *Journal of Mechanics of Continua Mathematical Sciences*, 16(6):13–27, jun 2021. ISSN 0973-8975. doi: https://doi.org/10.26782/jmcms.2021.06.00002.

[12] Farooq Ahmed Shah and Ehsan Ul Haq. Some new multi-step derivative-free iterative methods for solving nonlinear equations. *App. and Eng. Math.*, 10(4):951–963, 2020.

[13] U. K. Qureshi, S. Jamali, Z. A. Kalhoro, and A. G. Shaikh. Modified Quadrature Iterated Methods of Boole Rule and Weddle Rule for Solving non-Linear Equations. *Journal of Mechanics of continua and Mathematical sciences*, 16(2):87–101, feb 2021. ISSN 0973-8975. doi: https://doi.org/10.26782/jmcms.2021.02.00008.

[14] A. Suhadolnik. Superlinear bracketing method for solving nonlinear equations. *Appl. Math*, 219(14): 7369–7376, 2013. doi: https://doi.org/10.1016/j.amc.2012.12.064.

[15] F. Soleymani, D. K. R. Babajee, and M. Sharifi. Modified jarratt method without memory with twelfth-order convergence. *Annals of the University of Craiova, Mathematics and Computer Science Series*, 39(1): 21–34, 2012. ISSN 1223-6934.

[16] S. Jamali, Z. A. Kalhoro, A. W. Shaikh, M. S. Chandio, and S. Dehraj. A novel two point optimal derivative free method for numerical solution of nonlinear algebraic, transcendental equations and application problems using weight function. *VFAST Transactions on Mathematics*, 10(2):137–146, 2022. ISSN 2309-0022.

[17] M. S. Petković, S. Ilic, and J. Dzunic. Derivative free two-point methods with and without memory for solving nonlinear equations. *Appl. Math*, 217:1887–1895, 2010. doi: https://doi.org/10.1016/j.amc.2010.06.043.

[18] A. Cordero, J. L. Hueso, E. Martínez, and J. R. Torregrosa. Steffensen type methods for solving nonlinear equations . *J. Comput. Appl. Math*, 236(12):3058–3064, 2012. doi: https://doi.org/10.1016/j.cam.2010.08.043.

[19] B. Kong-ied. Two new eighth and twelfth order iterative methods for solving nonlinear equations. *Int. J. Math*, 16(1):333–344, 2021.

[20] D. Jain. Families of newton-like methods with fourth-order convergence. *Int. J. Comput. Math*, 90(5): 1072–1082, 2013. doi: https://doi.org/10.1080/00207160.2012.746677.

[21] P. Sivakumar and J. Jayaraman. Some new higher order weighted newton methods for solving nonlinear equation with applications. *Math. Comput. Appl.*, 24(2):1–16, 2019. doi: https://doi.org/10.3390/mca24020059.

[22] F. Soleymani. Efficient optimal eighth-order derivative-free methods for nonlinear equations. *Jpn. J. Ind. Appl. Math*, 30(2):287–306, June 2013. doi: https://doi.org/10.1007/s13160-013-0103-7.

[23] N. Rafiq, M. Shams, N. A. Mir, and Y. U. Gaba. A highly efficient computer method for solving polynomial equations appearing in engineering problems. *Math. Probl. Eng*, 2021, 2021. doi: https://doi.org/10.1155/2021/9826693.

[24] F. Zafar, A. Cordero, and J. R. Torregrosa. An efficient family of optimal eighth-order multiple root finders. *Mathematics*, 6(12):1–16, 2018. doi: https://doi.org/10.3390/math6120310.

[25] M. Shams, N. Ra, N. Kausar, N. A. Mir, and A. Alalyani. Computer oriented numerical scheme for solving engineering problems. *Comput. Syst. Sci*, 42(2):689–701, 2022. doi: https://doi.org/10.32604/csse.2022.022269.