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GENERAL LORENTZ TRANSFORMATIONS AND APPLICATIONS

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Abstract. *In this paper, we solve a formidable nonlinear problem presented by a veteran Physicist C.B.van Wyk in his paper on General Lorentz Transformations and applications [1].We arrived at the solution by first examining the problem in a non-invariant manner.This assured us that the solution of the problem exists.To solve the problem in an invariant fashion, we had to understand the geometry of the situation.This finally gave us a reformulation of the formidable nonlinear problem as a simple linear one. We have verified that the solution of the reformulated problem does solve the original problem.In fact,the original problem can be recovered from the reformulated one.*

Keywords: Lorentz transformation, Anti-symmetric matrix, Pseudo-vector, Pseudo-scalar, Vector formalism, Spinor formalism, Invariant solution, Non-invariant solution.

1. **Introduction.** In this paper, we present an interesting solution of a problem posed by Prof. C. B. van Wyk in his paper titled "General Lorentz Transformations and applications" which appeared in J. Math. Phys. **27** (5) May 1986.
In this paper, on page 1309, he mentioned:

"To solve (28) for $\hat{n}, \hat{v}$ and ϕ_1, when $\mathbf{a}$ and $\mathbf{b}$ are given, may be a formidable non-linear problem."

The relevant equations (28) are presented below:

$$\begin{aligned} \mathbf{a} &= (C_1^2 + S_1^2)\hat{n} - 2S_1^2(\hat{v}.\hat{n})\hat{v}, \\ \mathbf{b} &= 2S_1C_1\hat{\mathbf{n}} \wedge \hat{\mathbf{v}}. \end{aligned}$$

In the above $C_1 = cosh\frac{\phi_1}{2}$ and $S_1 = sinh\frac{\phi_1}{2}$. For $\mathbf{a}, \mathbf{b}, \hat{\mathbf{n}}$ and $\hat{\mathbf{v}}$, we remark that

1. $\mathbf{a}, \mathbf{b}$ are not independent. These satisfy
   $\mathbf{a}.\mathbf{b} = 0$ and $\mathbf{a}^2 - \mathbf{b}^2 = 1$. (Eq. (4) in origional paper)
2. $\hat{\mathbf{n}}$ and $\hat{\mathbf{v}}$ are unit vectors,
3. $\mathbf{a}$ and $\hat{\mathbf{n}}$ are pseudovectors,
4. $\mathbf{a} \wedge \mathbf{b}, \mathbf{b}, \hat{\mathbf{v}}$ and $\hat{\mathbf{n}} \wedge \hat{\mathbf{v}}$ are vectors,
5. $\mathbf{a}^2, \mathbf{b}^2$ are scalars whereas $\hat{\mathbf{n}}.\hat{\mathbf{v}}$ is a pseudoscalar.

Thus above two equations are relations between pseudo-vectors and vectors respectively.

We first tried to solve the problem by using non-invariant method just to assure ourselves that a solution was possible. We took $\hat{n}$ along the $x-$axis, $\hat{n} \wedge \hat{v}$ along the $z-$axis, consequently **b** is along the $z - axis$ and **a** as well as $\hat{v}$ are in the $xy-$plane. We found that we could solve the problem. The solution of problem by using above method will also be briefly presented.

After arriving at solution of problem by a non-invariant method, we felt that the problem could be solved by using an invariant method. We will present the complete solution to the problem by using this invariant approach.

2. **Background.** As mentioned earlier this paper is basically based on the work done by C.B. Van Wyk [1]. Motivation behind his work was the well known result that the most general proper vector Lorentz transformation can be generated by an antisymmetric 4×4 matrix. A practical obstacle in the handling of these vector transformations is the amount of labor required to multiply 4×4 matrices. This obstacle is surmounted by formulating the corresponding general transformation for the two component spinor that contains the same tensor.

2.1. **The Vector Formulation.** The vector form of general Lorentz transformation is given by:

$$\begin{aligned} \Lambda_1(\mathbf{a}, \mathbf{b}, \theta, \phi) &= e^{iU\theta + U^D\phi} \\ &= e^{iU\theta} e^{U^D\phi} \\ &= \Lambda_1(\mathbf{a}, \mathbf{b}, \theta, 0)\Lambda_1(\mathbf{a}, \mathbf{b}, 0, \phi), \end{aligned} \tag{1}$$

with θ and ϕ real parameters, satisfying

$$\Lambda_1 \Lambda_1^T = \Lambda_1^T \Lambda_1 = I, \det\, \Lambda_1 = 1. \tag{2}$$

Here U is 4×4 anti-symmetric matrix given by:

$$U = \begin{pmatrix} 0 & -ia_3 & ia_2 & b_1 \\ ia_3 & 0 & -ia_1 & b_2 \\ -ia_2 & ia_1 & 0 & b_3 \\ -b_1 & -b_2 & -b_3 & 0 \end{pmatrix},$$

and U^D is its dual defined by

$$U^D_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} U_{\rho\sigma}.$$

2.2. **The Spinor formalism for the general Lorentz transformation.** The general Lorentz transformation for the two-component spinor is given by:

$$\Lambda_2(\mathbf{a}, \mathbf{b}, \theta, \phi) = e^{i\frac{1}{2}\sigma.(\mathbf{a}+i\mathbf{b})\theta} e^{\frac{1}{2}\sigma.(\mathbf{a}+i\mathbf{b})\phi}. \tag{3}$$

The determinant of this matrix equals unity provided that

$$(\mathbf{a} + i\mathbf{b}).(\mathbf{a} + i\mathbf{b}) = 1,$$

2.3. **Application.** Let us consider the case in which the operator for transforming from the rest to the laboratory frame of reference is given by the similarity transformation

$$L_1^{-1}(\hat{\mathbf{v}}, \phi_1) R_1(\hat{\mathbf{n}}, \theta) L_1(\hat{\mathbf{v}}, \phi_1) = \Lambda_1(\mathbf{a}, \mathbf{b}, \theta, 0), \tag{4}$$

where

$$\mathbf{a} = (C_1^2 + S_1^2)\hat{\mathbf{n}} - 2S_1^2(\hat{\mathbf{v}}.\hat{\mathbf{n}})\hat{\mathbf{v}}, \tag{5}$$

$$\mathbf{b} = 2S_1C_1(\hat{\mathbf{n}} \wedge \hat{\mathbf{v}}). \tag{6}$$

In the above $C_1 = \cosh\frac{\phi_1}{2}$ and $S_1 = \sinh\frac{\phi_1}{2}$.

For above mentioned case the author "Prof. C. B. van Wyk" has made a remark that **to solve eqs. (5) and (6) for $\hat{\mathbf{n}}, \hat{\mathbf{v}}$ and ϕ_1 when a and b are given, may be a *formidable nonlinear problem*.**

Basic aim behind this paper is to present a solution of the above mentioned problem which was posed by Prof. C. B. van Wyk. The solution seems simple but it was arrived at, by attempts in different directions. By using the correct geometry, we are able to convert the formidable non-linear problem to a linear one.

3. **Non-invariant Solution to the problem.** Let us consider eqs. (5) and (6) given by

$$\begin{aligned} \mathbf{a} &= (C_1^2+S_1^2)\hat{\mathbf{n}} - 2S_1^2(\hat{\mathbf{v}}.\hat{\mathbf{n}})\hat{\mathbf{v}}, \\ \mathbf{b} &= 2S_1C_1(\hat{\mathbf{n}}\wedge\hat{\mathbf{v}}). \end{aligned}$$

These two equations are consistent with eq. (4)(of the original paper) given by:

$$\mathbf{a}.\mathbf{a} - \mathbf{b}.\mathbf{b} = 1, \mathbf{a}.\mathbf{b} = 0, \text{ or } (\mathbf{a}+i\mathbf{b})^2 = 1,$$

There are five variables on the right hand sides of the eqs. (5) and (6), namely, two for $\hat{\mathbf{v}}$, two for $\hat{\mathbf{n}}$ and one for S_1 or C_1 (Since $\hat{\mathbf{v}}$ and $\hat{\mathbf{n}}$ are unit vectors).

Now we want to solve eqs. (5) and (6) for finding $\hat{\mathbf{v}}$, $\hat{\mathbf{n}}$ and ϕ_1, when $\mathbf{a}$ and $\mathbf{b}$ are given to us.

Let us choose $\hat{\mathbf{n}}$ along the $x-$axis, $\hat{\mathbf{v}}$ in the $xy-$plane, and $\hat{\mathbf{n}}\wedge\hat{\mathbf{v}}$ is along the $z-$axis.This is clear from the following figure:

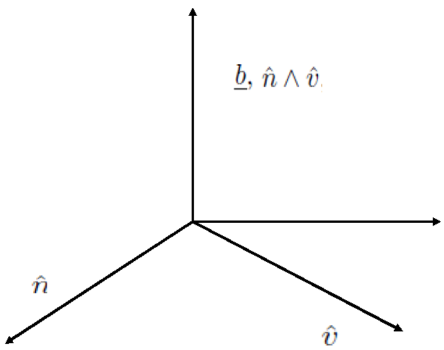

FIGURE 1. Geometrical interpretation of non-invariant solution

Hence we may write

$$\begin{aligned} \hat{\mathbf{n}} &= (1,0,0), \\ \hat{\mathbf{v}} &= (v_1,v_2,0), \ (where\ v_1^2+v_2^2=1) \\ \hat{\mathbf{n}}\wedge\hat{\mathbf{v}} &= (0,0,v_2), \\ \mathbf{a} &= (a_1,a_2,a_3), \\ \mathbf{b} &= (0,0,b). \end{aligned}$$

Eq. (5) can be rewritten as

$$(a_1,a_2,a_3) = (1,0,0)(C_1^2+S_1^2) - 2(v_1,v_2,0)(v_1)S_1^2,$$

which gives us

$$a_1 = C_1^2+S_1^2-2v_1^2S_1^2, \tag{7}$$

$$a_2 = -2v_1v_2S_1^2, \tag{8}$$

and

$$a_3 = 0.$$

Now eq. (6) says that

$$\mathbf{b} = 2S_1C_1(\hat{\mathbf{n}} \wedge \hat{\mathbf{v}}),$$

and in component form it is rewritten as

$$(0, 0, b) = 2S_1C_1(0, 0, v_2),$$

and from the above we finally arrive at

$$v_2 = \frac{b}{2S_1C_1}. \tag{9}$$

By using value of v_2 in eq.(8) we find

$$v_1 = \frac{-a_2C_1}{bS_1}. \tag{10}$$

By using eq. (10) in eq. (7) we have

$$a_1 = C_1^2 + S_1^2 - 2(\frac{-a_2C_1}{bS_1})^2 S_1^2,$$

which on simplification and by using condition $a^2 - b^2 = 1$, gives us

$$C_1^2 = \frac{a_1^2 + a_2^2 - 1}{2(a_1 - 1)},$$

and

$$S_1^2 = \frac{(a_1 - 1)^2 + a_2^2}{2(a_1 - 1)}.$$

Now by making use of the values of C_1, S_1 and the condition $a^2 - b^2 = 1$, in eqs. (9) and (10) we finally arrive at

$$\begin{aligned} v_1 &= \frac{-a_2}{\sqrt{(a_1 - 1)^2 + a_2^2}}, \\ v_2 &= \frac{a_2 - 1}{\sqrt{(a_1 - 1)^2 + a_2^2}}. \end{aligned}$$

One can easily check that these values of v_1 and v_2 infact satisfy the condition $v_1^2 + v_2^2 = 1$.

Hence we have finally reached the solution to the problem posed by Prof. van Wyk by using a non-invariant approach. On basis of this we can say that the given problem is solvable and in the next Section we will present the solution to the problem by using invariant method.

4. **Invariant Solution to the problem.** As mentioned before $\mathbf{a}$ is a pseudovector whereas $\mathbf{b}$ is a vector. From eqs. (5) and (6), examining consistency, we find that

1. $\mathbf{a}, \hat{\mathbf{n}}$ are pseudovectors.
2. $\mathbf{b}, \hat{\mathbf{v}}, \hat{\mathbf{n}} \wedge \hat{\mathbf{v}}, \mathbf{a} \wedge \mathbf{b}$ are vectors.

In the above, $\hat{\mathbf{n}}$ and $\hat{\mathbf{v}}$ are unit pseudovector and unit vector respectively. The content of eq. (4) is that the transformation of a rotation by a Lorentz boost is a rotation with the same angle as the given rotation. The R.H.S is a pure rotation where $\mathbf{a}, \mathbf{b}$ are linear combinations of $\hat{\mathbf{n}}$ and $\hat{\mathbf{v}}$. A natural question arises that

Why is the problem formidable?

This is because of the presence of $\hat{\mathbf{v}}.\hat{\mathbf{n}}$ and $\hat{\mathbf{n}} \wedge \hat{\mathbf{v}}$ in eqs. (5) and (6) respectively what makes the problem non-linear. Note that on the R.H.S's of these equations, ϕ_1 makes its appearance through $\mathbf{a}$ and $\mathbf{b}$. To solve the problem, understanding of the geometry of the problem plays an important role. Eq. (6) tells us that $\mathbf{b}$ and $\hat{\mathbf{n}} \wedge \hat{\mathbf{v}}$ are parallel. We also know that $\mathbf{a}.\mathbf{b} = 0$. Therefore $\mathbf{b}$ and $\hat{\mathbf{n}} \wedge \hat{\mathbf{v}}$ fix a direction. $\mathbf{a}, \hat{\mathbf{n}}$ and $\hat{\mathbf{v}}$ are in a plane perpendicular to this direction. Again we note that (this is a very crucial remark) that $\mathbf{a} \wedge \mathbf{b}$ is also perpendicular to this direction. Thus $\mathbf{a}, \hat{\mathbf{n}}, \hat{\mathbf{v}}$ and $\mathbf{a} \wedge \mathbf{b}$ are all in the same plane. This is exhibited in the following figure:

Thus if we can express the problem in terms of the four vectors in the plane, then the inversion

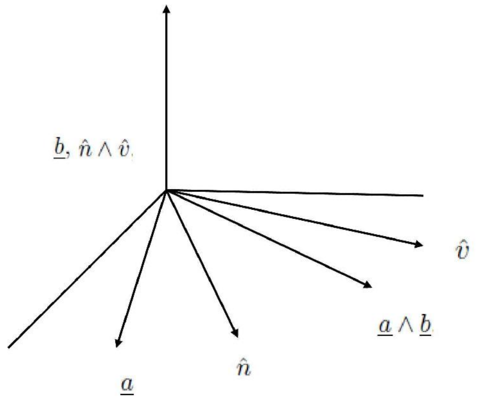

FIGURE 2. Geometrical interpretation of invariant solution

would be trivial.

The first equation (i.e. eq. (5)) is in terms of $\mathbf{a}, \hat{\mathbf{n}}, \hat{\mathbf{v}}$, which are all in the same plane, whereas the second equation (i.e. eq. (6)) involves $\mathbf{b}$ and $\hat{\mathbf{n}} \wedge \hat{\mathbf{v}}$ which are not in this plane. At this stage, we remark that the eqs. (5) and (6) are consistent with $\mathbf{a}.\mathbf{b} = 0$ and $\mathbf{a}^2 - \mathbf{b}^2 = 1$, as can easily be checked. From eq. (6) we have

$$(\hat{\mathbf{n}}.\hat{\mathbf{v}})^2 = 1 - \frac{\mathbf{b}^2}{4S_1^2 C_1^2}. \tag{11}$$

Now we calculate $\mathbf{a} \wedge \mathbf{b}$ from these two equations. Indeed

$$\frac{(\mathbf{a} \wedge \mathbf{b})}{2S_1 C_1} = (\hat{\mathbf{n}}.\hat{\mathbf{v}})\hat{\mathbf{n}} - (1 + \frac{\mathbf{b}^2}{2C_1^2})\hat{\mathbf{v}}. \tag{12}$$

Thus the problem is reformulated by using (5) and (12) in the form

$$\begin{aligned} \mathbf{a} &= (1 + 2S_1^2)\hat{\mathbf{n}} - 2S_1^2(\hat{\mathbf{n}}.\hat{\mathbf{v}})\hat{\mathbf{v}}, \\ \frac{\mathbf{a} \wedge \mathbf{b}}{2S_1 C_1} &= (\hat{\mathbf{n}}.\hat{\mathbf{v}})\hat{\mathbf{n}} - (1 + \frac{\mathbf{b}^2}{2C_1^2})\hat{\mathbf{v}}. \end{aligned}$$

Since $(\hat{\mathbf{n}}.\hat{\mathbf{v}})^2 = 1 - \frac{\mathbf{b}^2}{4S_1^2 C_1^2}$, in terms of vectors $\mathbf{a}, \mathbf{a} \wedge \mathbf{b}, \hat{\mathbf{n}}, \hat{\mathbf{v}}$. Now we have a **Linear Problem** with the coefficients depending upon $\mathbf{b}^2$ and ϕ_1. We can calculate the determinant of the R.H.S i.e.

$$\begin{vmatrix} 1 + 2S_1^2 & -2S_1^2(\hat{\mathbf{n}}.\hat{\mathbf{v}}) \\ \hat{\mathbf{n}}.\hat{\mathbf{v}} & -(1 + \frac{\mathbf{b}^2}{2C_1^2}) \end{vmatrix},$$

which simplifies to
$-1 - \mathbf{b}^2 = -\mathbf{a}^2$ by using $\mathbf{a}^2 - \mathbf{b}^2 = 1$. Thus we can indeed invert the two equations to arrive at

$$\mathbf{a}^2\hat{\mathbf{n}} = \mathbf{a}(1 + \frac{\mathbf{b}^2}{2C_1^2}) - \frac{S_1}{C_1}(\hat{\mathbf{v}}.\hat{\mathbf{n}})(\mathbf{a} \wedge \mathbf{b}), \tag{13}$$

$$\mathbf{a}^2\hat{\mathbf{v}} = (\hat{\mathbf{n}}.\hat{\mathbf{v}})\mathbf{a} - \frac{1 + 2S_1^2}{2S_1 C_1}(\mathbf{a} \wedge \mathbf{b}). \tag{14}$$

We have finally obtained $\hat{\mathbf{n}}$ and $\hat{\mathbf{v}}$ as linear combinations of $\mathbf{a}$ and $\mathbf{a} \wedge \mathbf{b}$ with the coefficients depending upon the angle ϕ_1 and $\mathbf{b}^2$.

We need to check that the solution is correct. We did not solve the two equations directly (we solved eqs. (5) and (12) not eqs. (5) and (6)). Let us check that $\mathbf{b} = 2S_1C_1(\hat{\mathbf{n}} \wedge \hat{\mathbf{v}})$ is indeed satisfied by

$$\begin{aligned} \mathbf{a}^2\hat{\mathbf{n}} &= \mathbf{a}(1+\frac{\mathbf{b}^2}{2C_1^2}) - \frac{S_1}{C_1}(\hat{\mathbf{v}}.\hat{\mathbf{n}})(\mathbf{a}\wedge\mathbf{b}) \\ \mathbf{a}^2\hat{\mathbf{v}} &= (\hat{\mathbf{n}}.\hat{\mathbf{v}})\mathbf{a} - \frac{1+2S_1^2}{2S_1C_1}(\mathbf{a}\wedge\mathbf{b}). \end{aligned}$$

In order to compute $\hat{\mathbf{n}}\wedge\hat{\mathbf{v}}$, we need

$$\begin{aligned} \mathbf{a}\wedge(\mathbf{a}\wedge\mathbf{b}) &= -\mathbf{a}^2\mathbf{b} \text{ as } \mathbf{a}.\mathbf{b}=0, \\ \text{and } (\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{a} &= -\mathbf{a}\wedge(\mathbf{a}\wedge\mathbf{b}) = \mathbf{a}^2\mathbf{b}. \end{aligned}$$

Therefore the above equations (13) and (14) then result in

$$\mathbf{a}^4\hat{\mathbf{n}}\wedge\hat{\mathbf{v}} = (\frac{1+\mathbf{b}^2}{2S_1C_1})\mathbf{a}^2\mathbf{b} = \frac{\mathbf{a}^4\mathbf{b}}{2S_1C_1}.$$

Therefore

$$\hat{\mathbf{n}}\wedge\hat{\mathbf{v}} = \frac{\mathbf{b}}{2S_1C_1},$$

as required. Thus we indeed have a solution.

Have we completed the solution of the formidable non-linear problem posed by Van Wyk? Let us read again what he mentioned in his paper:

"To solve eqs. (5) and (6) for $\hat{\mathbf{n}}, \hat{\mathbf{v}}$ and ϕ_1 when a and b are given, may be a *formidable nonlinear problem*."

We still need to calculate ϕ_1, where $\mathbf{a}$ and $\mathbf{b}$ were chosen such that $\mathbf{a}.\mathbf{b} = 0$, whereas $\hat{\mathbf{n}}.\hat{\mathbf{v}} \neq 0$. Thus $\hat{\mathbf{n}}$ and $\hat{\mathbf{v}}$ are not orthogonal. However from (5), it is obvious that

$$(\mathbf{a}-\hat{\mathbf{n}}).\hat{\mathbf{v}} = 0.$$

But

$$\begin{aligned} (\mathbf{a}-\hat{\mathbf{n}}).\hat{\mathbf{n}} &= 2S_1^2 - 2S_1^2(\hat{\mathbf{n}}.\hat{\mathbf{v}})^2 \\ &= 2S_1^2 - 2S_1^2(1-\frac{\mathbf{b}^2}{4S_1^2C_1^2}) \\ &= \frac{\mathbf{b}^2}{2C_1^2}, \end{aligned}$$

which gives

$$C_1^2 = \frac{\frac{\mathbf{b}^2}{2}}{(\mathbf{a}-\hat{\mathbf{n}}).\hat{\mathbf{n}}} = \frac{\frac{\mathbf{b}^2}{2}}{(\mathbf{a}.\hat{\mathbf{n}}-1)}. \tag{15}$$

Have we now found C_1 in terms of $\hat{\mathbf{n}}, \mathbf{a}$ and $\mathbf{b}$, where $\hat{\mathbf{n}}$ was already obtained in terms of a and $\mathbf{a}\wedge\mathbf{b}$ after having inverted the problem? Let us calculate $\mathbf{a}.\hat{\mathbf{n}}$ from eq. (13) namely

$$\mathbf{a}^2\hat{\mathbf{n}} = \mathbf{a}(1+\frac{\mathbf{b}^2}{2C_1^2}) - \frac{S_1}{C_1}(\hat{\mathbf{v}}.\hat{\mathbf{n}})(\mathbf{a}\wedge\mathbf{b}),$$

which gives

$$\mathbf{a}.\hat{\mathbf{n}} = 1 + \frac{\mathbf{b}^2}{2C_1^2}.$$

Then

$$\frac{\frac{\mathbf{b}^2}{2}}{(\mathbf{a}.\hat{\mathbf{n}}-1)} = C_1^2.$$

Thus eq. (15) becomes $C_1^2 = C_1^2$, which is an identity. Thus we are unable to compute C_1^2 and as a result ϕ_1. Hence we have not achieved what Van Wyk desired after solving the formidable non-linear problem. In fact, we were making a wrong demand. Let us go back to our original equations

$$\begin{aligned}\mathbf{a} &= (C_1^2+S_1^2)\hat{\mathbf{n}} - 2S_1^2(\hat{\mathbf{v}}.\hat{\mathbf{n}})\hat{\mathbf{v}},\\ \mathbf{b} &= 2S_1C_1(\hat{\mathbf{n}}\wedge\hat{\mathbf{v}}).\end{aligned}$$

These are two vector equations. But $\mathbf{a}$ and $\mathbf{b}$ are restricted by

$$\mathbf{a}.\mathbf{b}=0, \mathbf{a}^2-\mathbf{b}^2=1.$$

Thus we have only four independent equations. The same conclusion we draw from the inverted form.

$$\begin{aligned}\mathbf{a}^2\hat{\mathbf{n}} &= \mathbf{a}(1+\frac{\mathbf{b}^2}{2C_1^2}) - \frac{S_1}{C_1}(\hat{\mathbf{v}}.\hat{\mathbf{n}})(\mathbf{a}\wedge\mathbf{b}),\\ \mathbf{a}^2\hat{\mathbf{v}} &= (\hat{\mathbf{n}}.\hat{\mathbf{v}})\mathbf{a} - \frac{1+2S_1^2}{2S_1C_1}(\mathbf{a}\wedge\mathbf{b}).\end{aligned}$$

Again we have two vector equations where $\hat{\mathbf{n}}$ and $\hat{\mathbf{v}}$ are unit vectors which shows that we have four independent inverted equations. Both the original equations and the inverted ones involve ϕ_1 in the form of $\cosh\frac{\phi_1}{2}=C_1$ and $\sinh\frac{\phi_1}{2}=S_1$. To demand for ϕ_1 in terms of $\mathbf{a}$ and $\mathbf{b}$ is really asking for too much. The basis for all this analysis was

$$L_1^{-1}(\hat{\mathbf{v}},\phi_1)R_1(\hat{\mathbf{n}},\theta)L_1(\hat{\mathbf{v}},\phi_1)=\Lambda_1(\mathbf{a},\mathbf{b},\theta,0).$$

Equations (5) and (6) are based on this equation. The angle θ is appearing on both sides of the above equation, whereas ϕ_1 is appearing only on the L.H.S. ϕ_1 is appearing in coefficients of the four independent equations and not manifestly on R.H.S. The reason again is that on L.H.S we have the element which is conjugate to $R_1(\hat{\mathbf{n}},\theta)$. This is the background why ϕ_1 remains as a parameter in the equation we are manipulating.

5. **Conclusion.** By examining the problem in terms of $\mathbf{a}$ and $\mathbf{a}\wedge\mathbf{b}$ and not in terms of $\mathbf{a}$ and $\mathbf{b}$, we have been able to calculate $\hat{\mathbf{n}}$ and $\hat{\mathbf{v}}$ as linear combinations of $\mathbf{a}$ and $\mathbf{a}\wedge\mathbf{b}$, where the coefficients depend upon $\mathbf{a}^2, \mathbf{b}^2$ and the angle ϕ_1. The problem was indeed formidable when posed in terms of $\mathbf{a}$ and $\mathbf{b}$ but in terms of $\mathbf{a}$ and $\mathbf{a}\wedge\mathbf{b}$, it became essentially a trivial problem of linear algebra. The message is:

"Choose the correct frame in which the problem may be simple."

All this resulted from the understanding of the geometry of the problem. Let us mention that we did not arrive at the solution immediately. At first we attempted the problem in a non-invariant fashion to see if the problem has a solution. We took $\hat{\mathbf{n}}$ along the $x-$axis as $(1,0,0)$ being a unit vector. Then $\mathbf{b}=(0,0,b)$ was taken along the $z-$axis. Eq. (6) says that $\mathbf{b}=2S_1C_1(\hat{\mathbf{n}}\wedge\hat{\mathbf{v}})$. Thus $\mathbf{b}\parallel\hat{\mathbf{v}}$. Since $\mathbf{a}.\mathbf{b}=0$, we choose $\mathbf{a}=(a_1,a_2,0)$ in the $xy-$plane. Then $\hat{\mathbf{v}}=(v_1,v_2,0)$ where $v_1^2+v_2^2=1$. Then we were able to solve for v_1,v_2 in terms of b,a_1,a_2. This solution gave us a hope that the problem is solvable. Understanding the correct geometry was the next step which finally enabled us to solve the problem. We also checked that the solution presented of the reformulated problem in terms of eqs. (5) and (12)

$$\begin{aligned}\mathbf{a} &= (1+2S_1^2)\hat{\mathbf{n}} - 2S_1^2(\hat{\mathbf{n}}.\hat{\mathbf{v}})\hat{\mathbf{v}}\\ \frac{\mathbf{a}\wedge\mathbf{b}}{2S_1C_1} &= (\hat{\mathbf{n}}.\hat{\mathbf{v}})\hat{\mathbf{n}} - (1+\frac{\mathbf{b}^2}{2C_1^2})\hat{\mathbf{v}}.\end{aligned}$$

indeed solves the original problem in terms of eqs. (5) and (6) given by

$$\begin{aligned}\mathbf{a} &= (C_1^2+S_1^2)\hat{\mathbf{n}} - 2S_1^2(\hat{\mathbf{v}}.\hat{\mathbf{n}})\hat{\mathbf{v}},\\ \mathbf{b} &= 2S_1C_1(\hat{\mathbf{n}}\wedge\hat{\mathbf{v}}).\end{aligned}$$

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