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A novel two point optimal derivative free method for numerical solution of nonlinear algebraic, transcendental Equations and application problems using weight function

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Abstract It's a big challenge for researchers to locate the root of nonlinear equations with minimum cost, lot of methods are already exist in literature to find root but their cost are very high In this regard we introduce a two-step fourth order method by using weight function. And proposed method is optimal and derivative free for solution of nonlinear algebraic and transcendental and application problems. MATLAB, Mathematica and Maple software are used to solve the convergence and numerical problems of proposed and their counterpart methods.

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Introduction

Finding the zeros of nonlinear functions quickly and accurately is a fundamental task in scientific computation. In general, this is the problem of solving a nonlinear equation $\phi(\xi) = 0$, where $\phi : \delta \subseteq \Re \to \Re$. There are certain analytical methods to solve nonlinear algebraic equations only $\phi(\xi) = 0$, upto 4^{th} degree. There is no any analytical method for solution of transcendental equations and nonlinear algebraic equations of higher order. As a result, the only option to get sufficient numerical solution is through numerical methods based on iterative procedures. One of the most well-known and renowned iterative approach for finding solution to nonlinear equation is Newton's method [1].

$$\xi_{n+1} = \xi_n - \frac{\phi(\xi_n)}{\phi'(\xi_n)} \tag{1}$$

However the convergence of Newton method is quadratic and there are two function evaluation required i.e $\phi(\xi)$ and $\phi'(\xi)$, if $\phi'(\xi) = 0$than this method not work at all and the computation cost of methods involved derivative are increased as compare to derivative free method that's way now the researchers are preferred derivative free methods. Steffensen developed a derivative-free iterative method [2, 3].

$$\omega^n = \xi^n + \phi(\xi^n), \;\; \xi^{n+1} = \xi^n - \frac{\phi(\xi^n)}{[\xi^n, \omega^n]} \tag{2}$$

where

$$[\xi^n, \omega^n] = \frac{\phi(\xi^n) - \phi(\omega^n)}{\xi^n - \omega^n}$$

It maintains the Newton's method's convergence order and efficiency index. The primary goal of this research is to develop efficient derivative-free techniques for solving nonlinear equation and We obtained a two point optimal iterative methods that will support the conjecture [4]. Multi-point iteration methods based on functional evaluations p could achieve an optimal convergence order2^{p-1}, according to Kung-Traub.

Before proceeding to the proposed method. First we start a short literature review of existing derivative free methods. Various derivative free algorithms of different order for solution of nonlinear equations introduced by [5–10] which are based on different interpolating technique and two new Chebyshev–Halley type derivative-free methods have been presented for numerical solution nonlinear equations. Both families require just three and four functional evaluations, respectively, to reach optimum fourth and eighth order of convergence [11, 12] Several techniques have been developed that are based on combinations of bisection, regula falsi, and parabolic interpolation named combined bracketing methods for solution of the nonlinear equations. A new approach for solving nonlinear equations based on Muller's algorithm has been developed by [13]. The Muller's approach is based on an interpolating polynomial constructed from the last three points of an iterative sequence. [14, 15] develop two hybrid method for solution of nonlinear equation. [16] P. Sivakumar and J. Jayaraman introduce three iterative method with two step fifth order with four function evaluation, three step eight order with five function evaluation and two step fourth order methods using weight function. C. Chun. *et al.*[17] Introduce the choosing weight functions, the main objective using weight functions in a iterative scheme is to increase the order of convergence and enhance the behavior of the scheme. For comparison we take ZLM see in [18], KTM see in [4], AKKB see in [11] and JLM see in [19].

Derivation of proposed method

In [20] mentioned a two-step fourth order optimal method with four function evaluation (two function and two first derivative). i.e.

$$\begin{aligned} &\text{Step.1 } \psi_n = \xi_n - \frac{\phi(\xi_n)}{\phi'(\xi_n)} \\ &\text{Step.2 } \xi_{n+1} = \psi_n - \frac{\phi(\psi_n)}{\phi'(\psi_n)} \end{aligned} \tag{3}$$

we replace $\phi'(\xi_n) \approx \phi[\kappa_n, \xi_n]$, where [9]

$$\kappa_n = \xi_n + \lambda\phi^m(\xi_n)\ \forall\, \lambda \neq 0 \wedge m \geq 2 \tag{4}$$

Taking $\lambda = 1 \wedge m = 3$ for two and three point methods.

$$\phi'(\xi_n) \approx \phi[\kappa_n, \xi_n] = \frac{\phi(\kappa_n) - \phi(\xi_n)}{\kappa_n - \xi_n} \tag{5}$$

And we replace

$$\phi'(\psi_n) \approx \phi[\xi_n, \psi_n] = \frac{\phi(\xi_n) - \phi(\psi_n)}{\xi_n - \psi_n} \tag{6}$$

Finally, we got

$$\begin{aligned} &\text{Step.1 } \psi_n = \xi_n - \frac{\phi(\xi_n)}{\phi[\kappa_n, \xi_n]} \\ &\text{Step.2 } \xi_{n+1} = \psi_n - \frac{\phi(\psi_n)}{\phi[\xi_n, \psi_n]} \end{aligned} \tag{7}$$

In equation (7), we introduced the weight function to enhance the. For weight function $A(t_1)$ about 1 we expand the Taylor series.

$$A(t_1) = A(1) + (t_1 - 1)A'(1) + \frac{1}{2}(t_1 - 1)^2 A''(1) + \frac{1}{6}(t_1 - 1)^3 A'''(1) + \dots.$$

By taking A(1)=2, $A'(1) = 1$, $\left|A''(1)\right| < \infty$

We get $A(t_1) = 1 + t_1$.

where

$$t_1 = \frac{\phi(\psi_n)}{\phi(\xi_n)}$$

.

Finally we got.

$$\begin{aligned} &\text{Step.1 } \psi_n = \xi_n - \frac{\phi(\xi_n)}{\phi[\kappa_n, \xi_n]} \\ &\text{Step.2 } \xi_{n+1} = \psi_n - A(t_1)\frac{\phi(\psi_n)}{\phi[\xi_n, \psi_n]} \end{aligned} \tag{8}$$

Convergence Analysis

Theorem I: Let $a \in D$ be a simple zero of a sufficiently differentiable function $\phi : D \subset \Re \to \Re$ in an open interval$D$, which contains ξ_0 as an initial approximation of a. Then the method in equation (8) is of order fourth and includes only three function evaluations per full iteration, and no derivatives used.

Proof.

We write down the Taylor's series expansion of the function$\phi(\xi_n)$.

$$\phi(\xi_n) = \sum_{m=0}^{\infty} \frac{\phi^m(a)}{m!}(\xi_n - a)^m = \phi(a) + \phi'(a)(\xi_n - a) + \frac{\phi''(a)}{2!}(\xi_n - a)^2 + \frac{\phi'''(a)}{3!}(\xi_n - a)^3 + \ldots \tag{9}$$

For simplicity, we assume that $A_k = \left(\frac{1}{k!}\right)\frac{\phi^k(a)}{\phi'(a)}$, $k \geq 2$.

And assume that $\varepsilon_n = \xi_n - a$. Thus, we have

$$\phi(\xi_n) = \phi'(a)\left[\varepsilon_n + A_2\varepsilon_n^2 + A_3\varepsilon_n^3 + A_4\varepsilon_n^4 + \ldots\right] \tag{10}$$

Furthermore, we have

$$\phi[\kappa_n, \xi_n] = \frac{\phi(\kappa_n) - \phi(\xi_n)}{\kappa_n - \xi_n} = \phi'(a)\begin{pmatrix} 1 + 2A_2\varepsilon_n + 3A_3\varepsilon_n^2 + \left(A_2\phi'^3(a) + 4A_4\right)\varepsilon_n^3 + \\ 3\phi'^3(a)\left(A_2^2 + A_3\right)\varepsilon_n^4 + O\left(\varepsilon_n^5\right) \end{pmatrix} \tag{11}$$

Dividing (10) by (11) gives us

$$\frac{\phi(\xi_n)}{\phi[\kappa_n, \xi_n]} = \varepsilon_n - A_2\varepsilon_n^2 + 2\left(A_2^2 - A_3\right)\varepsilon_n^3 + \left(-4A_2^3 + 7A_2A_3 - A_2\phi'^3(a) - 3A_4\right)\varepsilon_n^4 + O\left(\varepsilon_n^5\right) \tag{12}$$

And hence, we have

$$\text{Step.1 } \psi_n = \xi_n - \frac{\phi(\xi_n)}{\phi[\kappa_n, \xi_n]} = A_2\varepsilon_n^2 + (-2A_2^2 + A_2\phi'^2(a) + 2A_3)\varepsilon_n^3 + (4A_2^3 - A_2^2\phi'^2(a) - 7A_2A_3 + 3A_3\phi'^2(a) + 3A_4)\varepsilon_n^4 + O(\varepsilon_n^5) \tag{13}$$

Again expanding $\phi(\psi_n)$ about a, we have

$$\phi(\psi_n) = \phi'(a)[\psi_n + A_2\psi_n^2 + A_3\psi_n^3 + A_4\psi_n^4 + O(\psi_n^5)] \tag{14}$$

And then from (14), we have

$$\phi(\psi_n) = \phi'(a)[A_2\varepsilon_n^2 + (-2A_2^2 + A_2\phi'^2 + 2A_3)\varepsilon_n^3 + (4A_2^3 - A_2^2\phi'^2(a) - 7A_2A_3 + 3A_3\phi'^2(a) + 3A_4)\varepsilon_n^4 + O(\varepsilon_n^5)] \tag{15}$$

From equation (10) and (15) we got

$$\phi[\xi_n, \psi_n] = \phi'(a)(1 + A_2\varepsilon_n + (A_2^2 + A_3)\varepsilon_n^2 + (-2A_2^3 + 3A_2A_3 + A_4)\varepsilon_n^3 + \ldots + O(\varepsilon_n^5)), \tag{16}$$

$$\frac{\phi(\psi_n)}{\phi[\xi_n, \psi_n]} = A_2\varepsilon_n^2 + (2A_3 - 3A_2^2)\varepsilon_n^3 + (7A_2^3 - 10A_2A_3 + A_2\phi'^3(a) + 3A_4)\varepsilon_n^4 + O(\varepsilon_n^5) \tag{17}$$

And

$$A(t_1)\frac{\phi(\psi_n)}{\phi[\xi_n, \psi_n]} = A_2\varepsilon_n^2 + (2A_3 - 2A_2^2)\varepsilon_n^3 + (A_2^3 - 6A_2A_3 + A_2\phi'^3(a) + 3A_4)\varepsilon_n^4 + O(\varepsilon_n^5) \tag{18}$$

Finally

$$\text{Step.2 } \xi_{n+1} = \psi_n - A(t_1)\frac{\phi(\psi_n)}{\phi[\xi_n, \psi_n]} = \left(3A_2^3 - A_2A_3\right)\varepsilon_n^4 + O\left(\varepsilon_n^5\right) \tag{19}$$

Since the order of convergence of proposed method is four, and there are three function evaluation used in each iteration with no derivative. Its efficiency index is $4^{1/3} = 1.587401$.

Numerical Results

Below problems are taken from literature [21–23] and tested in new method

Problem-1

Functions	Exact Root
$\phi_1(x) = \sin(x) + \cos(x) + x$	–4.566247045676308...
$\phi_2(x) = \log\left(x^2 + x + 2\right) - x + 1$	4.152590736757158...
$\phi_3(x) = x^3 + 2x + \sin(x)$	0
$\phi_4(x) = 1 + e^{x^2+x} - \cos\left(-x^2 + 1\right) + x^3$	–1
$\phi_5(x) = 3x + \cos(x) + e^x$	–0.4961113566254599...
$\phi_6(x) = -20x^5 - \frac{x}{2} + \frac{1}{2}$	0.4276772969310036...

Table 1. Shows the value of $|x_1 - x_0|$, $|x_2 - x_1|$, $|x_3 - x_2|$ of proposed method and its counterpart methods.

$(\phi_n(x), x_0)$	Proposed 4^{th}	ZLM 4^{th}	KTM 4^{th}	AKKB 4^{th}	JLM 4^{th}
$(\phi_1(x), -0.6)$					
$\|x_1 - x_0\|$	$1.43375e-1$	$1.43359e-1$	$1.43358e-1$	$1.433591e-1$	$1.433708e-1$
$\|x_2 - x_1\|$	$6.19537e-7$	$1.66868e-5$	$1.68884e-5$	$1.619300e-5$	$4.480367e-6$
$\|x_3 - x_2\|$	$1.78343e-27$	$8.85189e-21$	$1.01312e-20$	$6.182331e-21$	$4.127964e-24$
$(\phi_2(x), 14)$					
$\|x_1 - x_0\|$	9.83698	9.82568	9.82345	9.823818	9.439102
$\|x_2 - x_1\|$	$1.04272e-2$	$2.17324e-2$	$2.39597e-2$	$2.359085e-2$	$4.082951e-1$
$\|x_3 - x_2\|$	$1.24865e-11$	$2.14886e-11$	$4.29164e-11$	$3.616746e-11$	$1.221476e-5$
$(\phi_3(x), 2)$					
$\|x_1 - x_0\|$	1.90068	1.04025	$9.73336e-1$	Not Converge	$9.507624e-1$
$\|x_2 - x_1\|$	$9.93168e-2$	$8.30100e-1$	$7.58700e-1$	Not Converge	$7.178805e-1$
$\|x_3 - x_2\|$	$3.88163e-6$	$1.29461e-1$	$2.65633e-1$	Not Converge	$3.225571e-1$
$(\phi_4(x), 0.4)$					
$\|x_1 - x_0\|$	6.14734	$2.55573e-1$	Not Converge	Not Converge	$3.819999e-2$
$\|x_2 - x_1\|$	$1.47340e-2$	$3.87264e-1$	Not Converge	Not Converge	1.139059
$\|x_3 - x_2\|$	$1.84544e-8$	$4.28567e-2$	Not Converge	Not Converge	$3.188494e-1$
$(\phi_5(x), -10)$					
$\|x_1 - x_0\|$	9.43015	9.35838	9.36034	9.352975	9.334471
$\|x_2 - x_1\|$	$7.37348e-2$	$1.45507e-1$	$1.43547e-1$	$1.509121e-1$	$1.694167e-1$
$\|x_3 - x_2\|$	$2.29922e-7$	$2.40526e-6$	$2.49807e-6$	$1.661492e-6$	$1.140798e-6$
$(\phi_6(x), 0.4)$					
$\|x_1 - x_0\|$	$2.77925e-2$	$2.77681e-2$	$2.79588e-2$	$2.918221e-2$	Not Converge
$\|x_2 - x_1\|$	$1.15185e-4$	$9.08258e-4$	$2.81466e-4$	$1.504898e-3$	Not Converge
$\|x_3 - x_2\|$	$2.87253e-14$	$8.58004e-14$	$4.90329e-12$	$1.199854e-8$	Not Converge

Problem 2. Planck's radiation law. See in [11, 16, 24, 25] **(**Application problem**)**

$$\exp(-x) - 1 + \frac{x}{5} = 0 \tag{20}$$

Initial guess $x_0 = 4m$.

Table 2. Numerical results for problem 2 for first four iterations and their absolute function values at $x_0 = 4$

Method	Iterations	1st iteration	2nd iteration	3rd iteration	4th iteration
Proposed 4^{th}	μ	4.96558×10^{0}	4.96511×10^{0}	4.96511×10^{0}	4.96511×10^{0}
	$\lvert\zeta(\mu)\rvert$	9.02926×10^{-5}	1.16925×10^{-18}	3.29009×10^{-74}	2.06260×10^{-296}
ZLM 4^{th}	μ	4.96569×10^{0}	4.96511×10^{0}	4.96511×10^{0}	4.96511×10^{0}
	$\lvert\zeta(\mu)\rvert$	1.12180×10^{-4}	3.59465×10^{-18}	3.79305×10^{-72}	4.70236×10^{-288}
KTM 4^{th}	μ	4.96577×10^{0}	4.96511×10^{0}	4.96511×10^{0}	4.96511×10^{0}
	$\lvert\zeta(\mu)\rvert$	1.27161×10^{-4}	6.23920×10^{-18}	3.61957×10^{-71}	4.09988×10^{-284}
AKKB 4^{th}	μ	4.96568×10^{0}	4.96511×10^{0}	4.96511×10^{0}	4.96511×10^{0}
	$\lvert\zeta(\mu)\rvert$	1.09601×10^{-4}	3.24297×10^{-18}	2.48770×10^{-72}	8.61433×10^{-289}
JLM 4^{th}	μ	4.96548×10^{0}	4.96511×10^{0}	4.96511×10^{0}	4.96511×10^{0}
	$\lvert\zeta(\mu)\rvert$	7.07685×10^{-5}	4.00140×10^{-18}	4.09178×10^{-74}	4.47418×10^{-290}

Table 3. Numerical results for problem 2, error fixed at $\delta = 10^{-300}$.

Method	*IG*	*N*	*FE*	*CPU Time*
Proposed 4^{th}	4	5	15	9.4×10^{-2}
ZLM 4^{th}	4	5	15	1.56×10^{-1}
KTM 4^{th}	4	5	15	1.41×10^{-1}
AKKB 4^{th}	4	5	15	1.56×10^{-1}
JLM 4^{th}	4	5	15	1.41×10^{-1}

Problem 3 (Continuous Stirred Tank Reactor (CSTR)) (see in [26]) (Application Problem):

$$\phi_4(\mu) = \mu^4 + 11.50\mu^3 + 47.49\mu^2 + 83.06325\mu + 51.23266875. \tag{21}$$

Table 4. Numerical results for problem 3 for first four iterations and their absolute function values at $\mu_0 = 1.5$

Method	Iterations	1st iteration	2nd iteration	3rd iteration	4th iteration
Proposed 4^{th}	μ	-5.62508×10^{-1}	-1.44546×10^{0}	-1.45000×10^{0}	-1.45000×10^{0}
	$\lvert\zeta(\mu)\rvert$	1.75888×10^{1}	2.59891×10^{-2}	2.86465×10^{-8}	5.48354×10^{-32}
ZLM 4^{th}	μ	4.65476×10^{-1}	-2.90357×10^{-1}	-8.28932×10^{-1}	-1.19113×10^{0}
	$\lvert\zeta(\mu)\rvert$	1.01393×10^{2}	3.08440×10^{1}	8.93256×10^{0}	2.25033×10^{0}
KTM 4^{th}	μ	4.68743×10^{-1}	-2.82414×10^{-1}	-8.14243×10^{-1}	-1.16842×10^{0}
	$\lvert\zeta(\mu)\rvert$	1.01835×10^{2}	3.13095×10^{1}	9.31592×10^{0}	2.53325×10^{0}
AKKB 4^{th}	μ	Not Converge	Not Converge	Not Converge	Not Converge
	$\lvert\zeta(\mu)\rvert$				
JLM 4^{th}	μ	4.68746×10^{-1}	-2.82381×10^{-1}	-8.13960×10^{-1}	-1.16656×10^{0}
	$\lvert\zeta(\mu)\rvert$	1.01836×10^{2}	3.13114×10^{1}	9.32339×10^{0}	2.55716×10^{0}

Table 5. Numerical results for problem 3, error fixed at $\delta = 10^{-300}$.

Method	*IG*	*N*	*FE*	*CPU Time*
Proposed 4^{th}	1.5	6	24	2.81000×10^{-1}
ZLM 4^{th}	1.5	10	40	2.84000×10^{-1}
KTM 4^{th}	1.5	10	40	4.70000×10^{-1}
AKKB 4^{th}	1.5	Not Converge	Not Converge	Not Converge
JLM 4^{th}	1.5	10	40	1.37090×10^{-1}

Conclusion

We present two step, derivative free optimal method with fourth order of convergence for numerical solution of nonlinear algebraic & transcendental equations. Mathematica 2021 and maple 2021 were used to obtained the results of various problems solved by various methods which were shown in above tables and compared, results shows the accuracy, consistency and stability are much better than the existing methods.

Credit Author Statement

Sanaullah Jamali: Solution of the problem, **Zubair Ahmed Kalhoro.**: Conceptualization, Methodology, **Abdul Wasim Shaikh**: Visualization, Investigation. **Muhammad Saleem Chandio***:* Supervision. Software support, **Sanaullah Dehraj**: Software, Validation, Writing- Reviewing and Editing

Compliance with Ethical Standards

It is declare that all authors don't have any conflict of interest. Furthermore, informed consent was obtained from all individual participants included in the study.

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