VFAST Transactions on Mathematics
http://vfast.org/journals/index.php/VTM@ 2021, ISSN(e): 2309-0022, ISSN(p): 2411-6343
Volume 10, Number 2, July-December, 2022 pp: 102-117

Effect of the Arbitrary Coefficients on the convergence of numerical solution of General Second Order Linear Homogeneous Partial Differential Equation

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Keywords: *Second order partial differential equation, Finite difference method, coefficients of PDE, numerical analysis, computational analysis.*
Subject Classification: *Numercal Methods, Partial Differential Equations, Applied Mathematics*
Journal Info:
Submitted: November 15, 2022
Accepted: December 04, 2022
Published: December 09, 2022

Abstract In this study the effect of the coefficients on the convergence of numerical solution of general second order linear homogeneous partial differential equation has been investigated. The main objective was to determine the sensitivity of the coefficients of the PDE in relation to the domain and mesh size. The finite difference method was used to discretize the PDE and numerical solution was obtained by implementing the algorithm on MATLAB. The outcomes of the research have provided interesting facts about the stable values of the coefficients. From the results it is found that the arbitrary coefficients d and e are more sensitive as compared to a, b, c and f. The outcomes of this research study are expected to provide the ways to predict and control the numerical solution convergence behavior obtained by the general second order PDE based on the variable coefficients of the PDE.

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1 Introduction

The general second order linear homogeneous PDE in 2D Cartesian coordinates is given as follows,

$$a\frac{\partial^2 u(x,y)}{\partial x^2} + b\frac{\partial^2 u(x,y)}{\partial x\partial y} + c\frac{\partial^2 u(x,y)}{\partial y^2} + d\frac{\partial u(x,y)}{\partial x} + e\frac{\partial u(x,y)}{\partial y} + fu(x,y) = 0, \quad (1)$$

where $u(x,y)$ is the dependent variable, X and Y be the independent variables; a, b, c, d, e and f are the arbitrary coefficients. This equation has enormous applications in the solution of various physical problems and some other classical equations could be derived from Eq. (1). The numerical methods are often considered to solve the equation due to their straightforward implementation. The variation in the coefficients may affect the simulation patterns of the dependent variable in the computational domain. It has to yet investigate the effect of the coefficients involved in the Eq. (1) on the convergence of solution. In literature; few related works have been discussed by using the various solution strategies. In [10] the generalized finite difference techniques (GFDM) as well as the made-up time integration approach is used to investigate obstacle issues and non-linear free boundary issues are sometimes known as non-linear free boundary problems. This work uses the GFDM a recently discovered domain-type meshless approach for spatial discretization. GFDM eliminates the need for mesh creation and numerical integration while maintaining excellent numerical precision. [13] Showed that how the fundamental solutions method (MFS) and related methodologies have evolved over the previous three decades. MFS-type approaches are used in a variety of applications. Using a mesh of polygonal components a node-based smoothed finite element approach (NS-FEM) for solving upper limit structural analysis was introduced by [6]. The GFDM) was applied for determining the heat source in steady state heat conduction difficulties [14]. To ensure the originality of results the source of heat is considered to satisfy a second order partial-differential-equation transforming the problem into a Fourth order partial-differential equation that can be solved quickly and accurately using the GFDM. A basic solution approach was combined with the regularization approach to answer each numerical cycle corresponds to two mixed, well-posed, as well as straightforward problems [2]. With generalized cross-validation criteria, the ideal value of the parameter value is established for each direct issue examined. For a flowing boundary issues formed on a meshless hybrid-Cartesian of the grid-system a nodal selection approach and an SVD- designed modified generalized finite difference (GFD) scheme was presented in [21]. The SVD-built a technique that is more stable and precise than the traditional least square method based on generalize finite difference method, according to this study. It was found by [12] that for each node only tiny systems of linear equations with a range come to the total amount of nodes in sphere whose effect must be determined. As a result, the computing effort climbs almost linearly as the number of nodes grows. A novel meshless methodology to overcome the classic in-order to increase the accuracy, the finite difference approach just after the Dirichlet leap where the approximation characteristics appear to be similar was proposed in [26]. In [9] the appearance of oscillating Dirichlet and Neumann boundary conditions on an identified or unidentified section of a boundary is represented by Signorini problems. A boundary-type and non-singular meshless technique of two-dimensional problem was discussed in [4]. The solution is represented as a distribution of double layer potential kernel functions. The numerical facts of a planned meshfree method demonstrate the accuracy of the solutions upon looking at the results of exact solution with conventional MFS, and BEM for the Dirichlet, Neumann, and mix-type boundary conditions (BCs) of interior and exterior issues with simple and complex boundaries. Strong agreements have been reached with specific solutions. In [8] the method of unique boundaries, a breakthrough boundary collocation without mesh strategy, was used for the first time to solve 3D elasticity problems. In the singular boundary method, the regularized BEM and the basic solution approach are linked. The basic idea is to maintain the latter's meshless and integration-free features while completely inheriting the formers dimensionality and stability. A project [7] showed the remarkable performance of the modified finite difference approach (GFD), which uses irregular node topologies to compute second-order PDEs that capture the activities of various corporal processes. The approach rapidly generates the derivatives of the nodes by applying the equations in differences obtained to any form of each domain with a second-order differential equation with any sort of boundary constraint (Dirichlet, Neumann, and mixed). In geometric processing and physics-based modelling, differential variables such as normals, curvatures, primary directions, and related matrices play a crucial role [11]. It is vital and difficult to compute these differential values reliably on surface meshes, and some existing approaches often cause inconsistencies that require ad hoc corrections. It was shown that computing the gradient and Hessian of a height function lays the groundwork for computing differential values reliably. The requirement to study the dynamics of a system in terms of a discrete formulation occurs frequently [15]. In [24] numerous applications, such as image processing, surface processing, computer graphics, and computer aided geometric design, numerical integration of geometric partial differential equations is applied. In the numerical integrations, discrete approximations of numerous first and second-order geometric differential operators are used look at consistent discretized approximations of these operators using a quadratic fitting approach in this study. Similarly, many others [1, 3, 5, 16–20, 22, 23, 25] have attempted to solve the second order linear PDEs by a variety of analytical and numerical methods. However, the detailed computational analysis of the effect of all the coefficients of general second order linear partial differential equations has not been reported. If the variations in the coefficients of the PDE in relation to the stable ranges of the discretization parameters is not chosen properly then the smoothness and convergence of the solution could be affected. The choice of the values of the coefficients may provide the ways to speedup the convergence of the numerical solution and may also control the smoothness of the numerical solution.

2 Methodology

This study is aimed at the computational analysis of the 2D general second order linear homogeneous partial differential equation. To achieve this goal, a 2D Cartesian domain $(x, y) \in \Omega = [0, L] \times [0, W]$ is considered and the Eq. (1) be defined on this domain; where L and W be the length and width of the domain respectively. The boundary conditions are of Dirichlet type and are defined as follows, $u(x, 0) = f_b(x, 0),\ where\ 0 \leq x \leq L,\ u(x, W) = f_t(x, W),\ where\ 0 \leq x \leq L,$ $u(0, y) = g_l(0, y),\ where\ 0 \leq y \leq W,$ and $u(L, y) = g_r(L, y),\ where\ 0 \leq y \leq W,$ where the f_b, f_t, g_l and g_r are the bottom, top, left and right boundary conditions respectively. In order to obtain the numerical solution of Eq. (1) central finite difference schemes are applied and those yield the following Eq. (2)

$$\begin{aligned} &a\frac{u(i-1,j)-2u(i,j)+u(i+1,j)}{h_1^2} + b\frac{u(i+1,j+1)-u(i+1,j-1)-u(i-1,j+1)+u(i-1,j-1)}{4h_1h_2} + \\ &c\frac{u(i,j-1)-2u(i,j)+u(i,j+1)}{h_2^2} + d\frac{u(i+1,j)-u(i-1,j)}{2h_1} + e\frac{u(i,j+1)-u(i,j-1)}{2h_2} + fu(i,j) = 0, \end{aligned} \tag{2}$$

Where h_1 and h_2 are the increments along x and y directions respectively. The above Eq. (2) is simplified for the central node-wise iterative solution and the required form is given as follows,

$$u(i,j) = -\frac{1}{4}\frac{1}{(2ah_2^2+2ch_1^2-fh_1^2h_2^2)}\times \begin{pmatrix} 4\,a\,h_2^2\,[u\,(i+1,\,j)+u\,(i-1,\,j)]+b\,h_1h_2[u\,(i+1,\,j+1)-u\,(i-1,\,j+1)-u\,(i+1,\,j-1)+u\,(i-1,\,j-1) \\ +4c\,h_1^2[u\,(i,\,j+1)+u\,(i,\,j-1)]+2\,d.h_1h_2^2[u\,(i+1,\,j)-\,u\,(i-1,\,j)]+ \\ 2e.h_1^2h_2[u\,(i,\,j+1)-u\,(i,\,j-1)] \end{pmatrix} \tag{3}$$

where $h_1 \neq 0,\ h_2 \neq 0,$ in addition to this the following restriction must be applied to obtained the numerically realistic solution,

$$2a\,h_2^2 + 2c\ h_1^2 - fh_1^2h_2^2 \neq 0 \tag{4}$$

To test the numerical solution algorithm a user defined MATLAB code was written to implement the methodology. The varied values of the coefficients a, b, c, d, e and f have been tested for the convergence of the numerical solution in relation to the domain size. The data is collected for the computational analysis of the convergence of numerical solutions.

3 Results and Discussions

Since the main objective of this study is to investigate the effect of coefficients of the governing PDE on the convergence of numerical solution, therefore the step by step values of the coefficients were varied and their effect on the convergence and smoothness was noted.It is quite computationally intensive to take all real values of the coefficients, however for testing purpose the discrete values of the coefficients a, b, c, d, e and f were chosen from the set $S = \{-10, -5, 0, 5, 10\}$. Similarly, for the domain size the values of L and W were taken from the set $T = \{1, 2, 3, 4\}$. If any single coefficient is varied by taking values from S and keeping all other coefficients fixed to the default value 1 then there can be 80 different possibilities or combinations for the L, W and the respective varied coefficient, such that $O(S)O(T)O(T) = 5X4X4 = 80$. Thus the significant data from the implementation was collected. The data is shown in the following Tables 1-6. Table 1 lists the number of iterations and computational time required to converge the numerical solution of Eq. (1) for all combinations of the parameters $L, W, h1, h2$ and the arbitrary constant coefficient a. In the Table 1 the combinations at the missing serial numbers indicate the divergence of the numerical solution. The results depicted in Figure 1 show that the convergence of the numerical solution is quite slow and possibly diverges for most of the positive values of a when all other parameters $b = c = d = e = f = 1$. The maximum iterations are observed when $L = 3, W = 2, h1 = 0.3, h2 = 0.2$ and $a = 0$. It can be deduced from the figure that numerical solution diverges for the positive values of. In relation to the number iterations required to obtain the numerical solution the computational time is also important to analyze. The computational time (sec) taken to converge the numerical solution of Eq. (1) fluctuates in a similar fashion as the number of iterations. Meaning to say that higher the number of iterations higher the computational time and lower the number of iterations lower the computational time.

After analyzing the effect of the parameter a, the effect of the parameter b is tested for the numerical solution. For all the combinations of the parameters $L, W, h1, h2$ and the arbitrary constant coefficient b as listed in the Table 2 and depicted in Figure 2 it appears that the convergence of the numerical solution for the values of b is some how faster than that of at a when all other parameters $a = c = d = e = f = 1$. The maximum iterations are observed when $L = 1, W = 4, b = \{-5, 5\}$. It can be deduced from the figure that the numerical solution converges for all the values of b except for $b = 0$ when all other parameters

$a = c = d = e = f = 1$.

Similarly, the effect of the parameter c is analyzed for the numerical solution for all combinations of the parameters $L, W, h1, h2$ and the arbitrary constant coefficient c as listed in Table 3 and depicted in Figure 3. The results reveal that the convergence of the numerical solution for the values of c is sensitive because the convergence is very slow at c when all other parameters $a = b = d = e = f = 1$. At most of the positive values of c such that $c > 0$ the numerical solution diverges. The maximum iterations are observed when $L = 2, W = 3, c = 0$. It can be deduced from the figure that the numerical solution converges for most of the values of c except for $c > 0$ with exception of any specific combination.

Then the effect of the parameter d is analyzed for the numerical solution for all the combinations of the parameters $L, W, h1, h2$ and the arbitrary constant coefficient d as listed in the Table 4 and exhibited in Figure 4. It can be observed that the convergence of the numerical solution for the values of d is very much sensitive as out of 80 combinations only at three combinations the solution converges. The maximum iterations are observed when $L = 4, W = 4$ and $d = \{-10, 10\}$, and the smooth solution is obtained when $d = 0$. Thus, the numerical solution diverges for most of the values of d except for $d = 0$ when $L = W = 4$.

The effect of the parameter e is analyzed for the numerical solution for all combinations of the parameters $L, W, h1, h2$ and the arbitrary constant coefficient e as listed in the Table 5 and shown in Figure 5. The results depicted show that the convergence of the numerical solution for the values of e is also very much sensitive as out of 80 combinations only at seven combinations the solution converges. The maximum iterations were observed when L=3, W=4 and e={-5}. It can be deduced that the numerical solution diverges for most of the values of e, whereas when $e = 0$ the solution diverges strictly. Finally, the effect of the parameter f is analyzed for the convergence behavior numerical solution of) for all combinations of the parameters $L, W, h1, h2$ and the arbitrary constant coefficient f as listed in the Table 6 and depicted in Figure 6. It is revealed that the convergence of the numerical solution for the values of f is not more sensitive as compared to the effect of d and e. The maximum iterations are observed when $L = 2$, $W = 2$ and $f = 10$. However, the numerical solution diverges for most of the values of f when $f \leq 0$.

From the above analysis of the effect of the coefficients on the numerical solution of Eq. (1) one can easily decide bounds for t he coefficients a, b, c, d, e and f. Also it can be presumed the specific values of coefficients on which the stable and smooth solution of the general second order linear homogeneous PDE could be obtained.

Table 1. Computational analysis of solution of Eq. (1) when $b = c = d = e = f = 1$ and $L, W, h1, h2, a$ are varied

S. No	L	W	h1	h2	a	Time, t (sec)	Iterations, It
01	1	1	0.1000	0.1000	-10	0.2346	4731
02	1	1	0.1000	0.1000	-5	0.0944	2005
03	1	1	0.1000	0.1000	0	0.5666	12283
06	1	2	0.1000	0.2000	-10	0.6106	13576
07	1	2	0.1000	0.2000	-5	0.6659	14562
08	1	2	0.1000	0.2000	0	0.1083	2310
12	1	3	0.1000	0.3000	-5	0.0708	1547
13	1	3	0.1000	0.3000	0	0.0563	1188
18	1	4	0.1000	0.4000	0	0.0391	802
21	2	1	0.2000	0.1000	-10	0.0500	896
26	2	2	0.2000	0.2000	-10	0.2156	4795

Table 1 continued from previous page

S. No	L	W	h1	h2	a	Time, t (sec)	Iterations, It
27	2	2	0.2000	0.2000	-5	0.0976	2031
28	2	2	0.2000	0.2000	0	0.4176	9261
31	2	3	0.2000	0.3000	10	0.8790	19572
32	2	3	0.2000	0.3000	-5	0.2657	5774
33	2	3	0.2000	0.3000	0	0.1420	3126
36	2	4	0.2000	0.4000	-10	0.1827	4035
37	2	4	0.2000	0.4000	-5	0.7241	15836
38	2	4	0.2000	0.4000	0	0.0807	1696
46	3	2	0.3000	0.2000	-10	0.085	1771
47	3	2	0.3000	0.2000	-5	0.0408	789
48	3	2	0.3000	0.2000	0	1.8731	39477
51	3	3	0.3000	0.3000	-10	0.2289	4908
52	3	3	0.3000	0.3000	-5	0.1050	2075
53	3	3	0.3000	0.3000	0	0.3116	6537
56	3	4	0.3000	0.4000	-10	0.6935	12522
57	3	4	0.3000	0.4000	-5	0.3711	4335
58	3	4	0.3000	0.4000	0	0.2095	2964
62	4	1	0.4000	0.1000	-5	0.0715	1179
66	4	2	0.4000	0.2000	-10	0.0546	909
68	4	2	0.4000	0.2000	0	0.0723	1485
71	4	3	0.4000	0.3000	-10	0.1116	2383
72	4	3	0.4000	0.3000	-5	0.0506	1068
73	4	3	0.4000	0.3000	0	0.5398	12136
76	4	4	0.4000	0.4000	-10	0.2298	5073
77	4	4	0.4000	0.4000	-5	0.0985	2139
78	4	4	0.4000	0.4000	0	0.2135	4597

Table 2. Computational analysis of solution of Eq. (1) when $a = c = d = e = f = 1$ and $L, W, h1, h2, b$ are varied

S. No	L	W	h1	h2	b	Time, t (sec)	Iterations, It
01	1	1	0.1000	0.1000	-10	0.1169	828
02	1	1	0.1000	0.1000	-5	0.1422	2458
03	1	1	0.1000	0.1000	0	0.0213	218
04	1	1	0.1000	0.1000	5	0.1325	2462
05	1	1	0.1000	0.1000	10	0.0483	828
06	1	2	0.1000	0.2000	-10	0.0657	1043
07	1	2	0.1000	0.2000	-5	0.2041	3665
09	1	2	0.1000	0.2000	5	0.1883	3674
10	1	2	0.1000	0.2000	10	0.0533	1043
11	1	3	0.1000	0.3000	-10	0.0847	1486

Table 2 continued from previous page

S. No	L	W	h1	h2	b	Time, t (sec)	Iterations, It
12	1	3	0.1000	0.3000	-5	0.4576	6890
14	1	3	0.1000	0.3000	5	0.6115	6918
15	1	3	0.1000	0.3000	10	0.1152	1486
16	1	4	0.1000	0.4000	-10	0.1431	2119
17	1	4	0.1000	0.4000	-5	0.7429	13965
19	1	4	0.1000	0.4000	5	0.7598	14033
20	1	4	0.1000	0.4000	10	0.1056	2121
21	2	1	0.2000	0.1000	-10	0.0533	1042
22	2	1	0.2000	0.1000	-5	0.2153	3664
24	2	1	0.2000	0.1000	5	0.2693	3674
25	2	1	0.2000	0.1000	10	0.0855	1042
26	2	2	0.2000	0.2000	-10	0.0695	820
27	2	2	0.2000	0.2000	-5	0.1865	2386
28	2	2	0.2000	0.2000	0	0.0127	238
29	2	2	0.2000	0.2000	5	0.174	2391
30	2	2	0.2000	0.2000	10	0.0684	821
31	2	3	0.2000	0.3000	10	0.0714	883
32	2	3	0.2000	0.3000	-5	0.2064	2684
34	2	3	0.2000	0.3000	5	0.2111	2692
35	2	3	0.2000	0.3000	10	0.0585	884
36	2	4	0.2000	0.4000	-10	0.0544	1023
37	2	4	0.2000	0.4000	-5	0.1574	3426
39	2	4	0.2000	0.4000	5	0.1611	3437
40	2	4	0.2000	0.4000	10	0.0516	1025
41	3	1	0.3000	0.1000	-10	0.0709	1484
42	3	1	0.3000	0.1000	-5	0.3230	6888
44	3	1	0.3000	0.1000	5	0.3492	6915
45	3	1	0.3000	0.1000	10	0.0702	1485
46	3	2	0.3000	0.2000	-10	0.0433	883
47	3	2	0.3000	0.2000	-5	0.1249	2688
49	3	2	0.3000	0.2000	5	0.1295	2694
50	3	2	0.3000	0.2000	10	0.0458	884
51	3	3	0.3000	0.3000	-10	0.0413	808
52	3	3	0.3000	0.3000	-5	0.1072	2273
54	3	3	0.3000	0.3000	5	0.1098	2278
55	3	3	0.3000	0.3000	10	0.0529	809
56	3	4	0.3000	0.4000	-10	0.0426	833
57	3	4	0.3000	0.4000	-5	0.1180	2365
58	3	4	0.3000	0.4000	0	0.0197	307
59	3	4	0.3000	0.4000	5	0.1508	2371

Table 2 continued from previous page

S. No	L	W	h1	h2	b	Time, t (sec)	Iterations, It
60	3	4	0.3000	0.4000	10	0.0774	834
61	4	1	0.4000	0.1000	-10	0.1784	2117
62	4	1	0.4000	0.1000	-5	1.1738	13970
64	4	1	0.4000	0.1000	5	0.6668	14034
65	4	1	0.4000	0.1000	10	0.1025	2120
66	4	2	0.4000	0.2000	-10	0.0557	1023
67	4	2	0.4000	0.2000	-5	0.1722	3426
69	4	2	0.4000	0.2000	5	0.1780	3437
70	4	2	0.4000	0.2000	10	0.0719	1024
71	4	3	0.4000	0.3000	-10	0.0421	833
72	4	3	0.4000	0.3000	-5	0.1177	2368
74	4	3	0.4000	0.3000	5	0.1143	2372
75	4	3	0.4000	0.3000	10	0.0419	834
76	4	4	0.4000	0.4000	-10	0.0394	791
77	4	4	0.4000	0.4000	-5	0.1066	2130
79	4	4	0.4000	0.4000	5	0.1099	2134
80	4	4	0.4000	0.4000	10	0.0414	791

Table 3. Computational analysis of solution of Eq. (1) when $a = b = d = e = f = 1$ and $L, W, h1, h2, c$ are varied

S. No	L	W	h1	h2	c	Time, t (sec)	Iterations, It
01	1	1	0.1000	0.1000	-10	0.2378	4732
02	1	1	0.1000	0.1000	-5	0.0977	2006
03	1	1	0.1000	0.1000	0	0.588	12364
06	1	2	0.1000	0.2000	-10	0.0450	897
12	1	3	0.1000	0.3000	-5	0.0297	587
17	1	4	0.1000	0.4000	-5	0.0589	1179
21	2	1	0.2000	0.1000	-10	0.6616	13829
22	2	1	0.2000	0.1000	-5	0.7118	14565
23	2	1	0.2000	0.1000	0	0.1182	2316
26	2	2	0.2000	0.2000	-10	0.2295	4797
27	2	2	0.2000	0.2000	-5	0.1021	2031
28	2	2	0.2000	0.2000	0	0.4328	9310
31	2	3	0.2000	0.3000	10	0.0848	1771
32	2	3	0.2000	0.3000	-5	0.0410	790
33	2	3	0.2000	0.3000	0	1.8398	39716
36	2	4	0.2000	0.4000	-10	0.0485	909
38	2	4	0.2000	0.4000	0	0.0490	1022
39	2	4	0.2000	0.4000	5	0.0132	247
42	3	1	0.3000	0.1000	-5	0.0740	1575

Table 3 continued from previous page

S. No	L	W	h1	h2	c	Time, t (sec)	Iterations, It
43	3	1	0.3000	0.1000	0	0.0572	1194
46	3	2	0.3000	0.2000	-10	1.1049	19569
47	3	2	0.3000	0.2000	-5	0.3967	5774
48	3	2	0.3000	0.2000	0	0.2143	3137
51	3	3	0.3000	0.3000	-10	0.2295	4909
52	3	3	0.3000	0.3000	-5	0.0990	2075
53	3	3	0.3000	0.3000	0	0.3036	6566
56	3	4	0.3000	0.4000	-10	0.1126	2384
57	3	4	0.3000	0.4000	-5	0.0514	1068
58	3	4	0.3000	0.4000	0	0.5639	12197
63	4	1	0.4000	0.1000	0	0.0402	805
66	4	2	0.4000	0.2000	-10	0.1884	4102
67	4	2	0.4000	0.2000	-5	0.7442	15833
68	4	2	0.4000	0.2000	0	0.0812	1700
71	4	3	0.4000	0.3000	-10	0.5775	12519
72	4	3	0.4000	0.3000	-5	0.2059	4334
73	4	3	0.4000	0.3000	0	0.1409	2974
76	4	4	0.4000	0.4000	-10	0.2326	5073
77	4	4	0.4000	0.4000	-5	0.1025	2138
78	4	4	0.4000	0.4000	0	0.2150	4614

Table 4. Computational analysis of solution of Eq. (1) when $a = b = c = e = f = 1$ and $L, W, h1, h2, d$ are varied

S. No	L	W	h1	h2	d	Time, t (sec)	Iterations, It
76	4	4	0.4000	0.4000	-10	0.7175	11122
78	4	4	0.4000	0.4000	0	0.0336	572
80	4	4	0.4000	0.4000	10	0.5713	11193

Table 5. Computational analysis of solution of Eq. (1) when $a = b = c = d = f = 1$ and $L, W, h1, h2, e$ are varied

S. No	L	W	h1	h2	e	Time, t (sec)	Iterations, It
57	3	4	0.3000	0.4000	-5	0.6454	13988
60	3	4	0.3000	0.4000	10	0.0636	1293
72	4	3	0.4000	0.3000	-5	0.0690	1412
75	4	3	0.4000	0.3000	10	0.6476	14009
76	4	4	0.4000	0.4000	-10	0.5326	11128
77	4	4	0.4000	0.4000	-5	0.1497	3130
80	4	4	0.4000	0.4000	10	0.152	3135

Table 6. Computational analysis of solution of Eq. (1) when $a = b = c = d = e = 1$ and $L, W, h1, h2, f$ are varied

S. No	L	W	h1	h2	f	Time, t (sec)	Iterations, It
20	1	4	0.1000	0.4000	10	0.0240	420
29	2	2	0.2000	0.2000	10	124.4773	2622144
30	2	2	0.2000	0.2000	10	0.0328	633
34	2	3	0.2000	0.3000	5	1.7127	36844
39	2	4	0.2000	0.4000	5	1.1135	23856
40	2	4	0.2000	0.4000	10	0.1437	3097
49	3	2	0.3000	0.2000	5	1.7162	36811
50	3	2	0.3000	0.2000	10	0.0550	1130
54	3	3	0.3000	0.3000	5	0.5054	10982
55	3	3	0.3000	0.3000	10	0.0393	775
59	3	4	0.3000	0.4000	5	0.3386	7198
60	3	4	0.3000	0.4000	10	0.6808	14621
69	4	2	0.4000	0.2000	5	1.0863	23845
70	4	2	0.4000	0.2000	10	0.0771	1605
74	4	3	0.4000	0.3000	5	0.3318	7197
75	4	3	0.4000	0.3000	10	0.1246	2584
79	4	4	0.4000	0.4000	5	0.2093	4358
80	4	4	0.4000	0.4000	10	0.4541	5851

Figure 1. Iterations for convergent solution of Eq. (1) when $b = c = d = e = f = 1$ and $L, W, h1, h2, a$ are varied.

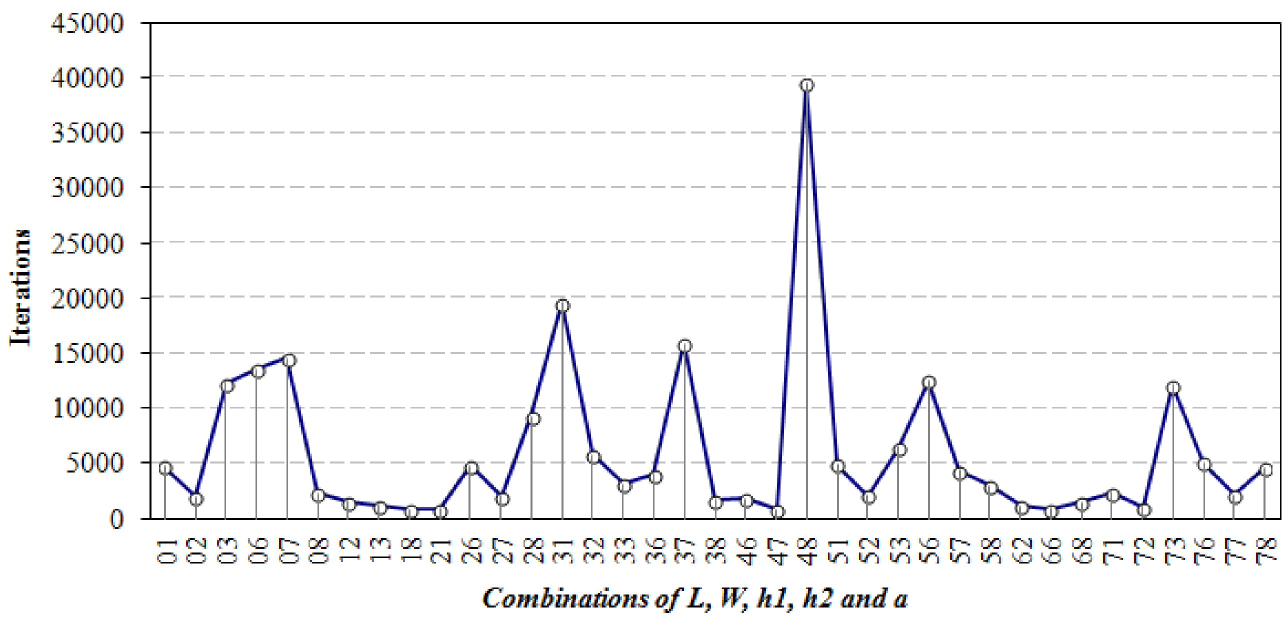


Figure 2. Iterations for convergent solution of Eq. (1) when $a = c = d = e = f = 1$ and $L, W, h1, h2, b$ are varied.

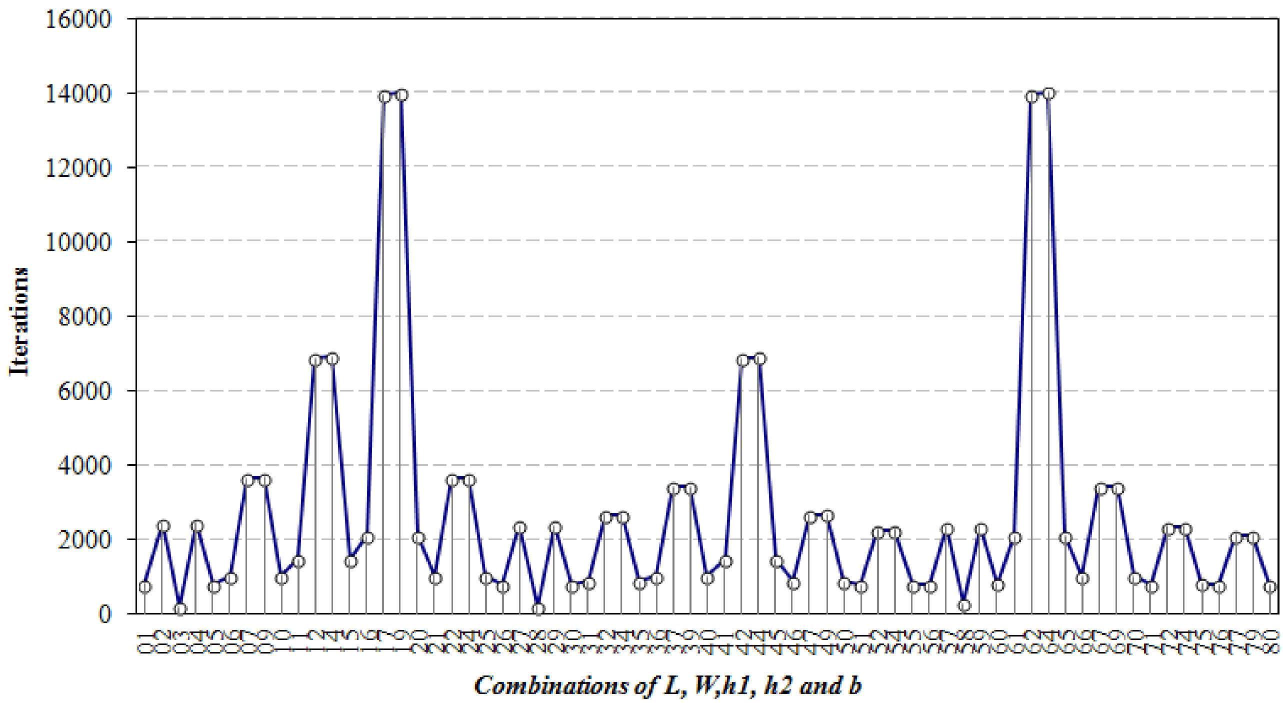


Figure 3. Iterations for convergent solution of Eq. (1) when $a = b = d = e = f = 1$ and $L, W, h1, h2, c$ are varied.

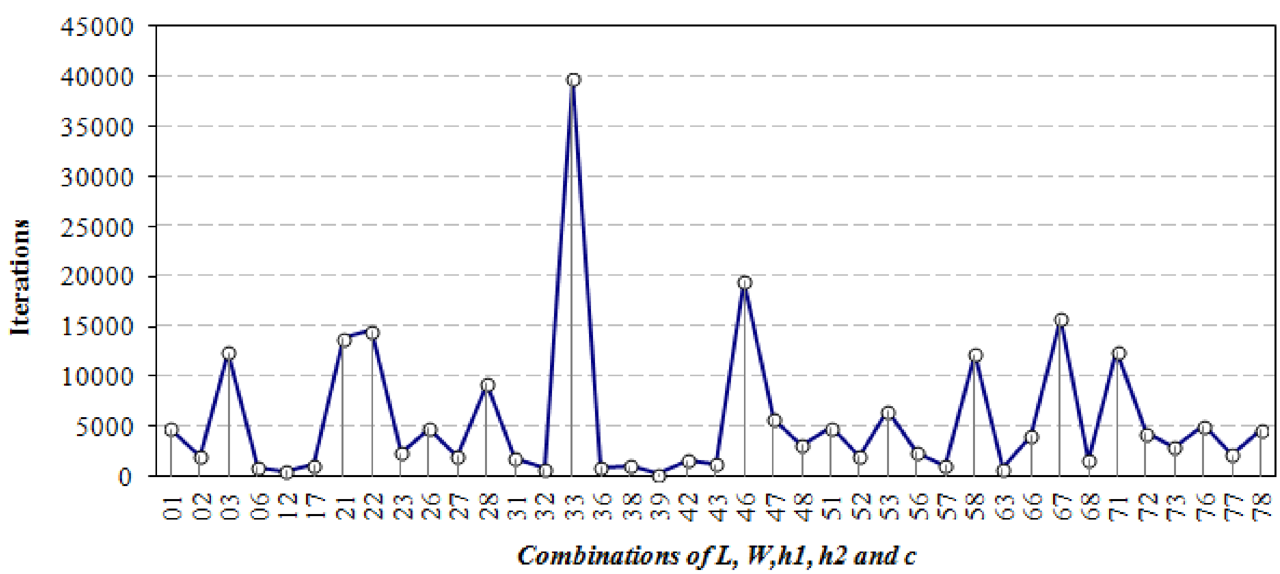


Figure 4. Iterations for convergent solution of Eq. (1) when $a = b = c = e = f = 1$ and $L, W, h1, h2, d$ are varied.

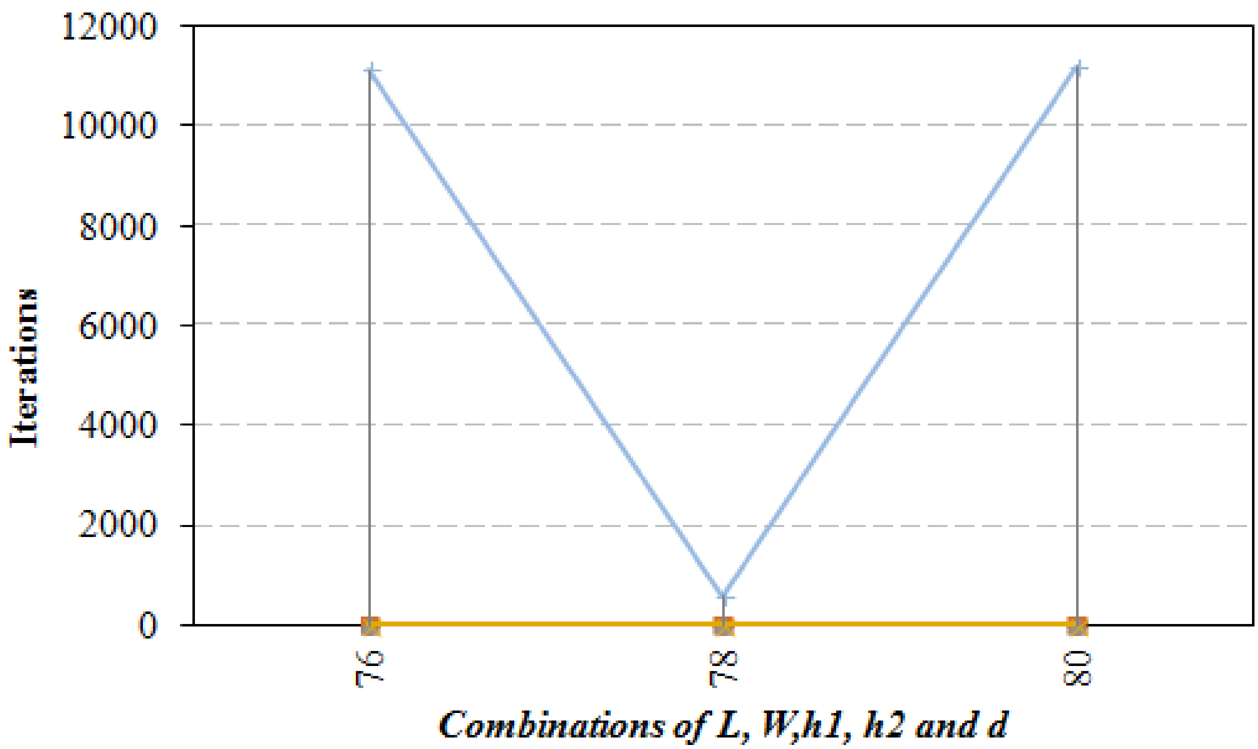


Figure 5. Iterations for convergent solution of Eq. (1) when $a = b = c = d = f = 1$ and $L, W, h1, h2, e$ are varied.

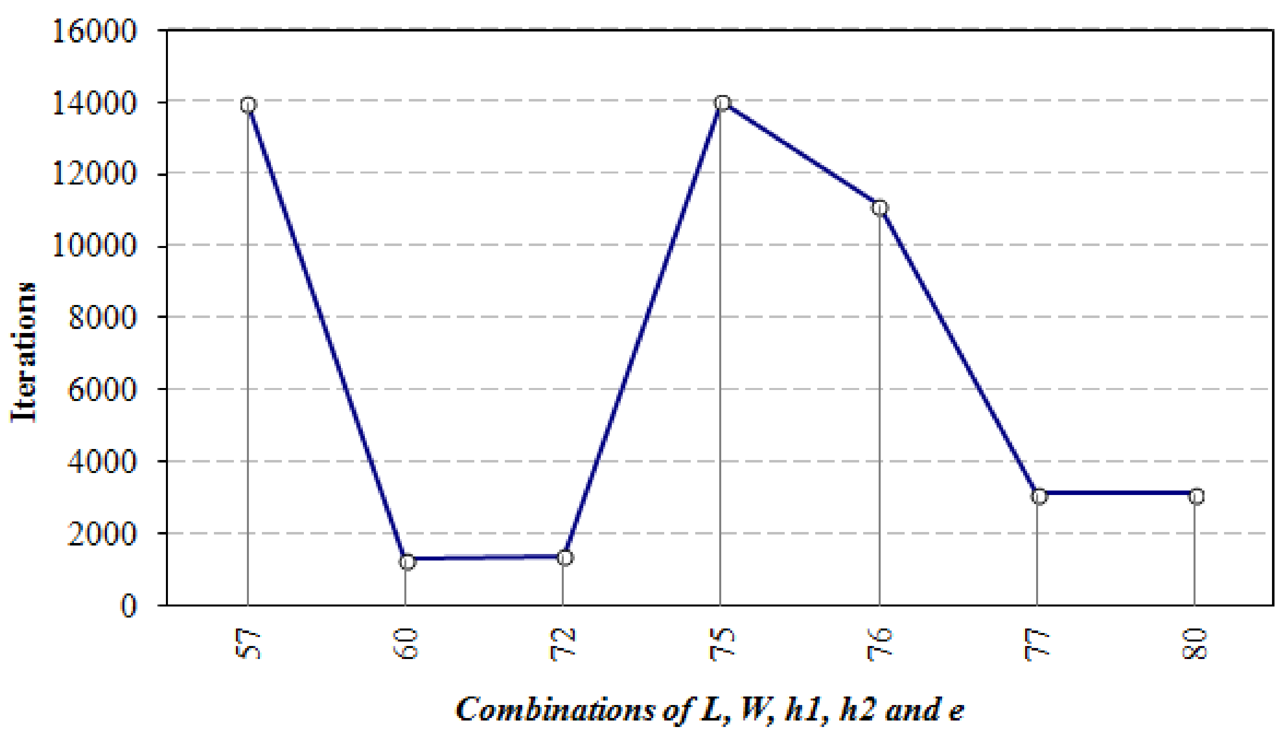
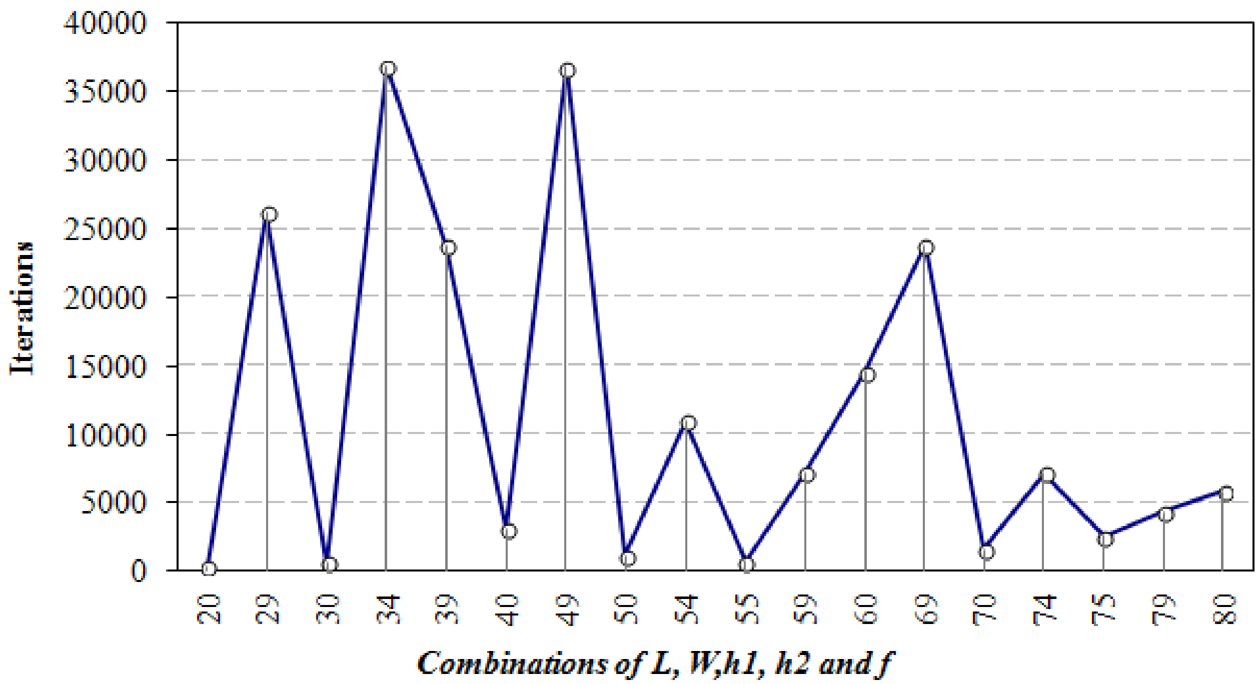


Figure 6. Iterations for convergent solution of Eq. (1) when $a = b = c = d = e = 1$ and $L, W, h1, h2, f$ are varied..



4 Conclusions

In this study a computational analysis of the general linear homogeneous second order PDE in relation to the varied domain size and the arbitrary coefficients was carried out. From the analysis of numerical solution it was revealed that the numerical solution diverges for the positive values of a when all other parameters $b = c = d = e = f = 1$, the solution remains stable for $a \leq 0$. Also, the numerical solution converges for all the values of the b except for $b = 0$ in some cases when all other parameters $a = c = d = e = f = 1$. The numerical solution of converges for most of the values of c except for $c > 0$ with exception of any specific combination of L, W, $h1$, $h2$ and c. Similarly, the numerical solution diverges for most of the values of d except for $d = 0$ when $L = W = 4$. The convergence of the numerical solution for the values of e is very much sensitive as out of 80 combinations only at seven combinations the solution converges. The convergence of the numerical solution for the values of f is not more sensitive as compared to the effect of d and e. The numerical solution of Eq. (1) diverges for most of the values of f when $f \leq 0$. The outcomes of the research have provided interesting facts about the stable values of the coefficients of general second order linear homogeneous equation. It is still worthwhile to further generalize the results of this study to more levels but the computational complexity may confine the analysis unexpectedly.

5 Acknowledgements

The authors are highly grateful to the Quaid-e-Awam University of Engineering, Science and Technology for providing facilities to conduct this research.

6 Author Contributions

Liaquat Ali Zardari Writing- Original draft preparation. **Shakeel Ahmed Kamboh** Conceptualization, Methodology. **Abbas Ali Ghoto** Validation and investigation. **Dr. Kirshan Kumar Luhana** technical reviewing, Editing, and software **Dr. Shah Zaman Nizamani** Writing- Reviewing.

7 Compliance with Ethical Standards

It is declare that all authors don't have any conflict of interest.

8 Funding Information

No funding.

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