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Delta Perturbation Method for Couette-Poiseuille flows in Third grade fluids

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Abstract This work theoretically investigate the steady plane Couette-Poiseuille flow between two parallel plates for third-grade fluid by using delta perturbation method, which is the kind of perturbation approach and was deliveredwith the aid of Bender and his colleagues in the 1980s. Utilizing DPM (delta perturbation method), analytical solutions have been found from the governing continuity and momentum equations subject to the necessary boundary conditions. In this proposed model, the Newtonian solution is obtained through the substitution $\beta_2 + \beta_3 = 0$. It is possible to measure the velocity field, temperature distribution, volumetric flow rate, and average velocity of the fluid flow. We derived that the third-grade fluid's velocity will change in response to an increasing material constant from the visual and table representations of the impacts of different parameters on the velocity and temperature profiles.The suggested model additionally mentions temperature distribution losses with increases in thermal conductivity k and rises as a result of increases of dynamic viscosity η , constant parameters β_2 and β_3 and material constant α. Here we have also find out that temperature distribution and velocity profile enhance with higher magnitude of pressure gradient.

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1 Introduction

Shear stress and rate of deformation are typically related nonlinearly in biological sciences, industries, and other sciences that are employed in our daily lives, that kind of fluid is referred to non-Newtonian fluid. These fluids are generally more difficult to gather, both numerically and analytically [17], [13]. Although a second-grade fluid model depicts common stress effects for steady flow. It lacks the shear thickening or thinning characteristic that many fluids do [21].
In any case, third-grade fluids can be used to explain this kind of phenomenon [7]. Third-grade fluid model behavior that deviates from Newtonian behavior represents a more advanced, if unsuccessful, attempt at a more complete representation. Due of its importance in day-to-day living, the third-order fluid model will be taken into consideration in this work. Fosdick and Rajagopal created the theory relating to the stability and thermodynamics of third-grade fluids [7]. Researchers were able to successfully explore and solve highly nonlinear differential equations determined by the flow of third-order fluid after a tricky effort [21]. The perturbation approach is frequently used to solve nonlinear differential equations with small/large parameters in order to calculate analytical solutions; however, because small/large parameter is involved, it is not guaranteed to work for all nonlinear differential equations. As a result, there are strong preconditions for the development of novel analytical techniques; [8] outlined the approaches' exploration.
The "delta perturbation approach" was a novel technique introduced by Bender and associates in the latter half of the 1980s. The group of perturbation methods actually includes this technique [2],[3], [4],[6],[5], [14]. With this approach, a nonlinearity already present in a nonlinear differential equation is developed [20],[19],[10]. This theory found considerable use across a wide range of scientific fields, particularly for nonlinear differential equations [2],[3], [4],[6],[5],[20],[19],[10]. It was initially applied to difficulties relating to the theory of the quantum field. The third-grade plane Couette-Poiseuille flow problem was examined in this work utilizing the delta perturbation approach. Numerous scholars have investigated this type of issue using a variety of techniques [15],[9],[16],[18],[12],[11]], particularly the homotopy perturbation method HPM, the Adomian decomposition method ADM, the optimal homotopy asymptotic approach OHAM, and various numerical techniques. Our primary goal is to use the DPM (delta perturbation method) to solve the problem. We obtain the solutions to the ensuing differential equations that are subject to boundary conditions using scientific arrangements, and we also find the Newtonian liquid's solution [1]. Additionally, expressions for flow rate, average velocity, temperature profile, and velocity profile are computed. The solution to the problem hasn't been mentioned in the literature, to the best of our knowledge. The following skills are used to help with this paper:
The Navier-Stokes equation and the energy equation's fundamental equations are provided in Section Number 2. Problem formulation is provided in Section 3. The problem's solution is provided in Section 4. Results and discussions are provided in Section 5, and final thoughts are provided in Section 6.

2 Basic Equations

The fundamental equations regulating an incompressible fluid's behavior, which ignore thermal effects and physical forces, are

$$\nabla \cdot \mathbf{V} = 0 \tag{1}$$

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{b} - \nabla \cdot \mathbf{T} - \nabla p \tag{2}$$

$$\rho C_p \frac{D\Theta}{Dt} = \frac{1}{2} tr(\mathbf{T}\mathbf{A}_1) + k\nabla^2\Theta \tag{3}$$

Where ρ denotes the density of fluid and **V** denotes velocity field, p denotes pressure, I denotesthe stress tensor, **b** denotes the body force, θ denotes the temperature distribution of the fluid, C_p denotes fluid specific heat , k denotes thermal conductivity and $\frac{D}{Dt}$ denotes the material derivative. Third-grade fluid models' extra stress tensors are defined by,

$$\mathbf{T} = \sum_{i=0}^{3} \mathbf{S}_i \tag{4}$$

Where

$$\mathbf{S}_1 = \eta \mathbf{A}_1$$
$$\mathbf{S}_2 = \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2$$
$$\mathbf{S}_3 = \beta_1 \mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_2)\mathbf{A}_1$$

Where η is the coefficient of viscosity and α_1 , α_2, β_1, β_2,and β_3 are material constants. The Rivilin-Ericksen tensor,$\mathbf{A}_n$ are defined by $\mathbf{A}_0 = \mathbf{I}$ (the identity tensor) and

$$\mathbf{A}_n = \frac{D\mathbf{A}_{n-1}}{Dt} + (\nabla\mathbf{V})^T\mathbf{A}_{n-1} + \mathbf{A}_{n-1}(\nabla\mathbf{V}). \qquad 1 \leq n \tag{5}$$

In cartesion coordinates gradient of velocity vector and its transpose can be described as:

$$\nabla\mathbf{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

and

$$(\nabla\mathbf{V})^T = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}$$

3 Problem Formulation

Consider the steady plane Couette-Poiseuille flow of third grade fluid between two unending parallel plates distance α apart. The lower plate is stationary and the upper plate is shifting with consistent speed v. The temperature of the higher plate is maintained at Θ_1 and that of lower plate is at Θ_0. The lower and higher plates are located in the plane $y = 0$ and $y = a$ respectively, of an orthogonal coordinate system with $x-$axis in the direction of flow. Furthermore, the geometry of the problem is given in Figure 1, and we take the velocity field, stress tensor, and energy distribution of the form:

$$\mathbf{V} = [u, v, w] = (u(y), 0, 0), \qquad \mathbf{T} = \mathbf{T}(y) \qquad \Theta = \Theta(y) \tag{6}$$

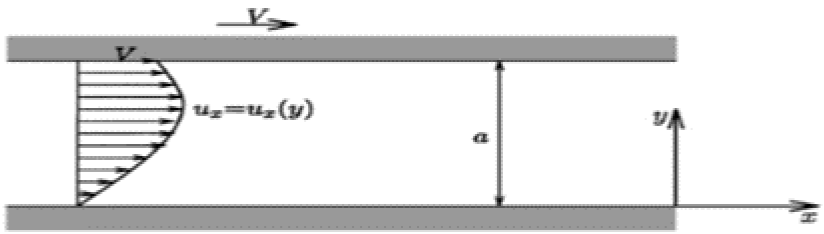

Figure 1. Plane Couette-Poiseuille flow geometry for third-grade fluid [15]

(1) is satisfied identically by using (6) the continuity equation, and from (4) and (5), we obtain the momentum components (2) in the form

x-component

$$-\frac{dp}{dx} + \eta\frac{d^2u}{dy^2} + 6(\beta_2 + \beta_3)\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} = 0 \tag{7}$$

y-component

$$-\frac{dp}{dy} + 2(\alpha_1 + \alpha_2)\frac{d}{dy}\left(\frac{du}{dy}\right)^2 = 0 \tag{8}$$

Introducing the generalized pressure p^* by the relation

$$p^* = -p + 2(\alpha_1 + \alpha_2)\left(\frac{du}{dy}\right)^2 \tag{9}$$

and substituting p^* in (8), we find that

$$\frac{dp^*}{dy} = 0 \tag{10}$$

Showing that $p^* = p^*(x)$. Consequently, (7) reduces to the single equation

$$-\frac{dp^*}{dx} + +\eta\frac{d^2u}{dy^2} + 6(\beta_2 + \beta_3)\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} = 0 \tag{11}$$

This equation is an ordinary differential equation of second order. Making use of (6) in the energy equation (3), we get

$$k\frac{d^2\Theta}{dy^2} + \left(\eta\frac{du}{dy} + 2(\beta_2 + \beta_3)\left(\frac{du}{dy}\right)^3\right)\frac{du}{dy} = 0 \tag{12}$$

The related boundary conditions are:

$$\text{at } y = 0,\ u = 0,\ \Theta = \Theta_0 \text{ Lower plate} \tag{13}$$

$$\text{at } y = a,\ u = v,\ \Theta = \Theta_1 \text{ Upper plate} \tag{14}$$

Where v represent the speed Integrating equation (11), we get

$$\frac{dp^*}{dx}y + A = \eta\frac{du}{dy} + 2(\beta_2 + \beta_3)\left(\frac{du}{dy}\right)^3 \tag{15}$$

Where A is constant of integration, now by substituting equation (15) in (12), the energy equation will simplified to,

$$k\frac{d\Theta^2}{dy^2} + \left(A + y\frac{dp^*}{dx}\right)\frac{du}{dy} = 0 \tag{16}$$

From here, we pointed out that equation (15) has no any contribution of $\mathbf{S}_2$.

4 Problems Solving Using the Delta Perturbation Method

we will attack (15) using the delta expansion. We replace $\left(\frac{du}{dy}\right)^3$ by $\left(\frac{du}{dy}\right)^{1+\delta}$ and consider the differential equation

$$\frac{dp^*}{dx}y + A = \eta\frac{du}{dy} + 2(\beta_2 + \beta_3)\frac{d}{dy}\left(\frac{du}{dy}\right)^{1+\delta} \tag{17}$$

We assume that $u(y)$ has a series expansion in power of δ:

$$u(y) = u_0(y) + \delta u_1(y) + \delta^2 u_2(y) + \ldots \tag{18}$$

We obtain the following problems of different order by substituting (18) in (14) and (17):

Zero order problems

$$\delta^0 : \frac{du_0}{dy} + 2\left(\frac{\beta_2 + \beta_3}{\eta}\right)\left(\frac{du_0}{dy}\right) = \frac{1}{\eta}\left(\frac{dp^*}{dx}y + A\right), \tag{19}$$

$$\text{with } u_0 = 0, \quad \text{at } y = 0$$

$$u_0 = v, \quad \text{at } y = a$$

First order problems

$$\delta^1 : \frac{du_1}{dy} + 2\left(\frac{\beta_2 + \beta_3}{\eta}\right)\left(\frac{du_1}{dy} + \frac{du_0}{dy}\ln\left(\frac{du_0}{dy}\right)\right) = A_1, \tag{20}$$

with

$$\text{with } u_1 = 0, \quad \text{at } y = 0$$

$$u_1 = 0, \quad \text{at } y = a$$

Second order problems

$$\delta^2 : \frac{du_2}{dy} + 2\left(\frac{\beta_2 + \beta_3}{\eta}\right)\left(\frac{du_2}{dy} + \frac{du_1}{dy}\left(1 + \ln\left(\frac{du_0}{dy}\right)\right) + \frac{1}{2}\left(\frac{du_0}{dy}\right)\ln^2\left(\frac{du_0}{dy}\right)\right) = A_2, \tag{21}$$

with

$$\text{with } u_2 = 0, \quad \text{at } y = 0$$

$$u_2 = 0, \quad \text{at } y = a$$

The solution of the problem up to first order with associated condition is as follows:

$$u_0 = \left(\frac{y(y-a)}{2(\beta_2+\beta_3)}\frac{dp^*}{dx}\right) + \frac{v}{a}y \tag{22}$$

$$\begin{aligned}u_1 =& \frac{(\beta_2+\beta_3)}{4a^3(\eta+2(\beta_2+\beta_3))}\frac{dp^*}{dx}\left\{(2a^3\left(\frac{dp^*}{dx}\right)^2 y(y-a) + (a-y)\ln\left(\frac{v}{a} - \frac{a\frac{dp^*}{dx}}{2\eta+4(\beta_2+\beta_3)}\right)\left(a^2\frac{dp^*}{dx} - v(2\eta+4(\beta_2+\beta_3))\right)^2\right.\\ &- a\ln\left(\frac{v}{a} - \frac{(a-2y)\frac{dp^*}{dx}}{2(\eta+2(\beta_2+\beta_3))}\right)\left(a(a-2y)\frac{dp^*}{dx} - v(2\eta+4(\beta_2+\beta_3))\right)^2\\ &\left.+ y\ln\left(\frac{v}{a} + \frac{a\frac{dp^*}{dx}}{2(2\eta+4(\beta_2+\beta_3))}\left(a^2\frac{dp^*}{dx} - v(2\eta+4(\beta_2+\beta_3))\right)^2\right\}\end{aligned} \tag{23}$$

By substituting the solutions of u_0 and u_1 and disregarding the second and higher order solutions, we can derive the series solution up to the first order.

$$\begin{aligned}u(y) =& \left(\frac{y(y-a)}{2(\eta+2(\beta_2+\beta_3))}\frac{dp^*}{dx}\right) + \frac{v}{a}y + \delta\left(\frac{(\beta_2+\beta_3)}{4a^3(\eta+2(\beta_2+\beta_3))}\frac{dp^*}{dx}\right)\\ &\left\{2a^3\left(\frac{dp^*}{dx}\right)^2 y(y-a) + (a-y)\ln\left(\frac{v}{a} - \frac{a\frac{dp^*}{dx}}{2\eta+4(\beta_2+\beta_3)}\right)\left((a^2\frac{dp^*}{dx} - v(2\eta+4(\beta_2+\beta_3))\right)^2\right.\\ &- a\ln\left(\frac{v}{a} - \frac{(a-2y)\frac{dp^*}{dx}}{2(\eta+2(\beta_2+\beta_3))}\right)\left(a(a-2y)\frac{dp^*}{dx} - v(2\eta+4(\beta_2+\beta_3))\right)^2\\ &+ y\ln\left(\frac{v}{a} - \frac{a\frac{dp^*}{dx}}{(2\eta+4(\beta_2+\beta_3))}\right)\left(a^2\frac{dp^*}{dx} - v(2\eta+4(\beta_2+\beta_3))\right)^2\end{aligned} \tag{24}$$

Temperature profile can be obtained for the use of equation (24) in (16), we obtained

$$\begin{aligned}\Theta(y) =& \\ &\frac{1}{240k}\left(+\frac{120((a-y)(2v^4y(\beta_2+\beta_3) + a^2(v^2y\eta + 2ak\Theta_0)) + 2a^3ky\Theta_1)}{a^4}\right.\\ &- \frac{40v(a-2y)(a-y)y(a^2\eta + 4v^2(\beta_2+\beta_3))\frac{dp^*}{dx}}{a^2(\eta+2(\beta_2+\beta_3))} + \frac{10(a-y)y(a^2-2ay+2y^3)(a^3\eta+12v^2(\beta_2+\beta_3))\left(\frac{dp^*}{dx}\right)^2}{a(\eta+2(\beta_2+\beta_3))}\\ &+ \frac{(a-y)y(3a^2-6ay+4y^2)(a^2-2ay+4y^2)(\beta_2+\beta_3)\left(\frac{dp^*}{dx}\right)^4}{(\eta+2(\beta_2+\beta_3))^4}\end{aligned} \tag{25}$$

Flow Rate and Average Film Velocity

Volumetric flow rate Q can be obtained by the use of formula, which is:

$$Q = 2\int_0^a u(y)dy \tag{26}$$

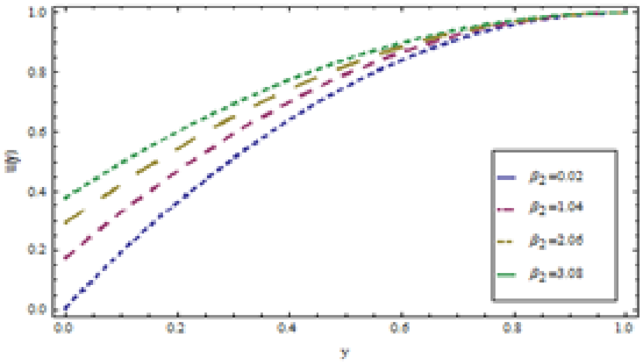

Figure 4. Effects of β_2 on velocity profile when $a = 1cm, \rho = 0.78\frac{g}{cm^3}, \delta = 2, \frac{dp^*}{dx} = -2Barye, \eta_3 = 0.01$ and $v = 1cm\ s^-1$.

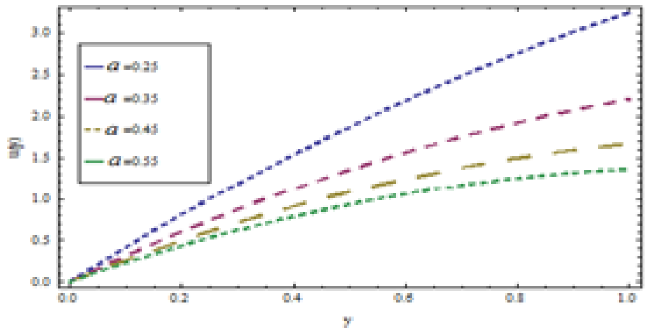

Figure 5. Effects of a on velocity profile when $\eta = 11.5$ poise, $\rho = 0.78\frac{g}{cm^3}, \delta = 2, \frac{dp^*}{dx} = -2Barye, \beta_2 = \beta_3 = 0.01$ and $v = 1cm\ s^-1$.

By making use of (26) we obtain:

$$Q = av - \frac{a^3 \frac{dp^*}{dx}}{6(\eta + 2(\beta_2 + \beta_3))} + \frac{\frac{dp^*}{dx}\delta(\beta_2 + \beta_3)}{2a^2(\eta + 2(\beta_2 + \beta_3))} \tag{27}$$

The average velocity of the fluid is given by the formula $\overline{V} = \frac{Q}{a}$ in following equation:

$$v - \frac{a^3 \frac{dp^*}{dx}}{6(\eta + 2(\beta_2 + \beta_3))} + \frac{\frac{dp^*}{dx}\delta(\beta_2 + \beta_3)}{a^2(2\eta + 4(\beta_2 + \beta_3))} \tag{28}$$

Remarks: Here, we have demonstrated that, if $\beta_2 + \beta_3 = 0$,in the consequences Newtonian solution will be Riverview, which is already mentioned in [1].

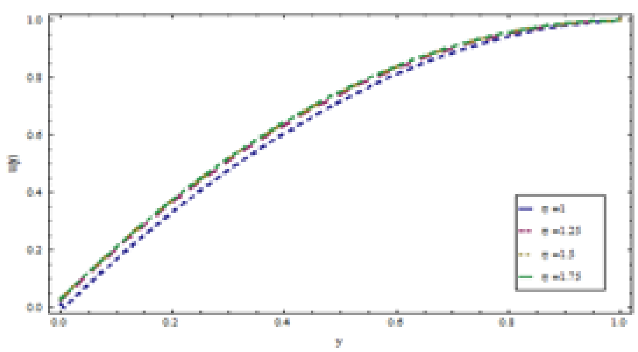

Figure 2. Effects of η on ve;ocity profile when $a = 1cm, \rho = 0.78\frac{g}{cm^3}, \delta = 2, \frac{dp^*}{dx} = -2Barye, \beta_2 = \beta_2 = 0.01$ and $v = 1cm\ s^-1$.

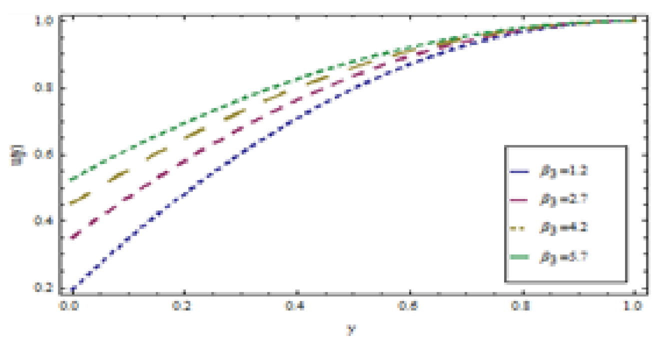

Figure 3. Effects of β_3 on velocity profile when $a = 1cm, \rho = 0.78\frac{g}{cm^3}, \delta = 2, \frac{dp^*}{dx} = -2Barye, \eta_1 1.5, \beta_2 = 0.01$ and $v = 1cm\ s^-1$.

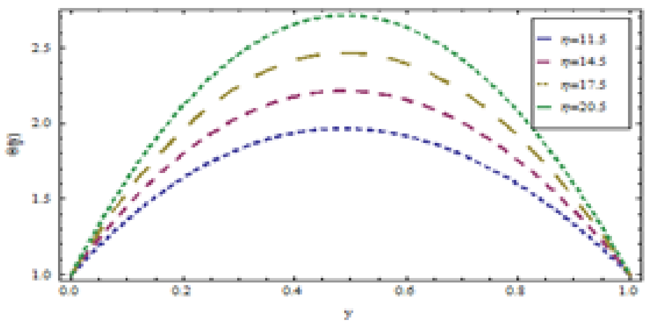

Figure 6. Effects of η on energy profile when $a = 1cm, \rho = 0.78\frac{g}{cm^3}$, $k = 1.5\frac{w}{cmc^0}$, $\frac{dp^*}{dx} = -2Barye$, $\beta_2 = \beta_3 = 0.01$, $v = 1cm\ s^{-}1$ and $\Theta_0 = \Theta_1 = 1c^0$.

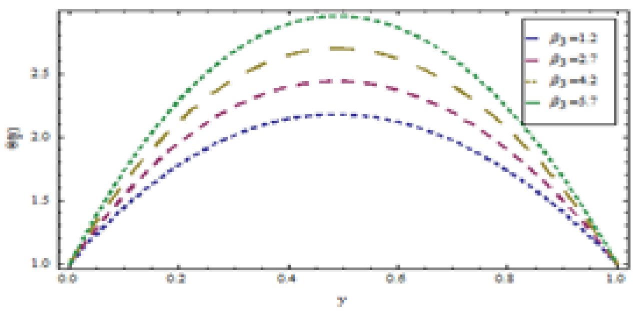

Figure 7. Effects of β_3 on energy profile when $a = 1cm, \rho = 0.78\frac{g}{cm^3}$, $k = 1.5\frac{w}{cmc^0}$, $\frac{dp^*}{dx} = -2Barye$, $\beta_2 = \beta_3 = 0.01$, $v = 1cm\ s^{-}1$, $\eta = 11.5$ poise and $\Theta_0 = \Theta_1 = 1c^0$.

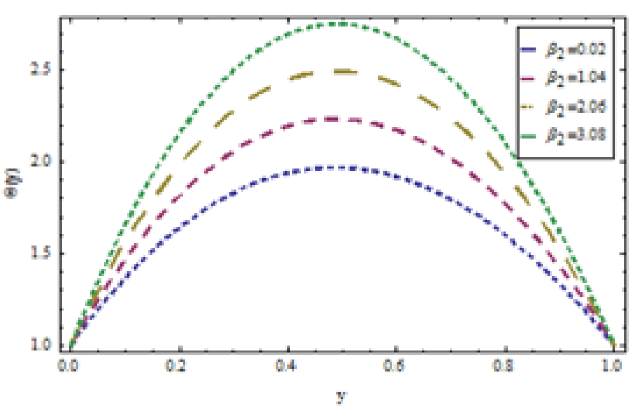

Figure 8. Effects of β_2 on energy profile when $a = 1cm, \rho = 0.78\frac{g}{cm^3}$, $k = 1.5\frac{w}{cmc^0}$, $\frac{dp^*}{dx} = -2Barye$, $\beta_3 = 0.01$, $v = 1cm\ s^{-}1$, $\eta = 11.5$ poise and $\Theta_0 = \Theta_1 = 1c^0$.

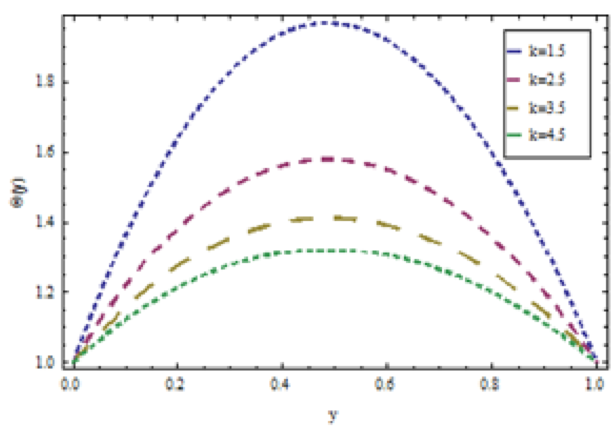

Figure 9. Effects of k on energy profile when $a = 1cm, \rho = 0.78\frac{g}{cm^3}$, $k = 1.5\frac{w}{cmc^0}$, $\frac{dp^*}{dx} = -2Barye$, $\eta = 11.5$ poise, $\beta_2 + \beta_3 = 0.01$, $v = 1cm\ s^{-}1$ and $\Theta_0 = \Theta_1 = 1c^0$.

y	$\frac{dp^*}{dx} = -$	$\frac{dp^*}{dx} = -2$	$\frac{dp^*}{dx} = -3$	$\frac{dp^*}{dx} = -4$
0	0.00474921	0.00399815	0.00421381	0.0043968
0.25	0.34618	0.439735	0.533263	0.626767
0.5	0.626366	0.750982	0.875582	1.00017
0.75	0.844306	0.937739	1.03116	1.12458
1	1	1	1	1

Table 1. Effect of pressure on the velocity distribution when $a = 1cm$,$\rho = 0.78\frac{g}{cm^3}$, $\delta = 2$, $\beta_2 = 0.01$, and $\eta = 11.5$ poise.

y	$\frac{dp^*}{dx} = -1$	$\frac{dp^*}{dx} = -2$	$\frac{dp^*}{dx} = -3$	$\frac{dp^*}{dx} = -4$
0	1	1	1	1
0.25	1.73501	1.74567	1.75645	1.76737
0.5	1.9652	1.96542	1.96578	1.96628
0.75	1.71411	1.70386	1.69375	1.68378
1	1	1	1	1

Table 2. The Effect of pressure on energy profile when $a = 1cm$,$\rho = 0.78\frac{g}{cm^3}$, $k = 1.5\frac{w}{cmc^0}$, $\eta = 11.5$ poise, $\beta_2 = 0.01$, $v = 1cm\ s^{-}1$ and $\Theta_0 = \Theta_1 = 1c^0$.

y	$a = 0.5$	$a = 0.35$	$a = 0.45$	$a = 0.55$
0	1	1	1	1
0.25	1	1.8021	1.396169	1.96329
0.5	7.00357	1.38193	0.525372	1.31604
0.75	22.8383	8.43372	3.21791	0.867183
1	46.3438	19.2436	9.18444	4.51842

Table 3. The Effect of a on energy profile when $\frac{dp^*}{dx} = -2$ Barye,$\rho = 0.78\frac{g}{cm^3}$, $k = 1.2$, $\beta_2 = 0.01$, $\delta = 2$, $\eta = 11.5$ poise, and $\Theta_0 = \Theta_1 = 1c^0$.

5 Results

The steady plane Couette-Poiseuille flow between two infinite parallel plates for third grade fluid was explored in the aforementioned sections using the delta perturbation method, a sort of perturbation methodology. Since the fluid is stable, uniform, and incompressible, an analytical solution to the nonlinear ordinary differential equation is obtained, providing the fluid's velocity profile and temperature distribution. On different parameters, the variation of the velocity profile and the temperature profile has been studied. The effects of dynamic viscosity η and material constant α constant parameters β_2 and β_3 on velocity profile are observed through figures (2) - (5) and effect of thermal conductivity k and other parameters such as α, β_2 ,β_3 and η are seen in the temperature profile in the figures (6) - (9). Impact of the magnitude of presesure gradient for velcoity profile and as well as temprature pfofile is given in Table 1-2 and effect of material constant α on temperature distribution is given in Table 3. From figures (2) - (5) it is detected that for rise inconstant parameters β_2, β_3 and dynamic viscosity η, the velocity of the fluid increases and velocity decreases for the increase of material constant α. It is noticed that from figures (6) - (9) and Table 3 that the temperature distribution decreases as thermal conductivity *k* rises and increase as dynamic viscosity η, constant parameters β_2 and β_3 and also material constant α.From Table 1 and 2, it is observed that the fluid's velocity and temperature distribution both increased as the magnitude of the pressure gradient increased.

6 Conclusions

Taking equation for incompressible, uniform and steady on plane Couette-Poiseuille flow between two parallel plates problem on behalf of third grade fluid, Analytical solution have been obtained by using delta perturbation method for non-linear ordinary differential equations, specifically for the purpose of velocity profile and the variation of temperature. Rate of flow and also average velocity have been obtained after using the value velocity profile. Here we have exactly retrieved the From figures (2)-(5), it is found that for increase of dynamic viscosity η and parameters β_2, β_3, the velocity of the fluid increases and velocity decreases for the increase of material constant α. It is noticed that from figures (6)-(9) and Table 3, losses in temperature distribution as thermal conductivity rises and increases as a result of increases in dynamic viscosity η , constant parameters β_2,β_3 and material constant α.From Tables 1 and 2, it has been observed that fluid's velocity increases as the magnitude of pressure gradient increases and moreover, the distribution of temperature rises.Additionally, we have identified a Newtonian solution for the fluid parameter setting $\beta_2 + \beta_3 = 0$. It should be noted that the fluid's third-grade velocity will raise and drop for the increase material constant, also for proposed model that temperature losses as thermal conductivity *k* increases and rises in response to growth of dynamic viscosity η , constant parameters β_2, β_3 and material constant α. Here we have also find out that temperature distribution and velocity profile enhance with higher magnitude of pressure gradient.

Author Contributions

Ahsan Mushtaque: Writing- Original draft preparation. **K. N. Memon**: Conceptualization, Methodology. **Fozia Shaikh**:Writing- Reviewing and Editing. **Abbas Ali Ghoto**: Validation, **A. M. Siddiqui**: Supervision.

Compliance with Ethical Standards

It is declare that all authors don't have any conflict of interest.

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