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EIGENVALUES, EIGENVECTORS AND THE CONJUGACY CLASSES OF THE PSEUDO-ORTHOGONAL GROUPS O(n, m)

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Abstract. *The eigenvalues, eigenvectors and the properties of the rotation part of the orthogonal group $O(3)$ are well-known. What can we say about the eigenvalues and eigenvectors of the Pseudo-orthogonal group $O(2,1)$? The answer to this question will be presented in this paper. Generalizations of the results for these groups in higher dimensions will also be discussed. The distinction between even and odd dimensions will be emphasized.*

Keywords: Lorentz transformation, Anti-symmetric matrix, Pseudo-vector, Pseudo-scalar, Vector formalism, Spinor formalism, Invariant solution, Non-invariant solution.

1. **Introduction.** The rotation groups appear everywhere in mathematical physics as well as applied mathematics. The Pseudo-orthogonal groups are equally relevant. The whole of relativity depends upon the Lorentz group $O(3,1)$. A special new branch called $O(2,1)$ gravity, is concerned with the group $O(2,1)$. The famous Laplace equation, when elliptic has $O(n)$ in the background and when hyperbolic, the corresponding relevant group is $O(n,m)$.

Recently Faiz and Riaz [3] used the eigenvalues of the rotation matrix $O(3)$ to obtain information on the eigenvalues and eigenvectors of the elasticity tensor c_{ijkl} with symmetries $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$. When expressed as a 6×6 symmetric matrix, the eigenvalues and eigenvectors give the structure of this matrix for the various geometrical configurations. We wish to deal with such interdisciplinary areas which arise using Pseudo-orthogonal groups.

In this paper, we discuss the eigenvalues and eigenvectors of the orthogonal groups $O(n)$ and the Pseudo-orthogonal groups $O(n,m)$. The rest of the paper is organized as follows. Only a brief introduction to each section is given.

In Section 2, we present results for $O(n)$. For $O(3)$, the results are very familiar and are available in books [4], [5] and [8]. Even for $O(n)$ we can find eigenvalues in literature, see for example [6]. We discuss these in order to prepare the ground-work for the Pseudo-orthogonal groups. In this Section, an inner product

$$(x,x) = x^{\dagger}x,$$

is considered which is **positive definite**.

Specializing to the case of $O(3)$, we write the general element in terms of the infinitesimal generators and show that $\lambda = 1$ is necessarily an eigenvalue not only for $O(3)$ but also for all odd dimensional orthogonal groups $O(2n+1)$. For $O(3)$, we also calculate the corresponding eigenvector which acts as the axis of the transformation about which the rotation is taking place [3].

In Section 3, we define the Pseudo-orthogonal groups $O(n, m)$ in terms of an $(n + m \times n + m)$ diagonal matrix g with n (-1) and m (+1) entries. We also define an inner product in this space of $(n + m)$ dimension by

$$< x, x >= x^\dagger g x.$$

We show that this inner product is **not necessarily positive definite**. This is the root cause of the difference which occurs for these Pseudo-orthogonal groups.

We find the eigenvalues for these matrices, and show that "1" is always an eigenvalue when $n+m$ is an odd integer. Then we specialize to $O(2, 1)$ and discuss this group in detail, emphasizing where it is different from $O(3)$. Again the eigenvalue $\lambda = 1$ is always present. Its eigenvector gives the direction of the axis about which the rotation or boost is taking place.

In Section 4, we discuss the conjugacy classes for $O(3)$ and $O(2, 1)$ groups. Again we point out the differences which are significant.

In the last Section, the position of the axes for $O(3)$ and $O(2, 1)$ is discussed. The question, where the axis can be for the group $O(3)$ is fully answered. For $O(2, 1)$, we present an answer which seems to be partial.

The purpose of the exercise:

In some of our papers, we have calculated the number of linearly independent invariant tensors for the groups $O(3)$ and $O(4)$ [1], [2] and [7].

It has been observed that for $O(2, 1)$ and $O(3, 1)$ the number of linearly independent invariant tensors is the same as for $O(3)$ and $O(4)$ respectively. This requires the calculation of the density functions for these groups. For $O(3)$, this is well known and for $O(4)$, we obtained it using the isomorphism $O(4) \simeq O(3) \times O(3)$.

We have results on the corresponding density matrices for $O(2, 1)$ which point to this equality. These results will appear in a later publication.

For us, the presence of the eigenvalue $\lambda = 1$ for Pseudo-orthogonal groups of odd dimensions seems to be very significant.

2. **The Rotation group $O(n)$.** In this section, we will consider the rotation group $O(n)$ consisting of real $n \times n$ matrices which satisfy

$$AA^T = A^T A = I. \tag{1}$$

It is then obvious that $(detA)^2 = 1$ or $detA = \pm 1$.
We will only consider the case where

$$detA = 1, \tag{2}$$

neglecting reflections.
We define an inner product on the space of complex n-vectors given by

$$(x, y) = x^\dagger y = \sum_{i=1}^{n} x_i^* y_i. \tag{3}$$

This is obviously an inner product and $(x, y) = \sum_{i=1}^{n} x_i^* x_i \geq 0$, and $(x, x) = 0 \Leftrightarrow x = 0$. Thus eq. (3) defines a **positive definite** inner product. Any matrix A which satisfies eq. (2) is obviously non-singular and hence $\lambda = 0$ cannot be an eigenvalue. Then A is unitary w.r.t the inner product, i.e,

$$(Ax, Ay) = (x, y).$$

Indeed

$$\begin{aligned}(Ax, Ay) = x^\dagger A^\dagger A y &= x^\dagger (A^T A) y \text{ (as A is real)} \\ &= x^\dagger y = (x, y)\end{aligned}$$

Suppose $x \neq 0$ is an eigenvector corresponding to the eigenvalue $\lambda \neq 0$ i.e,

$$Ax = \lambda x,$$

Then $(Ax, Ax) = (x, x)$ implies

$$|\lambda|^2 (x, x) = (x, x).$$

But $(x, x) \neq 0$, therefore,$|\lambda| = 1$. Thus, all the eigenvalues of A have modulus 1. Since $detA = 1$, the product of the eigenvalues is 1. The eigenvalue equation
$|\det A - \lambda I| = 0$ is a real polynomial of degree n and thus can have either real or pairs of complex conjugates as roots. This shows that for $A \in O(2n)$, the eigenvalues are of the form $e^{i\theta_1}, e^{-i\theta_1}, e^{i\theta_2}, e^{i\theta_1} e^{-i\theta_2}, ..., e^{i\theta_n},\ e^{-i\theta_n}$, whereas for $O(2n+1)$, one of the eigenvalues is 1. The corresponding eigenvector could be taken as the axis and the element $A \in O(2n+1)$ is actually a rotation about that axis. In particular for $O(3)$, the eigenvalues are $1, e^{i\theta}, e^{-i\theta}$, where $0 \leq \theta \leq 2\pi$.

Again if x, y are eigenvectors corresponding to the eigenvalues λ_1, λ_2 where $\lambda_1 \neq \lambda_2$, we have

$$(Ax, Ay) = \lambda_1^{\dagger} \lambda_2 (x, y) = (x, y),$$

wherein the last step follows from unitarity of $A \in O(3)$. Thus eigenvectors corresponding to different eigenvalues are orthogonal. The infinitesimal generators corresponding to rotations along the x, y, z axes respectively are given by

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

which are obtained from

$$e^{J_1\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},\ e^{J_2\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix},\ e^{J_3\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The $J_i's$ obviously satisfy the commutation relation

$$[J_i, J_j] = \epsilon_{ijk} J_k, \tag{5}$$

for $1 \leq i, j, k \leq 3$ where $\epsilon_{ijk} = 0$, if any two indices are equal and $\epsilon_{ijk} = 1$, if i, j, k is an even permutation of (123) and $\epsilon_{ijk} = -1$, if i, j, k is an odd permutation of 123, and where the summation over repeated indices is implied. A general element of $O(3)$ is expressible as:

$$\begin{aligned}
A &= e^{J_3\theta_3}\, e^{J_2\theta_2}\, e^{J_1\theta_1} \\
&= \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}. \\
&= \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & \sin\theta_2\sin\theta_1 & \sin\theta_2\cos\theta_1 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ -\sin\theta_2 & \cos\theta_2\sin\theta_1 & \cos\theta_2\cos\theta_1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta_3\cos\theta_2 & \cos\theta_3\sin\theta_2\sin\theta_1 - \sin\theta_3\cos\theta_1 & \cos\theta_3\sin\theta_2\cos\theta_1 + \sin\theta_3\sin\theta_1 \\ \sin\theta_3\cos\theta_2 & \sin\theta_3\sin\theta_2\sin\theta_1 + \cos\theta_3\cos\theta_1 & \sin\theta_3\sin\theta_2\cos\theta_1 - \cos\theta_3\sin\theta_1 \\ -\sin\theta_2 & \cos\theta_2\sin\theta_1 & \cos\theta_2\cos\theta_1 \end{pmatrix}.
\end{aligned}$$

The eigenvalue equation $|\det A - \lambda I| = 0$ becomes

$$\begin{aligned}
&\lambda^3 - \lambda^2[\cos\theta_3(\cos\theta_2 + \cos\theta_1) + \cos\theta_2\cos\theta_1 \\
&+ \sin\theta_3\sin\theta_2\sin\theta_1] + \lambda[\cos\theta_3(\cos\theta_2 + \cos\theta_1) \\
&+ \cos\theta_2\cos\theta_1 + \sin\theta_3\sin\theta_2\sin\theta_1] - 1 = 0,
\end{aligned} \tag{6}$$

which is factorized as:

$$\begin{aligned}
&(\lambda - 1)(\lambda^2 - [\cos\theta_3(\cos\theta_2 + \cos\theta_1) + \\
&\cos\theta_2\cos\theta_1 + \sin\theta_3\sin\theta_2\sin\theta_1 - 1]\lambda + 1) = 0.
\end{aligned} \tag{7}$$

This factorization must occur as $\lambda = 1$ is one of the eigenvalues. The eigenvalues of the matrix A are then $1, e^{i\theta}, e^{-i\theta}$, where

$$\begin{aligned}1 + 2\cos\theta &= \cos\theta_1\cos\theta_2 + \cos\theta_2\cos\theta_3 + \cos\theta_3\cos\theta_1 \\ &+ \sin\theta_1\sin\theta_2\sin\theta_3. \end{aligned} \tag{8}$$

One can easily see that the expression on the right hand side of the above equation always lies between -1 and 3 and can then be expressed as $1 + 2\cos\theta$ for some angle θ, in the range 0 to 2π. The extreme values -1 and 3 are obtained by taking $\theta_1 = \theta_2 = \theta_3 = \frac{3\pi}{2}$ and 0 respectively.

The presence of eigenvalue 1 is significant. The corresponding eigenvector can be taken as the axis and the matrix A corresponds to rotation about this axis through an angle θ.

To evaluate the direction of the axis, we use the eigenvector equation

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Then x, y, z satisfy

$$\begin{aligned}&(\cos\theta_1\cos\theta_2 + 1)x + (\cos\theta_3\sin\theta_2 sin\theta_1 - \sin\theta_3\cos\theta_1)y + \\ &(\cos\theta_3\sin\theta_2 cos\theta_1 + \sin\theta_3\sin\theta_1)z = 0.\end{aligned} \tag{9}$$

$$\begin{aligned}&sin\theta_3\cos\theta_2 x + (\sin\theta_3\sin\theta_2 sin\theta_1 + \cos\theta_3\cos\theta_1 - 1)y + \\ &(\sin\theta_3\sin\theta_2 cos\theta_1 - \cos\theta_3\sin\theta_1)z = 0.\end{aligned} \tag{10}$$

$$\sin\theta_2 x + (\cos\theta_2\sin\theta_1 y + (\cos\theta_2 cos\theta_1 - 1)z = 0. \tag{11}$$

We eliminate x from the last two equations to find

$$\frac{y}{z} = \frac{\cos\theta_3\sin\theta_2\sin\theta_1 + \sin\theta_3\cos\theta_2 - \sin\theta_3\cos\theta_1}{\cos\theta_3\sin\theta_2\cos\theta_1 + \sin\theta_3\sin\theta_1 - \sin\theta_2}. \tag{12}$$

Together with eq. (11), this equation then gives

$$\frac{x}{z} = \frac{-\sin\theta_3\sin\theta_2\sin\theta_1 + \cos\theta_3\cos\theta_2 - \cos\theta_3\cos\theta_1 - \cos\theta_2\cos\theta_1 + 1}{\cos\theta_3\sin\theta_2\cos\theta_1 + \sin\theta_3\sin\theta_1 - \sin\theta_2}. \tag{13}$$

In eqs. (12,13), we have the ratios of the components of the axes.

We may mention that the eigenvalues for $O(n)$ are well known. For example see [4]. We have presented them in order to prepare ground work for the Pseudo-orthogonal groups.

3. **The Pseudo-orthogonal group $O(n,m)$.** Analogously to the treatment of the rotation group $O(n)$ in the last section, we now define the group $O(n,m)$, consisting of real $(n+m)\times(n+m)$ matrices which satisfy the equation

$$AgA^T = A^T gA = g, \tag{14}$$

where g is the diagonal matrix with m (+1) elements and n (-1) elements. This is obvious that the expression

$$|x_1|^2 + |x_2|^2 + ... + |x_m|^2 - |x_{m+1}|^2 - ...... - |x_{m+n}|^2 = x^\dagger gx,$$

remains invariant under the linear transformation by the matrix A. Indeed $x^\dagger gx \to x^{'\dagger} gx^{'}$, where $x^{'} = Ax$. This follows from

$$\begin{aligned}x^{'\dagger} gx^{'} &= x^\dagger A^\dagger gAx = x^\dagger(A^T gA)x \text{ (as } A \text{ is real)} \\ &= x^\dagger gx \text{ (using eq. (14))}.\end{aligned}$$

The corresponding equation for $O(n)$ can be obtained by taking $g = I$ in the above equation. Eq. (14) shows that $O(n,m)$ has $\frac{N(N-1)}{2}$ independent parameters where $N = n+m$. In contrast to the group $O(n)$, the group $O(n,m)$ is non-compact as some of the parameters have infinite range.

In particular for the group $O(2,1)$, the matrix $g = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$, and $|x_1|^2 - |x_2|^2 - |x_3|^2$, remains invariant under the transformation by an element $A \in O(2,1)$.

From eq. (14), we immediately deduce that $(detA)^2 = 1$ or $detA = \pm 1$. We shall concentrate on matrices with $detA = 1$. Thus $O(n,m)$ will consist of real $(n+m) \times (n+m)$ non-singular unimodular matrices. The choice excludes all reflections.

We define an inner product of two, in general complex, $(n+m)$ vectors x and y by

$$< x, y >= x^\dagger g y.$$

This obviously satisfies linearity properties in x and y required for the inner product. However, this inner product is not **positive definite**. For example taking

$$x = y = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } g = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix},$$

we have

$$< x, x >= x^\dagger g x = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0.$$

However the transformations in $O(n,m)$ satisfy

$$< Ax, Ay >=< x, y >,$$

i.e. each member of $O(2,1)$ is unitary, which follows from

$$\begin{aligned} < Ax, Ay > &= (Ax)^\dagger g A y = x^\dagger A^\dagger g A y = x^\dagger A^T g A y \text{ (as } A \text{ is real)} \\ &= x^\dagger g y \\ &= < x, y > . \end{aligned}$$

Suppose now x is an eigenvector of a matrix $A \in O(n,m)$ with eigenvalue λ where $x \neq 0$, $Ax = \lambda x$, and $\lambda \neq 0$, as A is non-singular. Then

$$\begin{aligned} < Ax, Ax > &= < x, x > \text{ implies} \\ < x, x > &= |\lambda|^2 < x, x > \end{aligned}$$

Thus either λ is of magnitude 1 or $< x, x >= 0$. **This result is different from the corresponding one for the group $\boldsymbol{O(n)}$.** For $O(n)$, the corresponding inner product $(x,x) \neq 0$ for $x \neq 0$. The eigenvalue equation $|A - \lambda I| = 0$ is a real polynomial of degree $(n+m)$. Its roots are either all real or the complex ones appearing as complex conjugate pairs.

Now starting from $Ax = \lambda x, \lambda \neq 0, x \neq 0$, since $A^T g A = g$, $(A^T g A)x = gx$ becomes

$$A^T(gx) = \frac{1}{\lambda}(gx). \tag{15}$$

In other words gx (which is necessarily non-zero, in general, **complex vector**) is an eigenvector of A^T with eigenvalue $\frac{1}{\lambda}$. Since A and A^T have the same eigenvalues, so if the matrix A has an eigenvalue λ, $\frac{1}{\lambda}$ is also an eigenvalue. The eigenvalues of $O(n,m)$ are then either ± 1, or are of the form $e^{i\theta}, e^{-i\theta}$ (complex conjugate pairs) or pairs like $e^{\theta}, e^{-\theta}$. The eigenvalue equation is a real polynomial equation. Whenever $n+m$ is an odd integer, since we are concentrating on real matrices with determinant equal to 1, $\lambda = 1$ will necessarily be an eigenvalue. The corresponding eigenvector can then be taken as the axis about which the transformation is taking place. This is useful for $n+m =$ an odd integer only.

We remark that for $O(2n+1)$, if x is an eigenvector with eigenvalue $e^{i\theta}$, x^* is an eigenvector with eigenvalue $e^{-i\theta}$.

Indeed, let

$$Ax = e^{i\theta} x.$$

Then

$$A^* x^* = e^{-i\theta} x \Rightarrow A x^* = e^{-i\theta} x. \text{ (as } A \text{ is real)}$$

Now we concentrate on the group $O(2,1)$ with $g = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$. Any $A \in O(2,1)$ leaves $|x_1|^2 - |x_2|^2 - |x_3|^2$ invariant.
Some particular matrices in $O(2,1)$ are:

$$e^{J_1\theta_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}, \; e^{J_2\theta_2} = \begin{pmatrix} \cosh\theta_2 & 0 & \sinh\theta_2 \\ 0 & 1 & 0 \\ \sinh\theta_2 & 0 & \cosh\theta_2 \end{pmatrix},$$
$$e^{J_3\theta_3} = \begin{pmatrix} \cosh\theta_3 & \sinh\theta_3 & 0 \\ \sinh\theta_3 & \cosh\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

where

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; J_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{17}$$

and $0 \le \theta_1 \le 2\pi, -\infty \le \theta_2, \theta_3 \le \infty$, which satisfy the commutation relations

$$[J_1, J_2] = -J_3, \; [J_2, J_3] = J_1, \; [J_3, J_1] = -J_2. \tag{18}$$

Contrary to the case of $O(3)$, not only the matrices in eq. (16) are in $O(2,1)$, but the matrices

$$\begin{pmatrix} -\cosh\theta_3 & \sinh\theta_3 & 0 \\ \sinh\theta_3 & -\cosh\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} -\cosh\theta_2 & 0 & \sinh\theta_2 \\ 0 & 1 & 0 \\ \sinh\theta_2 & 0 & \cosh\theta_2 \end{pmatrix},$$

are also in $O(2,1)$.
Note that the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & -\cos\theta_1 \end{pmatrix},$$

is already included in the set, as it is obtained by replacing θ by $(\pi - \theta)$.
The eigenvalues and eigenvectors of the three matrices in eq. (16) above are:

Matrix	*Eigenvalues*	*Eigenvectors*
$e^{J_1\theta_1}$	$1, e^{i\theta}, e^{-i\theta}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$
$e^{J_2\theta_2}$	$1, e^{\theta}, e^{-\theta}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$
$e^{J_3\theta_3}$	$1, e^{\theta}, e^{-\theta}$	$\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$

These are normalized using the inner product $(x,x) = x^\dagger x$. The inner product $< x, x >= x^\dagger g x$ for the eigenvectors for $e^{J_2\theta_2}, e^{J_3\theta_3}$ is $-1, 0, 0$ respectively. The eigenvectors corresponding to different eigenvalues all satisfy $(x,y) = 0$. Though the eigenvectors corresponding to the eigenvalue 1 and any different eigenvalue satisfy $< x, y >= 0$ and $(x,y) = 0$, but $< x, y >\neq 0$ when we consider the matrices for $e^{J_2\theta_2}$ and $e^{J_3\theta_3}$ and eigenvalues $e^{\theta}, e^{-\theta}$. Because these are symmetric matrices, the eigenvectors for different eigenvalues for the two matrices $e^{J_2\theta_2}, e^{J_3\theta_3}$ do satisfy $(x,y) = 0$.

Next we find explicit expressions for the angle θ for the eigenvalues of the general $O(2,1)$ matrix

$$A = e^{J_3\theta_3} e^{J_2\theta_2} e^{J_1\theta_1}. \tag{19}$$

Using eq. (16), we have

$$A = \begin{pmatrix} \cosh\theta_3\cosh\theta_2 & \cosh\theta_3\sinh\theta_2\sin\theta_1 + \sinh\theta_3\cos\theta_1 & \cosh\theta_3\sinh\theta_2\cos\theta_1 - \sinh\theta_3\sin\theta_1 \\ \sinh\theta_3\cosh\theta_2 & \sinh\theta_3\sinh\theta_2\sin\theta_1 + \cosh\theta_3\cos\theta_1 & \sinh\theta_3\sinh\theta_2\cos\theta_1 - \cosh\theta_3\sin\theta_1 \\ \sinh\theta_2 & \cosh\theta_2\sin\theta_1 & \cosh\theta_2\cos\theta_1 \end{pmatrix}. \tag{20}$$

The eigenvalue equation $|A - \lambda I| = 0$ looks quite cumbersome. However, it does simplify to

$$\lambda^3 - \lambda^2[\sinh\theta_3\sinh\theta_2\sin\theta_1 + \cosh\theta_3\cosh\theta_2 + \cosh\theta_3\cos\theta_1 + \cosh\theta_2\cos\theta_1] + \lambda[\sinh\theta_3\sinh\theta_2\sin\theta_1 + \cosh\theta_3\cosh\theta_2 + \cosh\theta_3\cos\theta_1 + \cosh\theta_2\cos\theta_1] - 1 = 0. \tag{21}$$

This is similar to the corresponding result in section 2. **This is indeed a reciprocal equation as can be seen by comparing the coefficients of λ^3, λ^0 and λ^2, λ respectively, which are equal but of opposite sign**. Thus $\lambda = 1$ is indeed an eigenvalue. The eq. (21) factorizes in the form:

$$(\lambda - 1)[\lambda^2 - (\sinh\theta_3\sinh\theta_2\sin\theta_1 + \cosh\theta_3\cosh\theta_2 + \cosh\theta_3\cos\theta_1 + \cosh\theta_2\cos\theta_1 - 1)\lambda + 1] = 0. \tag{22}$$

The coefficient of $(-\lambda)$ in the factor not containing $(\lambda - 1)$, is indeed $\sinh\theta_3\sinh\theta_2\sin\theta_1 + \cosh\theta_3\cosh\theta_2 + \cosh\theta_3\cos\theta_1 + \cosh\theta_2\cos\theta_1 - 1$ which takes values in the region -1 to ∞. When it is in the region -1 to 1, it can be taken as $2\cos\theta$, and then the roots are complex conjugates. For real roots, it must be ≥ 2 when it can be expressed as $2\cosh\theta$ and then the trace of the matrix A will be $1 + 2\cosh\theta$. The condition for this is

$$\sinh\theta_3\sinh\theta_2\sin\theta_1 + \cosh\theta_3\cosh\theta_2 + (\cosh\theta_3 + \cosh\theta_2)\cos\theta_1 - 1 \geq 2,$$

where we can write

$$2\cosh\theta + 1 = \sinh\theta_3\sinh\theta_2\sin\theta_1 + \cosh\theta_3\cosh\theta_2 + (\cosh\theta_3 + \cosh\theta_2)\cos\theta_1. \tag{23}$$

However, when it is in the region (-2, 2), we may write

$$2\cos\theta + 1 = \sinh\theta_3\sinh\theta_2\sin\theta_1 + \cosh\theta_3\cosh\theta_2 + (\cosh\theta_3 + \cosh\theta_2)\cos\theta_1. \tag{24}$$

Eq. (23) represents the case where all the three roots of A are real. In fact these can be taken as $1, e^{\theta}, e^{-\theta}$. Analogously, eq. (24) represents the case where the roots are of the form $1, e^{i\theta}, e^{-i\theta}$, i.e, the two complex roots appear as complex conjugates. Exactly the way we obtained the $(\frac{x}{z}, \frac{y}{z})$ for the eigenvector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ of the eigenvalue "1" for $O(3)$, we arrive at

$$\frac{x}{z} = \frac{\cosh\theta_3\sinh\theta_2\sin\theta_1 - \sinh\theta_3\cosh\theta_2 - \sinh\theta_3\cos\theta_1}{\cosh\theta_3\sinh\theta_2\cos\theta_1 - \sinh\theta_3\sin\theta_1 - \sinh\theta_2}, \tag{25}$$

and

$$\frac{y}{z} = \frac{\sinh\theta_3\sinh\theta_2\sin\theta_1 - \cosh\theta_3\cosh\theta_2 + \cosh\theta_3\cos\theta_1 + \cosh\theta_2\cos\theta_1 - 1}{\cosh\theta_3\sinh\theta_2\cos\theta_1 - \sinh\theta_3\sin\theta_1 - \sinh\theta_2}. \tag{26}$$

At this stage, we wish to answer the question about the range $(-\infty$,-2) in the above calculations. The matrix

$$\begin{pmatrix} -\cosh\theta & \sinh\theta & 0 \\ \sinh\theta & -\cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is also in $O(2,1)$, has trace $1 - 2\cosh\theta$. In this case, the roots are $1, -e^{\theta}, -e^{-\theta}$. Thus $O(2,1)$ has matrices which can have any value of the trace. The traces of the form $1 - 2\cosh\theta$, $1 + 2\cosh\theta$ and $1 + 2\cos\theta$ are in the range (-∞, -1), (3, ∞) and (-1, 3) respectively. **Thus for**

***O*(2,1), the trace covers the whole real line**. In comparison, for $O(3)$, the trace can only be the form $1+2\cos\theta$ and it covers the range (-1, 3) only.

4. **Conjugacy Classes of Elements of *O*(3) and *O*(2,1).** The elements of $O(3)$ can be classified by the trace, i.e, $1+2\cos\theta$. For each number in the range (-1, 3), we can construct the matrix $A(\theta)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$ in $O(3)$ with trace $1+2\cos\theta$. Since all elements equivalent to this matrix must have the same trace, the conjugacy classes (hereafter called classes), are characterised by a number in the range
(-1, 3). All members of the class with the trace $1+2\cos\theta$ are transformable **by matrices in *O*(3),** to the form $A(\theta)$ above, i.e, given any other element B in $O(3)$ with the same value of the trace, we can find an element C in $O(3)$ such that $CA(\theta)C^{-1}=B$. This choice, however, is not unique. Whichever member we choose as C in $O(3)$, $TrA(\theta)=Tr(B)=1+2\cos\theta$.

The situation with $O(2,1)$ is not so simple. Again the classes are characterised by a single number on the real line. In this case when this number is in the ranges (-∞, -1), (-1, 3) and (3, ∞) respectively, the corresponding matrices with traces in these ranges are transformed by members (in general different) of $O(2,1)$ to the forms
$\begin{pmatrix} -\cosh\theta & \sinh\theta & 0 \\ \sinh\theta & -\cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$, respectively.
In particular, we calculate $C\in O(2,1)$ which transforms

$$\begin{pmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \; to \; \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix},$$

i.e, we wish to find C such that

$$\begin{aligned} C\begin{pmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}C^{-1} &= \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}, \\ \text{or } C\begin{pmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}C, \end{aligned}$$

i.e,

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}\begin{pmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix},$$

or we require

$$\begin{pmatrix} c_{11}\cosh\theta + c_{12}\sinh\theta & c_{11}\sinh\theta + c_{12}\cosh\theta & c_{13} \\ c_{21}\cosh\theta + c_{22}\sinh\theta & c_{21}\sinh\theta + c_{22}\cosh\theta & c_{23} \\ c_{31}\cosh\theta + c_{32}\sinh\theta & c_{31}\sinh\theta + c_{32}\cosh\theta & c_{33} \end{pmatrix}$$
$$= \begin{pmatrix} c_{11}\cosh\theta + c_{31}\sinh\theta & c_{12}\cosh\theta + c_{32}\sinh\theta & c_{13}\cosh\theta + c_{33}\sinh\theta \\ c_{21} & c_{22} & c_{23} \\ c_{11}\sinh\theta + c_{31}\cosh\theta & c_{12}\sinh\theta + c_{32}\cosh\theta & c_{13}\sinh\theta + c_{33}\cosh\theta \end{pmatrix}.$$

Comparing the elements in the last column,

$$\begin{aligned} c_{13}(\cosh\theta-1)+c_{33}\sinh\theta &= 0 \\ c_{13}\sinh\theta + c_{33}(\cosh\theta-1) &= 0. \end{aligned}$$

Since

$$\begin{vmatrix} \cosh\theta - 1 & \sinh\theta \\ \sinh\theta & \cosh\theta - 1 \end{vmatrix} = (\cosh\theta - 1)^2 - \sinh^2\theta = 2 - 2\cosh\theta \neq 0,$$

therefore,

$$c_{13} = c_{33} = 0.$$

Similarly

$$c_{21} = c_{22} = 0.$$

From the equality of the other members, $c_{12} = c_{31}, c_{11} = c_{32}$. Thus the matrix C must be of the form

$$\begin{pmatrix} c_{11} & c_{12} & 0 \\ 0 & 0 & c_{23} \\ c_{12} & c_{11} & 0 \end{pmatrix}.$$

Since it must lie in $O(2,1)$, we must choose it as

$$\begin{pmatrix} \cosh\theta_1 & \sinh\theta_1 & 0 \\ 0 & 0 & -1 \\ \sinh\theta_1 & \cosh\theta_1 & 0 \end{pmatrix},$$

which has determinant

$$\cosh^2\theta_1 - \sinh^2\theta_1 = 1.$$

We may verify that this is a correct solution by checking

$$\begin{pmatrix} \cosh\theta_1 & \sinh\theta_1 & 0 \\ 0 & 0 & -1 \\ \sinh\theta_1 & \cosh\theta_1 & 0 \end{pmatrix} \begin{pmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\theta_1\cosh\theta + \sinh\theta_1\sinh\theta & \cosh\theta_1\sinh\theta + \sinh\theta_1\cosh\theta & 0 \\ 0 & 0 & -1 \\ \sinh\theta_1\cosh\theta + \cosh\theta_1\sinh\theta & \sinh\theta_1\sinh\theta + \cosh\theta_1\cosh\theta & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix} \begin{pmatrix} \cosh\theta_1 & \sinh\theta_1 & 0 \\ 0 & 0 & -1 \\ \sinh\theta_1 & \cosh\theta_1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\theta\cosh\theta_1 + \sinh\theta\sinh\theta_1 & \cosh\theta\sinh\theta_1 + \sinh\theta\cosh\theta_1 & 0 \\ 0 & 0 & -1 \\ \sinh\theta\cosh\theta_1 + \cosh\theta\sinh\theta_1 & \sinh\theta\sinh\theta_1 + \cosh\theta\cosh\theta_1 & 0 \end{pmatrix}.$$

A **particular** C is given by

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = e^{J_1(\frac{\pi}{2})}$$

Thus, there are three types of classes in $O(2,1)$. Their prototypes are:

$$\begin{pmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -\cosh\theta & \sinh\theta & 0 \\ \sinh\theta & -\cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Every matrix in $O(2,1)$ is similar to exactly one of these. The angle in the above is exactly the one which will make its trace exactly equal to the trace of the matrix for which we are determining the class.

5. **The position of the axes for $O(3)$ and $O(2,1)$.** For both these groups $\lambda = 1$ is an eigenvalue and the corresponding eigenvector is called the axis of the element of the group.

The matrices of these groups are characterised by 3 independent elements. The position of the axis in space is given in terms of the two angles and the third is provided by the angle of rotation for $O(3)$ and angle of rotation or the one appearing in the boost for $O(2,1)$.

For $O(3)$, the rotation is characterised by an angle θ where $0 \leq \theta \leq 2\pi$. The axes in this case cover the whole of the space of the sphere of radius 2π, with axes emanating from the centre of the sphere.
The size of the axis is given by θ and (α, β) provides its direction as shown in fig.(1) below.

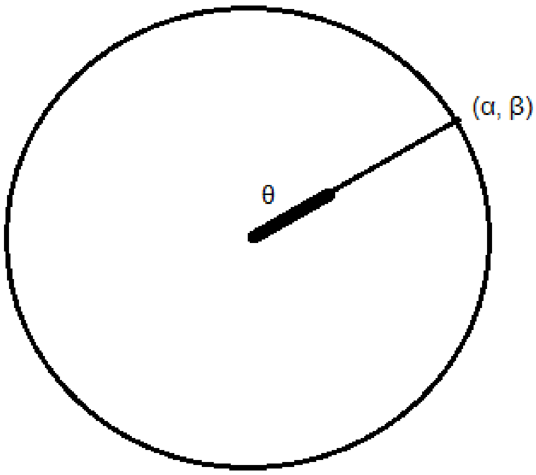

FIGURE 1. Sphere of radius 2π (the axis is shown as bold)

Any axis can be transformed to any other with the same rotation by a rotation in $O(3)$. This leads to the classification mentioned earlier. Each class is represented by a θ where $0 \leq \theta \leq 2\pi$.
For $O(2,1)$ if we take

$$
\begin{aligned}
e^{J_1\theta_1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}, e^{J_2\theta_2} = \begin{pmatrix} \cosh\theta_2 & 0 & \sinh\theta_2 \\ 0 & 1 & 0 \\ \sinh\theta_2 & 0 & \cosh\theta_2 \end{pmatrix} \\
e^{J_3\theta_3} &= \begin{pmatrix} \cosh\theta_3 & \sinh\theta_3 & 0 \\ \sinh\theta_3 & \cosh\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}
$$

the axes corresponding to these are

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
$$

respectively. If for a matrix $A \in O(2,1), x$ is the eigenvector for the eigenvalue λ, i.e, $Ax = \lambda x$, then for another matrix $B \in O(2,1)$

$$
(BAB^{-1})(Bx) = \lambda(Bx),
$$

i.e, Bx is the eigenvector for the same eigenvalue λ for the matrix BAB^{-1}, obtained from A by a similarity transformation. If $B \in O(2,1)$ we arrive at the class containing A.

In particular for

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix},$$

the axis is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.
Then with $e^{J_1\theta_1} = B, BAB^{-1} = A$ (as B commutes with A).
However with $B = e^{J_3\theta_3}e^{J_2\theta_2}$, we can calculate the axis as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cosh\theta_3 \cosh\theta_2 \\ \sinh\theta_3 \cosh\theta_2 \\ \sinh\theta_2 \end{pmatrix},$$

which is obtained by calculating $e^{J_3\theta_3}e^{J_2\theta_2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. (This is normalised by $x^2 - y^2 - z^2 = 1$ as expected.)
In the above, we note that

$$\left|\frac{y}{x}\right| = |\tanh\theta_3| \leq 1,$$

whereas

$$\left|\frac{z}{x}\right| = \left|\frac{\tanh\theta_2}{\cosh\theta_3}\right|,$$

also lies in this range. Thus in $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, x has magnitude larger than the ones of y and z. Thus these axes occupy the space for $x > 0$, as shown in fig.(2).

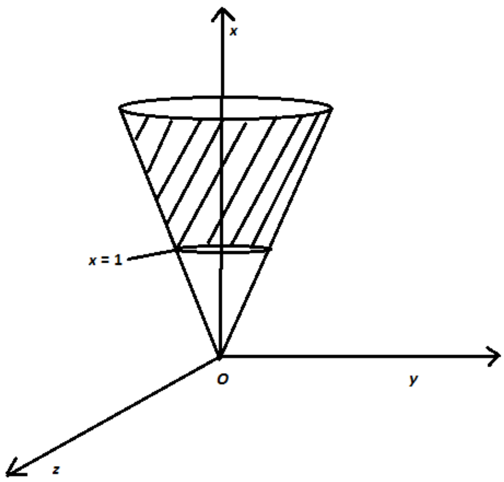

FIGURE 2. Part of the cone shown (the axes form part of the fulcurum)

Note that

$$\cosh\theta_3 \cosh\theta_2 \geq 1.$$

y and z can be both +ve and -ve.

Next we discuss $e^{J_2\theta_2}$ which has axis $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Then $e^{J_3\theta_3}e^{J_1\theta_1}$ will have axis

$$\begin{pmatrix} \sinh\theta_3\cos\theta_1 \\ \cosh\theta_3\cos\theta_1 \\ \sin\theta_1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and now $x^2 - y^2 - z^2 = -1$. (The same as for $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$)

and

$$\left|\frac{x}{y}\right| = |\tanh\theta_3| \leq 1,$$

and

$$\left|\frac{z}{y}\right| = \left|\frac{\tan\theta_1}{\cosh\theta_3}\right|,$$

can have any value. Note that x, y, z can take +ve and -ve values. Similar results are obtained when we discuss $e^{J_2\theta_2}e^{J_1\theta_1}$.

6. Conclusion. The importance of the study of Orthogonal groups $O(n)$ and the Pseudo-orthogonal groups $O(n, m)$ from the point of view of Applied mathematics and Physics cannot be over emphasised. In this paper, we have followed the study of eigenvalues and eigenvectors of the orthogonal groups to arrive at the corresponding results for the Pseudo-orthogonal groups. This requires the introduction of a non-positive definite inner product which results in different properties. We have particularly considered in detail the group $O(2, 1)$ for which we have calculated the eigenvalues and the eigenvector corresponding to the eigenvalue 1 which forms the axis of the transformation. We have also obtained the various conjugacy classes for this group.

This study is useful for the problem of calculating the density function. An application of this density function is in obtaining the number of linearly independent invariants of tensors of arbitrary rank.

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