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An Innovative Soft Rough Dual Hesitant Fuzzy Sets and Dual Hesitant Fuzzy Soft Rough Sets

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Abstract This article seeks to demonstrate the novel properties of soft rough dual hesitant fuzzy sets (DHFSRSs) and dual hesitant fuzzy soft rough sets (SRDHFSs). The fundamental characteristics of DHFSRSs and SRDHFSs are thoroughly investigated. Additionally, we present a portrayal hypothesis for the DHFSRSs and SRDHFSs, which demonstrates that the level arrangements of the DHFSRSs and SRDHFSs can be used to characterize both the lower and upper DHFSRSs and SRDHFSs estimates in an identical manner.

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1 Introduction

In the recent past, a thriving interest has been shown in the fuzzy set (FS) of various mathematics working in diverse domains of science and engineering due to the application of different fields. For example, facial pattern recognition, air conditioners, antiskid braking systems, vacuum cleaners, washing mechanics, transmission systems, unmanned helicopters, control of subway systems. First of all, the theory of the fuzzy set was explored in 1965 by Zadeh [16] which useful in various regions. FS has contained one function is named truth grade be the possession of the unit interval. FS has acquired wide attainment, and many scientists have employed it in numerous research areas [5, 7]. A lot of scientists and researchers point out that if we convert the range of FS, which is a real number as replacement of a complex number from a unit disc in a complex plane. What will be the consequence develop. To investigate the complex FS (CFS) [11] which covers the grade of positive in the way of a complex number the possession to the unit disc in a complex plane. CFS manages with two-dimension data in a single set. CFS is a vital method to designate the view of human organisms in the form of grades. CFS has received wide dedication in the environment of many fields [1, 2]. Recently, Torra [15] presented the idea of HFS as an augmentation of the FS wherein the truth level of a specific component, known as the hesitant fuzzy component (HFC), is characterized as a lot of potential qualities. The indicated circumstance can be initiated in a cooperative choice creation issue. To explain the need of presenting the HFS, consider a circumstance where two chiefs talk about the true level of a component x to a set A, one needs to allot 0.3, yet the other 0.5. Appropriately, the trouble of setting up a typical participation degree isn't hence there is room for giving and take or some chance dissemination esteems, but since there are a lot of potential qualities. In 2018, Alcantud and Torra [3] demonstrated decay hypotheses and augmentation standards for the HFS. In the HFS, the enrollment degree comprises a few potential quality's mirroring the epistemic assurance. However, the epistemic vulnerability degree is disregarded. In this manner, Zhu et al. [17] suggested an expansion of the HFSâ€“dual hesitant fuzzy set (DHFS), where both the enrollment and non-participation degrees cover a lot of potential qualities. Moreover, all the FS, the IVFS, and the IFS can be treated as the specific instances of the DHFS. The DHFS, by examination, can mirror the slow epistemic vulnerability to not well-known articles all the more granularly. While likelihood hypothesis, FS hypothesis, rough set hypothesis, and other numerical devices are notable and frequently helpful ways to deal with depicting vulnerability, every one of these speculations have its innate troubles as called attention to in [10]. The purpose behind these troubles is the insufficiency of the parametrization instrument of the hypothesis. In 1999, Molodtsov [8]presented the idea of a soft set that can be viewed as another numerical apparatus for managing vulnerabilities. This supposed soft set hypothesis is liberated after the challenges influencing current strategies. In 2018, Molodtsov [9] presented the idea of proportionality of soft sets and talked about the right activities and right connections for soft sets based on equality. Be that as it may, in the commonsense model, the boundaries in the soft set are an obscure decision include ambiguous arguments. Thinking about this opinion, Maji et al. [6] presented the thought of a fuzzy soft set through joining the fuzzy set and the soft set. Roy and Maji [12] introduced a fuzzy soft set hypothetical methodology with regard to dynamic issue. Further efforts in this area are mentioned in [13, 14]. Pawlak's rough set [16] can be depicted along a couple of fresh sets named as the lower estimate and the upper guess. The lower estimation is the best determinable set contained in the certain arrangement of items, while the upper guess is the littlest perceptible set having the specified set. Through utilizing the idea of lower and upper estimates in rough set hypothesis, information covered up in data frameworks might be uncovered and communicated as choice principles. The rough set has been reached out by random as well as summing up universes of talk from one to two, substances after ordinary sets to fuzzy sets, relations after proportionality relations to other parallel relations, and administrators from conjunctions and disjunctions to fuzzy legitimate administrators. The aims of this manuscript are forthwith as follows:

1. To provide the novelty of dual hesitant fuzzy soft rough sets. Furthermore, shown the results by using an example.
2. The lower and upper approximations of the dual hesitant fuzzy soft rough sets are also presented.
3. To explore the novel of soft rough dual hesitant fuzzy sets and also verified with the help of an example.
4. The lower and upper approximations of the soft rough dual hesitant fuzzy sets are also presented.

The summary of this article is monitored as, In section 2, we briefly assess some basic notions of DHFS, SSs, RSs, SRS, and their fundamental properties. In section 3, we explored the soft rough dual hesitant fuzzy sets and their lower and upper approximations, verified by using numerical examples. In section 4, we explored the dual hesitant fuzzy soft rough sets and their lower and upper estimates, shown by using numerical examples. The deduction of paper is debated in section 5.

2 Preliminaries

Some Basic concepts

This communication's goal is to analyze some existing concepts, including DHFS [17], SSs [8], RSs [10], and SRS [4], and their basic properties, which are extremely useful for the approaches that will be explored in the following sections.

2.0.1 Definition:

Definition 1. *:[17] A DHFS $\check{D}_{dh}$ is stated by:*

$$\check{D}_{dh} = \left\{ \left(\Theta, T_{\check{D}_{dh}}(\Theta), F_{\check{D}_{dh}}(\Theta) \right) : \Theta \in \Theta' \right\} \tag{1}$$

where $T_{\check{D}_{dh}}(\Theta) \subseteq [0,1]$ and $F_{\check{D}_{dh}}(\Theta) \subseteq [0,1]$ are express the grade of supporting and supporting against with conditions such that: $0 \leq \mu_{ms}, \eta_{nms} \leq 1$ and $0 \leq \mu^{+}_{ms} + \eta^{+}_{nms} \leq 1$. Where, $\mu_{ms} \in T_{\check{D}_{dh}}(\Theta), \eta_{nms} \in F_{\check{D}_{dh}}(\Theta)$, $\mu^{+}_{ms} = mix_{\mu_{ms} \in T_{\check{D}_{dh}}(\Theta)}(\mu_{ms})$ and $\eta^{+}_{ms} = mix_{\eta_{ms} \in F_{\check{D}_{dh}}(\Theta)}(\eta_{ms})$. For simplicity, we consider that the pair $\check{D}_{dh} = \left(T_{\check{D}_{dh}}(\Theta), F_{\check{D}_{dh}}(\Theta) \right)$ is named dual hesitant fuzzy element (DHFE).

Definition 2. *$\check{D}_{dh-1} = \left(T_{\check{D}_{dh-1}}, \AE'_{\check{D}_{dh-1}} \right)$ and $\check{D}_{dh-2} = \left(T_{\check{D}_{dh-2}}, \AE'_{\check{D}_{dh-2}} \right)$ with $\delta_{sc} \geq 0$, then*

$$\check{D}_{dh-1} \oplus_{dh} \check{D}_{dh-2} = \left\{ \cup_{\begin{pmatrix} \mu_{ms-1} \in T_{\check{D}_{dh-1}}, \\ \mu_{ms-2} \in T_{\check{D}_{dh-2}} \end{pmatrix}} \left(\mu_{ms-1} + \mu_{ms-2} - \mu_{ms-1}\mu_{ms-2} \right), \cup_{\begin{pmatrix} \eta_{ms-1} \in \AE'_{\check{D}_{dh-1}}, \\ \eta_{ms-2} \in \AE'_{\check{D}_{dh-2}} \end{pmatrix}} \left(\eta_{ms-1}\eta_{ms-2} \right) \right\} \tag{2}$$

$$\check{D}_{dh-1} \otimes_{dh} \check{D}_{dh-2} = \left\{ \cup_{\begin{pmatrix} \mu_{ms-1} \in T_{\check{D}_{dh-1}}, \\ \mu_{ms-2} \in T_{\check{D}_{dh-2}} \end{pmatrix}} \left(\mu_{ms-1}\mu_{ms-2} \right), \cup_{\begin{pmatrix} \eta_{ms-1} \in \AE'_{\check{D}_{dh-1}}, \\ \eta_{ms-2} \in \AE'_{\check{D}_{dh-2}} \end{pmatrix}} \left(\eta_{ms-1} + \eta_{ms-2} - \eta_{ms-1}\eta_{ms-2} \right) \right\} \tag{3}$$

$$\delta_{sc} \check{D}_{dh-1} = \left\{ \cup_{\left(\mu_{ms-1} \in T_{\check{D}_{dh-1}} \right)} \left(1 - \left(1 - \mu_{ms-1} \right)^{\delta_{sc}} \right), \cup_{\left(\eta_{ms-1} \in \AE'_{\check{D}_{dh-1}} \right)} \left(\eta^{\delta_{sc}}_{ms-1} \right) \right\} \tag{4}$$

$$\check{D}^{\delta_{sc}}_{dh-1} = \left\{ \cup_{\left(\mu_{ms-1} \in T_{\check{D}_{dh-1}} \right)} \left({\mu_{ms-1}}^{\delta_{sc}} \right), \cup_{\left(\eta_{ms-1} \in \AE'_{\check{D}_{dh-1}} \right)} \left(1 - \left(1 - \eta_{ms-1} \right)^{\delta_{sc}} \right) \right\} \tag{5}$$

$$\check{D}_{dh-1} \cup_{dh} \check{D}_{dh-2} = \left\{ \cup_{\begin{pmatrix} \mu_{ms-1} \in T_{\check{D}_{dh-1}}, \\ \mu_{ms-2} \in T_{\check{D}_{dh-2}} \end{pmatrix}} max \left(\mu_{ms-1}, \mu_{ms-2} \right), \cup_{\begin{pmatrix} \eta_{ms-1} \in \AE'_{\check{D}_{dh-1}}, \\ \eta_{ms-2} \in \AE'_{\check{D}_{dh-2}} \end{pmatrix}} min \left(\eta_{ms-1}\eta_{ms-2} \right) \right\} \tag{6}$$

$$\check{D}_{dh-1} \cap_{dh} \check{D}_{dh-2} = \left\{ \cup_{\begin{pmatrix} \mu_{ms-1} \in T_{\check{D}_{dh-1}}, \\ \mu_{ms-2} \in T_{\check{D}_{dh-2}} \end{pmatrix}} min \left(\mu_{ms-1}, \mu_{ms-2} \right), \cup_{\begin{pmatrix} \eta_{ms-1} \in \AE'_{\check{D}_{dh-1}}, \\ \eta_{ms-2} \in \AE'_{\check{D}_{dh-2}} \end{pmatrix}} max \left(\eta_{ms-1}\eta_{ms-2} \right) \right\} \tag{7}$$

Definition 3. *[8]: A pair* $\left(F_{fc}, E_p\right)$ *is called SS over* Θ *if* $F_{fc} : E_p \rightarrow P\left(\grave{E}\right)$*, where* $E_p \subseteq \Theta$ *and* $P(\Theta)$ *denoted the power set of* Θ*.*

Definition 4. *[8]: A SS* $\left(F_{fc}, E_p\right)$ *over* Θ*. Then the crisp soft relation* $\Theta \times E_p$ *is stated by:*

$$\Xi_r = \left\{\left((\Theta, \hat{a}), \mu_{\Xi_r}(\Theta, \hat{a})\right) : (\Theta, \hat{a}) \in \Theta \times E_p\right\} \qquad (8)$$

where $\mu_{\Xi_r} : \grave{E} \times E_p \rightarrow \{0, 1\}$ *and* $\mu_{\Xi_r}(\Theta, \hat{a}) = \begin{cases} 1 & (\Theta, \hat{a}) \in \Xi_r \\ 0 & (\Theta, \hat{a}) \notin \Xi_r \end{cases}.$

Definition 5. *: [14] A set-valued function* $\Xi_{sv} : \Theta \rightarrow P(\Theta)$ *is stated by:*

$$\Xi_{sv} = \left\{\hat{a} \in \{\Theta : (\Theta, \hat{a}) \in \{\Xi \subseteq \hat{} \times \hat{}\right\}, \Theta \in \Theta^{\prime}. \qquad (9)$$

The pair $(\Theta, \ \Xi)$ *denoted the crisp approximation space. Suppose any* $\xi \subseteq \Theta$*. The upper approximation (UA) and lower approximation (LA) are stated by:*

$$\overline{\Xi}(\xi) = \left\{\Theta \in \Theta : \Xi_{sv}(\Theta) \cap \xi \neq \emptyset\right\} \qquad (10)$$

$$\underline{\Xi}(\xi) = \{\Theta \in \Theta : \Xi_{sv}(\Theta) \subseteq \xi\} (11)$$

The pair $\left(\overline{\Xi}(\xi), \underline{\Xi}(\xi)\right)$ *denoted crisp RS with* $\overline{\Xi}, \underline{\Xi} : P(\Theta) \rightarrow P(\Theta)$ *is called UA and LA.*

Definition 6. *:[14] A set-valued function* $\Xi_{sv} : \Theta \rightarrow P\left(E_p\right)$ *is stated by:*

$$\Xi_{sv}\left(\hat{a}\right) = \left\{\hat{a} \in E_p : (\Theta, \hat{a}) \in \ddot{} \subseteq \Theta \times E_p\right\}, \Theta \in \Theta. \qquad (12)$$

The pair $\left(\Theta, E_p, \ \Xi\right)$ *denoted the crisp approximation space. Suppose any* $\xi \subseteq \Theta$*. The UA and LA are stated by:*

$$\overline{\Xi}(\xi) = \left\{\Theta \in \Theta : \Xi_{sv}(\Theta) \cap \xi \neq \emptyset\right\} \qquad (13)$$

$$\underline{\Xi}(\xi) = \{\Theta \in \Theta : \Xi_{sv}(\Theta) \subseteq \xi\} \qquad (14)$$

The pair $\left(\overline{\Xi}(\xi), \underline{\Xi}(\xi)\right)$ *denoted crisp soft RS with* $\overline{\Xi}, \underline{\Xi} : P\left(E_p\right) \rightarrow P(\Theta)$ *is called UA and LA operators.*

3 Soft Rough Dual Hesitant Fuzzy Sets

The main motivation of this study is to investigate the novelty approach of soft rough DHFS, which is the mixture of DHFS, SS, and RS. To enhance the quality of the study, some unique characteristics of the ideas that are being presented are also investigated and demonstrated with the aid of several numerical models.

Definition 7. *Suppose* $\left(\Theta, E_p, \Xi\right)$ *denoted the crisp soft approximation space (AS) with any DHFE* $\check{D}_{dh} = \left(T_{\check{D}_{dh}}(\Theta), F_{\check{D}_{dh}}(\Theta)\right)$*. Then the LA and UA of* $\check{D}_{dh}$ *based on* $\left(\Theta, E_p, \ \Xi\right)$ *are stated by:*

$$\overline{\Xi}(\xi) = \left\{\left(T_{\overline{\Xi}(\xi)}(\Theta), F_{\overline{\Xi}(\xi)}(\Theta)\right) : \grave{E} \in \Theta\right\} \qquad (15)$$

$$\underline{\Xi}(\xi) = \left\{\left(T_{\underline{\Xi}(\xi)}(\Theta), F_{\underline{\Xi}(\xi)}(\Theta)\right) : \Theta \in \Theta\right\} \qquad (16)$$

where $T_{\overline{\Xi}(\xi)}(\Theta) = \cup_{\mu_{ms} \in T_{\overline{\Xi}(\xi)}}\left(\vee_{? \in \Xi_{sv}(\Theta)} \mu_{ms}(?)\right), F_{\overline{\Xi}(\xi)}(\Theta) = \cup_{\eta_{ms} \in F_{\overline{\Xi}(\xi)}}\left(\wedge_{? \in \Xi_{sv}(\Theta)} \eta_{ms}\left(\hat{a}\right)\right)$ *and* $T_{\underline{\Xi}(\hat{\imath})}(\Theta) = \cup_{\mu_{ms} \in T_{\underline{\Xi}(\xi)}}\left(\wedge_{\hat{a} \in \Xi_{sv}(\Theta)} \mu_{ms}\left(\hat{a}\right)\right), F_{\underline{\Xi}(\xi)}(\Theta) = \cup_{\eta_{ms} \in F_{\underline{\Xi}(\xi)}}\left(\vee_{\hat{a} \in \Xi_{sv}(\Theta)} \eta_{ms}\left(\hat{a}\right)\right)$*. Then the pair* $\left(\overline{\Xi}(\xi), \underline{\Xi}(\xi)\right)$ *denoted soft rough DHFS with* $\overline{\Xi}, \underline{\Xi} : DHF\left(E_p\right) \rightarrow DHF(\Theta)$ *is called upper and lower soft rough dual hesitant fuzzy approximation operators.*

We consider that the set of universal and the set of the parameter are stated by: $\Theta = \left\{\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5\right\}$ and $E_p = \left\{\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4\right\}$. We present soft set based on Θ is stated by: $F_{fc}\left(\hat{a}_1\right) = \left\{\Theta_1, \Theta_3, \Theta_4\right\}, F_{fc}\left(\hat{a}_2\right) = \{\Theta_2, \Theta_4\}, F_{fc}\left(\hat{a}_3\right) = \emptyset$ and $F_{fc}\left(\hat{a}_4\right) = \Theta$. Further, we define the crisp soft relation $\Theta \times E_p$ is stated by:

$$\Xi = \left\{\left(\Theta_1, \hat{a}_1\right), \left(\Theta_3, \hat{a}_1\right), \left(\grave{\mathrm{E}}_4, \hat{a}_1\right), \left(\Theta_2, \hat{a}_2\right), \left(\Theta_4, \hat{a}_2\right), \left(\Theta_1, \hat{a}_4\right), \left(\Theta_2, \hat{a}_4\right), \left(\Theta_3, \hat{a}_4\right), \left(\Theta_4, \hat{a}_4\right), \left(\Theta_5, \hat{a}_4\right)\right\}$$

Moreover, we present the DHFSs, which is followed as:

$$\xi = \check{\mathrm{D}}_{dh-1} = \left\{\begin{array}{c} \left(\hat{a}_1, (\{0.2, 0.3, 0.4\}, \{0.4, 0.3, 0.1\})\right), \left(\hat{a}_2, (\{0.21, 0.31, 0.41\}, \{0.41, 0.31, 0.11\})\right), \\ \left(\hat{a}_3, (\{0.22, 0.32, 0.42\}, \{0.42, 0.32, 0.12\})\right), \left(\hat{a}_4, (\{0.23, 0.33, 0.43\}, \{0.43, 0.33, 0.13\})\right) \end{array}\right\}$$

By using Eq. (15) and Eq. (16), we get

$$\mathrm{T}_{\overline{\Xi}(\xi)}(\Theta_1) = \{0.23, 0.33, 0.43\}, \mathrm{F}_{\overline{\Xi}(\xi)}(\Theta_1) = \{0.4, 0.3, 0.1\}$$

$$\mathrm{T}_{\overline{\Xi}(\xi)}(\Theta_2) = \{0.23, 0.33, 0.43\}, \mathrm{F}_{\overline{\Xi}(\xi)}(\Theta_2) = \{0.41, 0.31, 0.11\}$$

$$\mathrm{T}_{\overline{\Xi}(\xi)}\left(\Theta_3\right) = \{0.23, 0.33, 0.43\}, \mathrm{F}_{\overline{\Xi}(\xi)}\left(\Theta_3\right) = \{0.4, 0.3, 0.1\}$$

$$\mathrm{T}_{\overline{\Xi}(\xi)}(\Theta_4) = \{0.23, 0.33, 0.43\}, \mathrm{F}_{\overline{\Xi}(\xi)}(\Theta_4) = \{0.4, 0.3, 0.1\}$$

$$\mathrm{T}_{\overline{\Xi}(\xi)}\left(\Theta_5\right) = \{0.23, 0.33, 0.43\}, \mathrm{F}_{\overline{\Xi}(\xi)}\left(\Theta_5\right) = \{0.43, 0.33, 0.13\}$$

$$\mathrm{T}_{\underline{\Xi}(\xi)}(\Theta_1) = \{0.2, 0.3, 0.4\}, \mathrm{F}_{\underline{\Xi}(\xi)}(\Theta_1) = \{0.43, 0.33, 0.13\}$$

$$\mathrm{T}_{\underline{\Xi}(\xi)}(\Theta_2) = \{0.21, 0.31, 0.41\}, \mathrm{F}_{\underline{\Xi}(\xi)}(\Theta_2) = \{0.43, 0.33, 0.13\}$$

$$\mathrm{T}_{\underline{\Xi}(\xi)}\left(\Theta_3\right) = \{0.2, 0.3, 0.4\}, \mathrm{F}_{\underline{\Xi}(\xi)}\left(\Theta_3\right) = \{0.43, 0.33, 0.13\}$$

$$\mathrm{T}_{\underline{\Xi}(\xi)}(\Theta_4) = \{0.2, 0.3, 0.4\}, \mathrm{F}_{\underline{\Xi}(\xi)}(\Theta_4) = \{0.43, 0.33, 0.13\}$$

$$\mathrm{T}_{\underline{\Xi}(\xi)}\left(\Theta_5\right) = \{0.23, 0.33, 0.43\}, \mathrm{F}_{\underline{\Xi}(\xi)}\left(\Theta_5\right) = \{0.43, 0.33, 0.13\}$$

Thus,

$$\overline{\Xi}\left(\check{\mathrm{D}}_{dh-1}\right) = \left\{\begin{array}{c} (\Theta_1, (\{0.23, 0.33, 0.43\}, \{0.4, 0.3, 0.1\})), (\Theta_2, (\{0.23, 0.33, 0.43\}, \{0.41, 0.31, 0.11\})), \\ \left(\Theta_3, (\{0.23, 0.33, 0.43\}, \{0.4, 0.3, 0.1\})\right) \\ (\Theta_4, (\{0.23, 0.33, 0.43\}, \{0.4, 0.3, 0.1\})), (\Theta_5, (\{0.23, 0.33, 0.43\}, \{0.43, 0.33, 0.13\})) \end{array}\right\}$$

$$\underline{\Xi}\left(\check{\mathrm{D}}_{dh-1}\right) = \left\{\begin{array}{c} (\Theta_1, (\{0.2, 0.3, 0.4\}, \{0.43, 0.33, 0.13\})), (\Theta_2, (\{0.21, 0.31, 0.41\}, \{0.43, 0.33, 0.13\})), \\ \left(\Theta_3, (\{0.2, 0.3, 0.4\}, \{0.43, 0.33, 0.13\})\right) \\ (\Theta_4, (\{0.2, 0.3, 0.4\}, \{0.43, 0.33, 0.13\})), (\Theta_5, (\{0.23, 0.33, 0.43\}, \{0.43, 0.33, 0.13\})) \end{array}\right\}$$

3.1 Theorems

Theorem 1. *Suppose* $\left(\Theta, E_p, \Xi\right)$ *denoted the crisp soft approximation space with any DHFE* $\check{D}_{dh} = \left(T_{\check{D}_{dh}}(\Theta), F_{\check{D}_{dh}}(\Theta)\right)$. *Then the upper and lower soft rough dual hesitant fuzzy approximation operators* $\overline{\Xi}(\xi)$ *and* $\underline{\Xi}(\xi)$ *in Def. (7) are holds the following conditions:*

$$\overline{\Xi}(\xi) =\sim \underline{\Xi}(\sim \xi)(17)$$

$$\underline{\Xi}(\xi_1 \cap \xi_2) = \underline{\Xi}(\xi_1) \cap \underline{\Xi}(\xi_2) \tag{18}$$

$$\xi_1 \subseteq \xi_2 \Rightarrow \underline{\Xi}(\xi_1) \subseteq \underline{\Xi}(\xi_2) \qquad (19)$$

$$\underline{\Xi}(\xi_1 \cup \xi_2) \supseteq \underline{\Xi}(\xi_1) \cup \underline{\Xi}(\xi_2) \qquad (20)$$

$$\underline{\Xi}(\xi) = \sim \overline{\Xi}(\sim \xi) \qquad (21)$$

$$\overline{\Xi}(\xi_1 \cup \xi_2) = \overline{\Xi}(\xi_1) \cup \underline{\Xi}(\xi_2) \qquad (22)$$

$$\xi_1 \subseteq \xi_2 \Rightarrow \overline{\Xi}(\xi_1) \subseteq \overline{\Xi}(\xi_2) \qquad (23)$$

$$\overline{\Xi}(\xi_1 \cap \xi_2) \supseteq \overline{\Xi}(\xi_1) \cap \overline{\Xi}(\xi_2) \qquad (24)$$

where $\sim \xi$ is the compliment of ξ.

Proof. Consider the Eq. (17) and by using the Def. (7), we have

$$\sim \underline{\Xi}(\sim \xi) = \left\{\left(\mathrm{F}_{\underline{\Xi}(\sim\xi)}(\Theta), \mathrm{T}_{\underline{\Xi}(\sim\xi)}(\Theta)\right) : \Theta \in \Theta\right\}$$

$$= \left\{\left(\cup_{\eta_{ms} \in \mathrm{F}_{\underline{\Xi}(\sim\xi)}}\left(\vee_{\hat{a} \in \Xi_{sv}(\Theta)} \eta_{ms}(\hat{a})\right), \cup_{\mu_{ms} \in \mathrm{T}_{\underline{\Xi}(\sim\xi)}}\left(\wedge_{\hat{a} \in \Xi_{sv}(\Theta)} \mu_{ms}(\hat{a})\right)\right) : \Theta \in \Theta\right\} = \left\{\left(\cup_{\mu_{ms} \in \mathrm{T}_{\underline{\Xi}(\xi)}}\left(\vee_{\hat{a} \in \Xi_{sv}(\Theta)} \mu_{ms}(\hat{a})\right), \cup_{\eta_{ms} \in \mathrm{F}\right.$$

$$= \left\{\left(\mathrm{T}_{\overline{\Xi}(\sim\xi)}(\Theta), \mathrm{F}_{\overline{\Xi}(\sim\xi)}(\Theta)\right) : \Theta \in \Theta\right\} = \overline{\Xi}(\xi)$$

Consider Eq. (18) and by using Eq. (7), we have

$$\underline{\Xi}(\xi_1 \cap \xi_2) = \left\{\left(\mathrm{T}_{\underline{\Xi}(\xi_1 \cap \xi_2)}(\Theta), \mathrm{F}_{\underline{\Xi}(\xi_1 \cap \xi_2)}(\Theta)\right) : \Theta \in \Theta\right\}$$

$$= \left\{\left(\cup_{\mu_{ms} \in \mathrm{T}_{\underline{\Xi}(\xi_1 \cap \xi_2)}}\left(\wedge_{\hat{a} \in \Xi_{sv}(\Theta)} \mu_{ms}(\hat{a})\right), \cup_{\eta_{ms} \in \mathrm{F}_{\underline{\Xi}(\xi_1 \cap \xi_2)}}\left(\vee_{\hat{a} \in \Xi_{sv}(\Theta)} \eta_{ms}(\hat{a})\right)\right) : \Theta \in \Theta\right\}$$

$$= \left\{\left(\begin{array}{cc} \cup\begin{pmatrix} \mu_{ms-1} \in \mathrm{T}_{\underline{\Xi}(\xi_1)}, \\ \mu_{ms-2} \in \mathrm{T}_{\underline{\Xi}(\xi_2)} \end{pmatrix} & \left(\wedge_{\hat{a} \in \Xi_{sv}(\Theta)}\left(\mu_{ms-1}(\hat{a}) \wedge \mu_{ms-2}(\hat{a})\right)\right), \\ \cup\begin{pmatrix} \eta_{ms-1} \in \mathrm{F}_{\underline{\Xi}(\xi_1)}, \\ \eta_{ms-2} \in \mathrm{F}_{\underline{\Xi}(\xi_2)} \end{pmatrix} & \left(\vee_{\hat{a} \in \Xi_{sv}(\Theta)}\left(\eta_{ms-1}(\hat{a}) \vee \eta_{ms-2}(\hat{a})\right)\right) \end{array}\right) : \Theta \in \Theta\right\}$$

$$= \left\{\left(\begin{array}{c} \cup_{\left(\mu_{ms-1} \in \mathrm{T}_{\underline{\Xi}(\xi_1)}\right)}\left(\wedge_{\hat{a} \in \Xi_{sv}(\text{È})} \mu_{ms-1}(\hat{a})\right) \wedge \cup_{\left(\mu_{ms-2} \in \mathrm{T}_{\underline{\Xi}(\xi_2)}\right)}\left(\wedge_{\hat{a} \in \Xi_{sv}(\Theta)} \mu_{ms-2}(\hat{a})\right), \\ \cup_{\left(\eta_{ms-1} \in \mathrm{F}_{\underline{\Xi}(\xi_1)}\right)}\left(\vee_{\hat{a} \in \Xi_{sv}(\Theta)} \eta_{ms-1}(\hat{a})\right) \vee \cup_{\left(\eta_{ms-2} \in \mathrm{F}_{\underline{\Xi}(\xi_2)}\right)}\left(\vee_{\hat{a} \in \Xi_{sv}(\Theta)} \eta_{ms-2}(\hat{a})\right) \end{array}\right) : \Theta \in \Theta\right\}$$

$$= \left\{\left(\mathrm{T}_{\underline{\Xi}(\xi_1)}(\Theta) \wedge \mathrm{T}_{\underline{\Xi}(\xi_2)}(\Theta), \mathrm{F}_{\underline{\Xi}(\xi_1)}(\Theta) \vee \mathrm{F}_{\underline{\Xi}(\xi_2)}(\Theta)\right) : \Theta \in \Theta\right\}$$

$$= \underline{\Xi}(\xi_1) \cap \underline{\Xi}(\xi_2)$$

The Eq. (18) to Eq. (24) is straightforward. For better understanding, the above equations are verified with the help of some examples.

$$\underline{\Xi}(\xi_1 \cap \hat{\imath}_2) = \left\{\left(\mathrm{T}_{\underline{\Xi}(\xi_1 \cap \xi_2)}(\Theta), \mathrm{F}_{\underline{\Xi}(\xi_1 \cap \xi_2)}(\Theta)\right) : \Theta \in \Theta\right\}$$

$$= \left\{\left(\cup_{\mu_{ms} \in \mathrm{T}_{\underline{\Xi}(\xi_1 \cap \xi_2)}}\left(\wedge_{\hat{a} \in \Xi_{sv}(\Theta)} \mu_{ms}(\hat{a})\right), \cup_{\eta_{ms} \in \mathrm{F}_{\underline{\Xi}(\xi_1 \cap \xi_2)}}\left(\vee_{\hat{a} \in \Xi_{sv}(\Theta)} \eta_{ms}(\hat{a})\right)\right) : \Theta \in \Theta\right\}$$

$$= \left\{\left(\begin{array}{cc} \cup\begin{pmatrix} \mu_{ms-1} \in \mathrm{T}_{\underline{\Xi}(\hat{\imath}_1)}, \\ \mu_{ms-2} \in \mathrm{T}_{\underline{\Xi}(\xi_2)} \end{pmatrix} & \left(\wedge_{\hat{a} \in \Xi_{sv}(\Theta)}\left(\mu_{ms-1}(\hat{a}) \wedge \mu_{ms-2}(\hat{a})\right)\right), \\ \cup\begin{pmatrix} \eta_{ms-1} \in \mathrm{F}_{\underline{\Xi}(\xi_1)}, \\ \eta_{ms-2} \in \mathrm{F}_{\underline{\Xi}(\xi_2)} \end{pmatrix} & \left(\vee_{\hat{a} \in \Xi_{sv}(\Theta)}\left(\eta_{ms-1}(\hat{a}) \vee \eta_{ms-2}(\hat{a})\right)\right) \end{array}\right) : \Theta \in \Theta\right\}$$

$$= \left\{ \left(\begin{array}{c} \cup_{\left(\mu_{ms-1} \in \mathrm{T}_{\underline{\Xi}(\xi_1)}\right)} \left(\wedge_{\hat{a} \in \Xi_{sv}(\Theta)} \mu_{ms-1} \left(\hat{a}\right) \right) \wedge \cup_{\left(\mu_{ms-2} \in \mathrm{T}_{\underline{\Xi}(\xi_2)}\right)} \left(\wedge_{\hat{a} \in \Xi_{sv}(\Theta)} \mu_{ms-2} \left(\hat{a}\right) \right), \\ \cup_{\left(\eta_{ms-1} \in \mathrm{F}_{\underline{\Xi}(\xi_1)}\right)} \left(\vee_{\hat{a} \in \Xi_{sv}(\Theta)} \eta_{ms-1} \left(\hat{a}\right) \right) \vee \cup_{\left(\eta_{ms-2} \in \mathrm{F}_{\underline{\Xi}(\xi_2)}\right)} \left(\vee_{\hat{a} \in \Xi_{sv}(\Theta)} \eta_{ms-2} \left(\hat{a}\right) \right) \end{array} \right) : \Theta \in \grave{\mathrm{E}} \right\}$$

$$= \left\{ \left(\mathrm{T}_{\underline{\Xi}(\xi_1)} (\Theta) \wedge \mathrm{T}_{\underline{\Xi}(\xi_2)} (\Theta), \mathrm{F}_{\underline{\Xi}(\xi_1)} (\Theta) \vee \mathrm{F}_{\underline{\Xi}(\xi_2)} (\Theta) \right) : \Theta \in \Theta \right\}$$

$$= \underline{\Xi}\left(\xi_1\right) \cap \underline{\Xi}\left(\xi_2\right)$$

The Eq. (18) to Eq. (24) is straightforward. For better understanding, the above equations are verified with the help of some examples. □

Consider then example (1), we have

$$\check{\mathrm{D}}_{dh-1} = \left\{ \begin{array}{c} \left(\hat{a}_1, (\{0.2, 0.3, 0.4\}, \{0.4, 0.3, 0.1\})\right), \left(\hat{a}_2, (\{0.21, 0.31, 0.41\}, \{0.41, 0.31, 0.11\})\right), \\ \left(\hat{a}_3, (\{0.22, 0.32, 0.42\}, \{0.42, 0.32, 0.12\})\right), \left(\hat{a}_4, (\{0.23, 0.33, 0.43\}, \{0.43, 0.33, 0.13\})\right) \end{array} \right\}$$

The compliment of $\check{\mathrm{D}}_{dh}$ is follow as:

$$\sim \check{\mathrm{D}}_{dh-1} = \left\{ \begin{array}{c} \left(\hat{a}_1, (\{0.4, 0.3, 0.1\}, \{0.2, 0.3, 0.4\})\right), \left(\hat{a}_2, (\{0.41, 0.31, 0.11\}, \{0.21, 0.31, 0.41\})\right), \\ \left(\hat{a}_3, (\{0.42, 0.32, 0.12\}, \{0.22, 0.32, 0.42\})\right), \left(\hat{a}_4, (\{0.43, 0.33, 0.13\}, \{0.23, 0.33, 0.43\})\right) \end{array} \right\}$$

Further, we have examined the compliment of the upper and lower soft rough dual hesitant fuzzy approximation operators, which are followed as:

$$\overline{\Xi}\left(\sim \check{\mathrm{D}}_{dh-1}\right) = \left\{ \begin{array}{c} (\Theta_1, (\{0.43, 0.33, 0.13\}, \{0.2, 0.3, 0.4\})), (\Theta_2, (\{0.43, 0.33, 0.13\}, \{0.21, 0.31, 0.41\})), \\ \left(\Theta_3, (\{0.43, 0.33, 0.13\}, \{0.2, 0.3, 0.4\})\right) \\ (\Theta_4, (\{0.43, 0.33, 0.13\}, \{0.2, 0.3, 0.4\})), \left(\Theta_5, (\{0.43, 0.33, 0.13\}, \{0.23, 0.33, 0.43\})\right) \end{array} \right\}$$

$$\sim \overline{\Xi}\left(\sim \check{\mathrm{D}}_{dh-1}\right) = \left\{ \begin{array}{c} (\Theta_1, (\{0.2, 0.3, 0.4\}, \{0.43, 0.33, 0.13\})), (\Theta_2, (\{0.21, 0.31, 0.41\}, \{0.43, 0.33, 0.13\})), \\ \left(\Theta_3, (\{0.2, 0.3, 0.4\}, \{0.43, 0.33, 0.13\})\right) \\ (\Theta_4, (\{0.2, 0.3, 0.4\}, \{0.43, 0.33, 0.13\})), \left(\Theta_5, (\{0.23, 0.33, 0.43\}, \{0.43, 0.33, 0.13\})\right) \end{array} \right\} = \underline{\Xi}\left(\check{\mathrm{D}}_{dh-1}\right)$$

Eq. (1), holds. Further, we have verified more equations to show the reliability and effectiveness of the explored approach. For this, we choose the set

$$\check{\mathrm{D}}_{dh-2} = \left\{ \begin{array}{c} \left(\hat{a}_1, (\{0.21, 0.31, 0.41\}, \{0.41, 0.31, 0.11\})\right), \left(\hat{a}_2, (\{0.22, 0.32, 0.42\}, \{0.42, 0.32, 0.12\})\right), \\ \left(\hat{a}_3, (\{0.23, 0.33, 0.43\}, \{0.43, 0.33, 0.13\})\right), \left(\hat{a}_4, (\{0.24, 0.34, 0.44\}, \{0.44, 0.34, 0.14\})\right) \end{array} \right\}$$

then

$$\check{\mathrm{D}}_{dh-1} \bigcap \check{\mathrm{D}}_{dh-2} = \left\{ \begin{array}{c} \left(\hat{a}_1, (\{0.2, 0.3, 0.4\}, \{0.41, 0.31, 0.11\})\right), \left(\hat{a}_2, (\{0.21, 0.31, 0.41\}, \{0.42, 0.32, 0.12\})\right), \\ \left(\hat{a}_3, (\{0.22, 0.32, 0.42\}, \{0.43, 0.33, 0.13\})\right), \left(\hat{a}_4, (\{0.23, 0.33, 0.43\}, \{0.44, 0.34, 0.14\})\right) \end{array} \right\}$$

$$\underline{\Xi}\left(\check{\mathrm{D}}_{dh-2}\right) = \left\{ \begin{array}{c} (\Theta_1, (\{0.21, 0.31, 0.41\}, \{0.44, 0.34, 0.14\})), (\Theta_2, (\{0.22, 0.32, 0.42\}, \{0.44, 0.34, 0.14\})), \\ \left(\Theta_3, (\{0.21, 0.31, 0.41\}, \{0.44, 0.34, 0.14\})\right) \\ (\Theta_4, (\{0.2, 0.3, 0.4\}, \{0.44, 0.34, 0.14\})), \left(\Theta_5, (\{0.23, 0.33, 0.43\}, \{0.43, 0.33, 0.13\})\right) \end{array} \right\}$$

$$\underline{\Xi}\left(\check{\mathrm{D}}_{dh-1} \bigcap \check{\mathrm{D}}_{dh-2}\right) = \left\{ \begin{array}{c} (\Theta_1, (\{0.2, 0.3, 0.4\}, \{0.44, 0.34, 0.14\})), (\Theta_2, (\{0.21, 0.31, 0.41\}, \{0.44, 0.34, 0.14\})), \\ \left(\Theta_3, (\{0.2, 0.3, 0.4\}, \{0.44, 0.34, 0.14\})\right) \\ (\Theta_4, (\{0.2, 0.3, 0.4\}, \{0.44, 0.34, 0.14\})), \left(\Theta_5, (\{0.23, 0.33, 0.43\}, \{0.43, 0.33, 0.13\})\right) \end{array} \right\}$$

$$\Xi\left(\check{\mathrm{D}}_{dh-1}\right) = \left\{ \begin{array}{c} \left(\Theta_1, \left(\{0.2, 0.3, 0.4\}, \{0.43, 0.33, 0.13\}\right)\right), \left(\Theta_2, \left(\{0.21, 0.31, 0.41\}, \{0.43, 0.33, 0.13\}\right)\right), \\ \left(\Theta_3, \left(\{0.2, 0.3, 0.4\}, \{0.43, 0.33, 0.13\}\right)\right) \\ \left(\Theta_4, \left(\{0.2, 0.3, 0.4\}, \{0.43, 0.33, 0.13\}\right)\right), \left(\Theta_5, \left(\{0.23, 0.33, 0.43\}, \{0.43, 0.33, 0.13\}\right)\right) \end{array} \right\}$$

$$\Xi\left(\check{\mathrm{D}}_{dh-1}\right) \bigcap \Xi\left(\check{\mathrm{D}}_{dh-2}\right) = \left\{ \begin{array}{c} \left(\Theta_1, \left(\{0.2, 0.3, 0.4\}, \{0.44, 0.34, 0.14\}\right)\right), \left(\Theta_2, \left(\{0.21, 0.31, 0.41\}, \{0.44, 0.34, 0.14\}\right)\right), \\ \left(\Theta_3, \left(\{0.2, 0.3, 0.4\}, \{0.44, 0.34, 0.14\}\right)\right) \\ \left(\Theta_4, \left(\{0.2, 0.3, 0.4\}, \{0.44, 0.34, 0.14\}\right)\right), \left(\Theta_5, \left(\{0.23, 0.33, 0.43\}, \{0.43, 0.33, 0.13\}\right)\right) \end{array} \right\}$$

Hence, Eq. (18) is proved. Similarly, we can prove that the Eq. (19) to Eq. (24). Further, we have explored that the DHFSRS and their fundamental properties.

4 Dual Hesitant Fuzzy Soft Rough Sets

This study's primary driving force is to examine the unique combination of DHFS, SS, and RS known as DHFSRS. Some special properties of the presented ideas are also explored and proved with using several numerical results to improve the quality of the research work.

Definition 8. *Suppose* (Θ, E_p, Ξ) *denoted the crisp soft AS with any DHFE* $\check{D}_{dh} = \left(T_{\check{D}_{dh}}(\Theta), F_{\check{D}_{dh}}(\Theta)\right)$. *Then the UA and LA of* $\check{D}_{dh}$ *based on* (Θ, E_p, Ξ) *based on* $\Theta \times E_p$*, are stated by:*

$$\overline{\Xi}(\xi) = \left\{\left(T_{\overline{\Xi}(\xi)}(\Theta), F_{\overline{\Xi}(\xi)}(\Theta)\right) : \Theta \in \Theta\right\} \qquad (25)$$

$$\underline{\Xi}(\xi) = \left\{\left(T_{\underline{\Xi}(\xi)}(\Theta), F_{\underline{\Xi}(\xi)}(\Theta)\right) : \Theta \in \Theta\right\} \qquad (26)$$

where $T_{\overline{\Xi}(\xi)}(\Theta) = \cup_{\mu_{ms} \in T_{\overline{\Xi}(\xi)}}\left(\vee_{\hat{a} \in E_p}\left[\mu_{ms}(\Theta, \hat{a}) \wedge \mu_{ms}(\hat{a})\right]\right), F_{\overline{\Xi}(\xi)}(\Theta) = \cup_{\eta_{ms} \in F_{\overline{\Xi}(\xi)}}\left(\wedge_{\hat{a} \in E_p}\left[\left(1 - \eta_{ms}(\Theta, \hat{a})\right) \vee \eta_{ms}(\hat{a})\right]\right)$ *and* $T_{\underline{\Xi}(\xi)}(\Theta) = \cup_{\mu_{ms} \in T_{\underline{\Xi}(\xi)}}\left(\wedge_{\hat{a} \in E_p}\left[\left(1 - \mu_{ms}(\Theta, \hat{a})\right) \vee \mu_{ms}(\hat{a})\right]\right), F_{\underline{\Xi}(\xi)}(\Theta) = \cup_{\eta_{ms} \in F_{\underline{\Xi}(\xi)}}\left(\vee_{\hat{a} \in E_p}\left[\eta_{ms}(\Theta, \hat{a}) \wedge \eta_{ms}(\hat{a})\right]\right)$.
Then the pair $\left(\overline{\Xi}(\xi), \underline{\Xi}(\xi)\right)$ *denoted DHFSRS of* ξ *concerning* (Θ, E_p, Ξ).
Further, we have verified that the above analysis is correct. So, we have to consider

$$T_{\overline{\Xi}(\xi)}(\Theta) + F_{\overline{\Xi}(\xi)}(\Theta)$$

$$= \cup_{\mu_{ms} \in T_{\overline{\Xi}(\xi)}}\left(\vee_{\hat{a} \in E_p}\left[\mu_{ms}(\Theta, \hat{a}) \wedge \mu_{ms}(\hat{a})\right]\right) + \cup_{\eta_{ms} \in F_{\overline{\Xi}(\xi)}}\left(\wedge_{\hat{a} \in E_p}\left[\left(1 - \eta_{ms}(\Theta, \hat{a})\right) \vee \eta_{ms}(\hat{a})\right]\right)$$

$$= \cup_{\left(\begin{array}{c}\mu_{ms} \in T_{\overline{\Xi}(\xi)}, \\ \eta_{ms} \in F_{\overline{\Xi}(\xi)}\end{array}\right)}\left(\vee_{\hat{a} \in E_p}\left[\mu_{ms}(\Theta, \hat{a}) \wedge \mu_{ms}(\hat{a})\right] + 1 - \vee_{\hat{a} \in E_p}\left[\mu_{ms}(\Theta, \hat{a}) \wedge \left(1 - \eta_{ms}(\hat{a})\right)\right]\right)$$

$$\leq \cup_{\left(\begin{array}{c}\mu_{ms} \in T_{\overline{\Xi}(\xi)}, \\ \eta_{ms} \in F_{\overline{\Xi}(\xi)}\end{array}\right)}\left(\vee_{\hat{a} \in E_p}\left[\mu_{ms}(\Theta, \hat{a}) \wedge \left(1 - \eta_{ms}(\hat{a})\right)\right] + 1 - \vee_{\hat{a} \in E_p}\left[\mu_{ms}(\Theta, \hat{a}) \wedge \left(1 - \eta_{ms}(\hat{a})\right)\right]\right) = 1$$

Then, we call that the $\overline{\Xi}, \underline{\Xi}$: *DHF* $(E_p) \rightarrow$ *DHF* (Θ) *is named upper and lower dual hesitant fuzzy soft rough approximation (DHFSR) operators.*

Theorem 2. *Suppose* (Θ, E_p, Ξ) *denoted the crisp soft AS with any DHFE* $\check{D}_{dh} = \left(T_{\check{D}_{dh}}(\Theta), F_{\check{D}_{dh}}(\Theta)\right)$. *Then the upper and lower soft rough dual hesitant fuzzy approximation operators* $\overline{\Xi}(\xi)$ *and* $\underline{\Xi}(\xi)$ *in Def. (8) are holds the following conditions:*

$$\overline{\Xi}(\xi) = \sim \underline{\Xi}(\sim \xi) \, (27)$$

$$\underline{\Xi}(\xi_1 \cap \xi_2) = \underline{\Xi}(\xi_1) \cap \underline{\Xi}(\xi_2) \qquad (28)$$

$$\xi_1 \subseteq \xi_2 \Rightarrow \underline{\Xi}(\xi_1) \subseteq \underline{\Xi}(\xi_2) \qquad (29)$$

$$\underline{\Xi}(\xi_1 \cup \xi_2) \supseteq \underline{\Xi}(\xi_1) \cup \underline{\Xi}(\xi_2) (30)$$

$$\underline{\Xi}(\xi) =\sim \overline{\Xi}(\sim \xi) (31)$$

$$\overline{\Xi}(\xi_1 \cup \xi_2) = \overline{\Xi}(\xi_1) \cup \underline{\Xi}(\xi_2) \qquad (32)$$

$$\xi_1 \subseteq \xi_2 \Rightarrow \overline{\Xi}(\xi_1) \subseteq \overline{\Xi}(\xi_2) \qquad (33)$$

$$\overline{\Xi}(\xi_1 \cap \xi_2) \supseteq \overline{\Xi}(\xi_1) \cap \overline{\Xi}(\xi_2) (34)$$

where $\sim \xi$ *is the compliment of* ξ*.*

Proof. The proof of this theorem is similar to the proof of the theorem (1) ☐

Theorem 3. *Suppose* (Θ, E_p, Ξ) *denoted the crisp soft AS with any DHFE* $\xi \in \breve{D}_{dh} = \left(T_{\breve{D}_{dh}}(\Theta), F_{\breve{D}_{dh}}(\Theta)\right)$*. Then the upper and lower soft rough dual hesitant fuzzy approximation operators* $\overline{\Xi}(\xi)$ *and* $\underline{\Xi}(\xi)$ *in Def. (8) are holds the following conditions:*

$$\underline{\Xi}(\theta) = \theta, \overline{\Xi}\left(E_p\right) = \Theta \qquad (35)$$

$$\underline{\Xi}(\xi) \subseteq \overline{\Xi}(\xi) \qquad (36)$$

Proof. Straightforward. (By using Def. (8))
Consider that Ξ is a hesitant fuzzy soft relation form Θ to the parameter set E_p, stated by

$$\Xi_{\propto} = \cup_{\mu_{ms} \in \mathrm{T}_{\overline{\Xi}(\xi)}} \left\{\left(\Theta, \hat{a}\right) : \mu_{ms}\left(\Theta, \hat{a}\right) \geq \propto\right\}$$

$$\Xi_{\propto}(\Theta) = \cup_{\mu_{ms} \in \mathrm{T}_{\overline{\Xi}(\xi)}} \left\{\hat{a} \in E_p : \mu_{ms}\left(\Theta, \hat{a}\right) \geq \propto\right\}$$

$$\Xi_{\propto+} = \cup_{\mu_{ms} \in \mathrm{T}_{\overline{\Xi}(\xi)}} \left\{\left(\Theta, \hat{a}\right) : \mu_{ms}\left(\Theta, \hat{a}\right) > \propto\right\}$$

$$\Xi_{\propto+}(\Theta) = \cup_{\mu_{ms} \in \mathrm{T}_{\overline{\Xi}(\xi)}} \left\{\hat{a} \in E_p : \mu_{ms}\left(\Theta, \hat{a}\right) > \propto\right\}$$

Then $\Xi_{\propto}$ and $\Xi_{\propto+}$ are two crisp hesitant soft relations on $\Theta \times E_p$. ☐

Theorem 4. *Suppose* (Θ, E_p, Ξ) *denoted the crisp soft AS with any DHFE* $\xi \in \breve{D}_{dh} = \left(T_{\breve{D}_{dh}}(\Theta), F_{\breve{D}_{dh}}(\Theta)\right)$*. Then the upper soft rough dual hesitant fuzzy approximation operators* $\overline{\Xi}(\xi)$ *in Def. () are holds the following conditions:*

$$T_{\overline{\Xi}(\xi)}(\Theta) = \vee_{\propto \in [0,1]}\left[\propto \wedge \overline{\Xi_{\propto}}(\xi_{\propto})(\Theta)\right] = \vee_{\propto \in [0,1]}\left[\propto \wedge \overline{\Xi_{\propto}}(\xi_{\propto+})(\Theta)\right] = \vee_{\propto \in [0,1]}\left[\propto \wedge \overline{\Xi_{\propto+}}(\xi_{\propto})(\Theta)\right] = \vee_{\propto \in [0,1]}\left[\propto \wedge \overline{\Xi_{\propto+}}(\xi_{\propto+})(\Theta)\right.$$

$$F_{\overline{\Xi}(\xi)}((\Theta)) = \wedge_{\propto \in [0,1]}\left[\propto \vee \left(1 - \overline{\Xi_{1-\propto}}\left(\xi^{\propto}\right)(\Theta)\right)\right] = \wedge_{\propto \in [0,1]}\left[\propto \vee \left(1 - \overline{\Xi_{1-\propto}}\left(\xi^{\propto+}\right)(\Theta)\right)\right] = \wedge_{\propto \in [0,1]}\left[\propto \vee \left(1 - \overline{\Xi_{(1-\propto)+}}\left(\xi^{\propto}\right)(\Theta)\right)\right] =$$

$$\left[\overline{\Xi}(\xi)\right]_{\propto+} \subseteq \overline{\Xi}_{\propto+}(\xi_{\propto+}) \subseteq \overline{\Xi}_{\propto+}(\xi_{\propto}) \subseteq \overline{\Xi}_{\propto}(\xi_{\propto}) \subseteq \left[\overline{\Xi}(\xi)\right]_{\propto} \qquad (39)$$

$$\left[\overline{\Xi}(\xi)\right]^{\propto+} \subseteq \overline{\Xi}_{(1-\propto)+}\left(\xi^{\propto+}\right) \subseteq \overline{\Xi}_{(1-\propto)+}\left(\xi^{\propto}\right) \subseteq \overline{\Xi}_{(1-\propto)}\left(\xi^{\propto}\right) \subseteq \left[\overline{\Xi}(\xi)\right]^{\propto} \qquad (40)$$

Proof. First, we have proved Eq. (37), such that

$\vee_{\propto\in[0,1]}\left[\propto\wedge\overline{\Xi_{\propto}}(\xi_{\propto})(\Theta)\right] = sup\left\{\propto\in[0,1]:\Theta\in\overline{\Xi_{\propto}}(\xi_{\propto})\right\} = sup\{\propto\in[0,1]:\Xi_{\propto}(\Theta)\cap\xi_{\propto}\not\not=\theta\} = sup\left\{\propto\in[0,1]:\exists\hat{a}\in E_p[\hat{a}\in\right.$

Similarly, we have proved that

$$\mathrm{T}_{\overline{\Xi}(\xi)}(\Theta) = \vee_{\propto\in[0,1]}\left[\propto\wedge\overline{\Xi_{\propto}}(\xi_{\propto+})(\Theta)\right] = \vee_{\propto\in[0,1]}\left[\propto\wedge\overline{\Xi_{\propto+}}(\xi_{\propto})(\Theta)\right] = \vee_{\propto\in[0,1]}\left[\propto\wedge\overline{\Xi_{\propto+}}(\xi_{\propto+})(\Theta)\right]$$

Eq. (38) is straightforward.
Further, we have proved that in Eq. (39), it is easily verified that

$$\overline{\Xi}_{\propto+}(\xi_{\propto+})\subseteq\overline{\Xi}_{\propto+}(\xi_{\propto})\subseteq\overline{\Xi}_{\propto}(\xi_{\propto})$$

We only prove that
$\left[\overline{\Xi}(\xi)\right]_{\propto+}\subseteq\overline{\Xi}_{\propto+}(\xi_{\propto+})$ and $\overline{\Xi}_{\propto}(\xi_{\propto})\subseteq\left[\overline{\Xi}(\xi)\right]_{\propto}$. For this for all $\Theta\in\left[\overline{\Xi}(\xi)\right]_{\propto+}$, we have $\cup_{\mu ms\in\mathrm{T}_{\overline{\Xi}(\xi)}}(\mu ms\geq\propto)$. Based on Def. (), we have $\cup_{\mu ms\in\mathrm{T}_{\overline{\Xi}(\xi)}}\left(\vee_{\hat{a}\in E_p}\left[\mu ms(\Theta,\hat{a})\wedge\mu ms(\hat{a})\right]>\propto\right)$ holds. Then there exists $\hat{a}_0\in E_p$, such that $\cup_{\mu ms\in\mathrm{T}_{\overline{\Xi}(\xi)}}\left(\mu ms\left(\Theta,\hat{a}_0\right)>\propto\right)$ and $\cup_{\mu ms\in\mathrm{T}_{\overline{\Xi}(\xi)}}\left(\mu ms\left(\hat{a}_0\right)>\propto\right)$. Thus $\hat{a}_0\in\Xi_{\propto+}(\Theta)$ and $\hat{a}_0\in\xi_{\propto+}$, then $\Xi_{\propto+}(\Theta)\cap\xi_{\propto+}\not\not=\theta$. Based on Def. (), we obtain $\Theta\in\overline{\Xi}_{\propto+}(\xi_{\propto+})$. Hence $\left[\overline{\Xi}(\xi)\right]_{\propto+}\subseteq\overline{\Xi}_{\propto+}(\xi_{\propto+})$.
Further, we have to examine $\overline{\Xi}_{\propto}(\xi_{\propto})\subseteq\left[\overline{\Xi}(\xi)\right]_{\propto}$, for this $\Theta\in\overline{\Xi}_{\propto}(\xi_{\propto})$, we have $\overline{\Xi}_{\propto}(\xi_{\propto})(\Theta)=1$. Since $\mathrm{T}_{\overline{\Xi}(\xi)}(\Theta)=\vee_{\propto\in[0,1]}\left[\propto\wedge\overline{\Xi_{\propto}}(\xi_{\propto})(\Theta)\right]\geq\propto\wedge\overline{\Xi_{\propto}}(\xi_{\propto})(\Theta)=\propto$, we obtain $\Theta\in\left[\overline{\Xi}(\xi)\right]_{\propto}$. Hence $\overline{\Xi}_{\propto}(\xi_{\propto})\subseteq\left[\overline{\Xi}(\xi)\right]_{\propto}$.
The result is proved.
Similarly, Eq. (40) is straightforward. □

Theorem 5. *Suppose* (Θ, E_p, Ξ) *denoted the crisp soft AS with any DHFE* $\xi\in\check{D}_{dh}=\left(T_{\check{D}_{dh}}(\Theta), F_{\check{D}_{dh}}(\Theta)\right)$. *Then the lower soft rough dual hesitant fuzzy approximation operators* $\underline{\Xi}(\xi)$ *in Def. () are holds the following conditions:*

$T_{\underline{\Xi}(\xi)}(\Theta)=\wedge_{\propto\in[0,1]}\left[\propto\vee\underline{\Xi_{1-\propto}}(\xi_{\propto+})(\Theta)\right]=\wedge_{\propto\in[0,1]}\left[\propto\vee\underline{\Xi_{(1-\propto)+}}(\xi_{\propto})(\Theta)\right]=\wedge_{\propto\in[0,1]}\left[\propto\vee\underline{\Xi_{(1-\propto)+}}(\xi_{\propto+})(\Theta)\right]=\wedge_{\propto\in[0,1]}\left[\propto\vee\underline{\Xi_{(}}\right.$

$F_{\underline{\Xi}(\xi)}((\Theta))=\vee_{\propto\in[0,1]}\left[\propto\wedge\left(1-\underline{\Xi_{\propto}}\left(\xi^{\propto}\right)(\Theta)\right)\right]=\vee_{\propto\in[0,1]}\left[\propto\wedge\left(1-\underline{\Xi_{\propto+}}\left(\xi^{\propto}\right)(\Theta)\right)\right]=\vee_{\propto\in[0,1]}\left[\propto\wedge\left(1-\underline{\Xi_{\propto+}}\left(\xi^{\propto+}\right)(\Theta)\right)\right]=\vee_{\propto\in}$

$$[\underline{\Xi}(\xi)]_{\propto+}\subseteq\underline{\Xi}_{1-\propto}(\xi_{\propto+})\subseteq\underline{\Xi}_{(1-\propto)+}(\xi_{\propto+})\subseteq\underline{\Xi}_{(1-\propto)+}(\xi_{\propto})\subseteq[\underline{\Xi}(\xi)]_{\propto} \tag{43}$$

$$[\underline{\Xi}(\xi)]^{\propto+}\subseteq\underline{\Xi}_{\propto}\left(\xi^{\propto+}\right)\subseteq\underline{\Xi}_{\propto+}\left(\xi^{\propto+}\right)\subseteq\underline{\Xi}_{\propto+}\left(\xi^{\propto}\right)\subseteq[\underline{\Xi}(\xi)]^{\propto} \tag{44}$$

Proof. The proof of this theorem is similar to the proof of the theorem (3) □

5 Conclusion

The trouble of building up a typical truth degree and falsity degree isn't because there is a safety buffer or some chance dispersion esteems, but since there is a lot of potential qualities. In this article we illustrate the novel characteristics of dual hesitant fuzzy soft rough sets and soft rough dual hesitant fuzzy sets. Fundamental properties of DHFSRSs and SRDHFSs are examined in detail. Moreover, we demonstrate a portrayal hypothesis for the DHFSRSs and SRDHFSs, which shows that the lower and upper DHFSRSs and SRDHFSs approximations can be identically characterized by utilizing level arrangements of the DHFSRSs and SRDHFSs.

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