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Power Hamy Mean Operators for managing Cubic Linguistic Spherical Fuzzy Sets and their Applications

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Abstract In modern social administrative economic activities, we are facing a considerable amount of multi-attribute group decision making problems (MAGDM). The methods and theory related to this method are very useful in the field of particular disciplines as well as in operational research, and a lot of achievements have been described. Obviously the real world is full of uncertainties and classical set theory cannot be used to describe different phenomena such as beauty, intelligence, height (tallness) and age etc. This thing leads mathematicians to develop the notion of fuzzy sets. Later Zadeh introduced the concept of membership and non-membership degree. Definitely human opinion about a phenomenon may be unidirectional or multi-directional, that's why Atanossov proposed the concept of another advance type of fuzzy sets, which is known as intuitionistic fuzzy sets. His concept is based on a degree of membership and degree of non-membership with a exquisite that their sum must not exceed 1. In our work we introduced cubic linguistic spherical fuzzy sets. Then, we proposed the fundamental operation law for CLSFVs and a series of their average operators (AOs), such as the CLSFPA (cubic linguistic spherical fuzzy power average), CLSFPWA (cubic linguistic spherical fuzzy power weighted average), CLSFPHM (cubic linguistic spherical fuzzy power hamy mean) and CLSFPWHM (cubic linguistic spherical fuzzy power weighted hamy mean) operators, was developed by combining the power average and hamy mean operators in cubic linguistic spherical fuzzy environment. Also we described some specific desirable properties of all these operators. In addition, we suggested a new MAGDM method

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1 Introduction

In modern social administrative economic activities, we are facing a considerable amount of multi-attribute group decision making problems. Methods and theories related to MAGDM(multi-attribute group decision making) are very useful in the region of particular disciplines like decision sciences, and a lot of achievements have been described. Obviously the real world is full of uncertainties and classical set theory cannot be used to describe different phenomena such as beauty, intelligence, height (tallness) and age etc. This thing leads mathematicians to develop the concept of fuzzy sets. Later the idea of membership degree and non-membership was introduced by Zadeh [21] Definitely human opinion about a phenomenon may be unidirectional or multi-directional, that's why Atanossov [3] produced the notion of intuitionistic fuzzy sets. Also he applied a requisite that sum of both degrees must not exceed 1. In many cases we face a situation when sum of both degrees exceeds from $[0,1]$ in [4]. To increase the range of intuionistic fuzzy sets Yager [18] proposed the concept of Pythagorean fuzzy sets. In Pythagorian the sum of squares of both degrees must not exceed than 1. The duplets that could not be considered in intuitionistic fuzzy sets were also considers in the Pythagorian fuzzy sets. But there exist some duplets that cannot be categorized as Pythagorian fuzzy sets. To overcome this situation Yager [19] acquainted the notion of "q-rung orthopair fuzzy sets", where any kind of duplet can be taken. Human opinion about uncertainties cannot be bounded to yes or no only as rendered by conventional Fuzzy sets [11] and intuitionistic "fuzzy sets" [15], "Pythagorian fuzzy sets" [12] or "q-rung orthopair fuzzy sets" [2], but it can be refusal and abstain as explained earlier. So the concept of "spherical fuzzy sets" was introduced [9] and "T-spherical fuzzy sets"[14]. The "linguistic Pythagorean fuzzy sets" (LSFSs)[7] can deal with decision maker's complicated evaluation values in "multi-attribute group decision making" (MAGDM) [6] because of linguistic terms to denote non-membership andmembership degrees. To improve the ability of "Linguistic spherical fuzzy sets" (LSFSs) in rendering uncertainties and fuzzy information, we generalized linguistic spherical fuzzy sets (LSFSs) to cubic linguistic spherical fuzzy set (CLSFSs) [13] and studied cubic linguistic spherical fuzzy sets (CLSFSs)-based "MAGDM" method. Firstly thebasic operational rules and definitions are investigated as well as comparison method and distance measure or Cubic linguistic spherical fuzzy sets (CLSFSs). The CLSFSs completely adsorb the advantages of both Linguistic spherical fuzzy sets and cubic fuzzy sets. Secondly two different operators, "cubic linguistic spherical fuzzy power average operator" [8] and the "cubic linguistic spherical fuzzy power Hamy mean operators" [1] are introduced. These operators are more effective and comprehensive for aggregation of attributes values in MAGDM problems. Finally, a new MAGDM method based on cubic linguistic "Spherical fuzzy sets" is presented as well as their aggregation properties. Examples are given to show the effectiveness. Comparative analysis is also given.

2 Preliminaries

Some Basic concepts

In this section some basic concepts are introduced. These concepts are consisting of "linguistic spherical fuzzy sets" (LSFSs), "uncertain linguistic spherical fuzzy sets" (ULPFSs), "cubic linguistic spherical fuzzy sets" (CLSFSs) and some other basic operators.

2.0.1 Definition:

Consider X as a finite universe of discourse. Let $S = \{S_\beta | S_\theta \leq S_\beta \leq S_t, \beta \in [0, t]\}$ be any continuous linguistic term set. Consider a "linguistic spherical fuzzy set" (LSFS) A described as:

$$A = \{(x, S_\theta(x), S_\sigma(x)) | x \in X\},$$

Which is defined on X and$S_\theta(x)$,$S_\phi(x)$,$S_\sigma(x) \in \tilde{S}$ are representing the "linguistic degree of membership" (MD) and the "linguistic degree of non-membership" (NMD) of the given $x \in X$ to LSFS, such that

$$\theta^2 + \phi^2 + \sigma^2 \leq t^2$$

To reduce difficulty, we called the ordered pair$(S_\theta(x), S_\phi(x) S_\sigma(x))$ a linguistic spherical fuzzy value (LSFV). We can express it a$(S_\theta, S_\sigma)$for easiness.

2.0.2 Definition:

Consider X as any finite universe of discourse.Let $S = \{S_\beta | S_\theta \leq S_\beta \leq S_t, \beta \in [0, t]\}$ be any continuous linguistic term set. Consider a "linguistic spherical fuzzy set" (LSFS) Adescribed as:

$$A = \{(x, [S_\theta(x), S_\phi(x), S_\tau(x)], [S_\delta(x), S(x), S_\varepsilon(x)])x \in X\},$$

Where $[S_\theta(x), S_\phi(x), S_\tau(x)], [S_\delta(x), S(x), S_\varepsilon(x)] \in \tilde{S}$ are two uncertain linguistic variables, representing the uncertain linguistic membership degree and the uncertain linguistic non-membership degree of the $x \in X$ to A. such that

$$\tau^2 + \varepsilon^2 \leq t^2 \quad and \quad \theta \leq \phi, \phi \leq \tau, \delta \leq , \leq \varepsilon$$

. For simplicity,

$$[S_\theta(x), S_\phi(x), S_\tau(x)], [S_\delta(x), S(x), S_\varepsilon(x)]$$

we called the ordered pair an uncertain linguistic spherical fuzzy value (ULSFV), which can be denoted as$([S_\theta, S_\phi, S_\tau], [S_\delta, S, S_\varepsilon])$for more simplicity. From this definition, it is obvious that uncertain linguistic spherical fuzzy sets (ULSFSs) are more effective and reliable than linguistic spherical fuzzy sets (LSFSs). Specially, if $S_\theta(x) = S_\tau$and$S_\sigma = S_\varepsilon$ then, the uncertain linguistic spherical fuzzy set (ULSFS) "A" converts to a linguistic spherical fuzzy set (LSFS).

2.0.3 Definition:

Suppose $a_1 = ([S_{\theta_1}, S_{\tau_1}], [S_{\sigma_1}, S_{\varepsilon_1}])$, $a_2 = ([S_{\theta_2}, S_{\tau_2}], [S_{\sigma_2}, S_{\varepsilon_2}])$ and $a = ([S_\theta, S_\tau], [S_\sigma, S_\varepsilon])$ be three uncertain linguistic spherical fuzzy variables (ULSFVs).Let$\tilde{S} = \{s_\beta | s_\theta \leq s_\beta \leq s_t, \beta \in [0, t]\}$be any linguistic term set but continuous and let λ be a non-negative real number, then

$$(1)\ \alpha_1 \oplus \alpha_2 = \left(\left[s_{t\sqrt{\frac{\theta_1^2}{t^2}+\frac{\theta_2^2}{t^2}-\frac{\theta_1^2\theta_2^2}{t^4}}}, s_{t\sqrt{\frac{\tau_1^2}{t^2}+\frac{\tau_2^2}{t^2}-\frac{\tau_1^2\tau_2^2}{t^4}}} \right], \left[s_{t\left(\frac{\sigma_1\sigma_2}{t^2}\right)} + s_{t\left(\frac{\varepsilon_1\varepsilon_2}{t^2}\right)} \right] \right)$$

$$(2)\ \alpha_1 \otimes \alpha_2 = \left(\left[s_{t\left(\frac{\theta_1\theta_2}{t^2}\right)}, s_{t\left(\frac{\sigma_1\sigma_2}{t^2}\right)} \right], \left[s_{t\sqrt{\frac{\sigma_1^2}{t^2}+\frac{\sigma_2^2}{t^2}-\frac{\sigma_1^2\sigma_2^2}{t^4}}}, s_{t\sqrt{\frac{\varepsilon_1^2}{t^2}+\frac{\varepsilon_2^2}{t^2}-\frac{\varepsilon_1^2\varepsilon_2^2}{t^4}}} \right] \right)$$

$$(3)\ \lambda\alpha = \left(\left[s_{t\sqrt{1-\left(1-\frac{\theta^2}{t^2}\right)^{\lambda}}}, s_{t\sqrt{1-\left(1-\frac{\tau^2}{t^2}\right)^{\lambda}}} \right], \left[s_{t\left(\frac{\sigma}{t}\right)^{\lambda}} + s_{t\left(\frac{\varepsilon}{t}\right)^{\lambda}} \right] \right), \quad \lambda > 0 \ ;$$

$$(4)\ \alpha^{\lambda} = \left(\left[s_{t\left(\frac{\theta}{t}\right)^{\lambda}}, s_{t\left(\frac{\tau}{t}\right)^{\lambda}} \right], \left[s_{t\sqrt{1-\left(1-\frac{\sigma^2}{t^2}\right)^{\lambda}}}, s_{t\sqrt{1-\left(1-\frac{\varepsilon^2}{t^2}\right)^{\lambda}}} \right] \right), \quad \lambda > 0;$$

2.0.4 Definition

Suppose$\alpha = ([s_\theta, s_\tau], [s_\sigma, s_\varepsilon])$be an uncertain linguistic spherical fuzzy variable (ULSFV). Which is defined on $\tilde{S} \cong \{s_\beta | s_\theta \le s_\beta \le s_t, \beta \in [0,t]\}$.The "score function of α" is expressed as $S(a) = s_{\sqrt{(t^2+\theta^2+\tau^2-\sigma^2-\varepsilon^2)/4}}$, where the accuracy function of a is described as,

$$H(\alpha) = S_{\sqrt{(t^2+\theta^2+\tau^2+\sigma^2+\varepsilon^2)/2}}.$$

For any two ULPFVs α_1 and α_2,

$$(1)\quad If \quad S(\alpha_1) > S(\alpha_2), \quad then \quad \alpha_1 > \alpha_2,$$

$$(2)\quad If \quad S(\alpha_1) = S(\alpha_2), \quad then \quad \alpha_1 = \alpha_2,$$

$$If \quad H(\alpha_1) > H(\alpha_2), \quad then \quad \alpha_1 > \alpha_2,$$

$$If \quad H(\alpha_1) = H(\alpha_2), then \quad \alpha_1 = \alpha_2.$$

2.0.5 Definition

Let R be the universe set. Then the set,

$$A = \{[r, P_A(r), I_A(r), N_A(r) | r \in R]\},$$

is known as spherical fuzzy set, where $P_A : R \to [0,1]$ is positive-membership degree, $I_A : R \to [0,1]$is neutral-membership degree of r in R and $N_A : R \to [0,1]$ said to be non-membership degree of r in R .P_A, I_A and $N_A$also satisfy the given condition;

$$(\forall r \in R)(0 \le ((P_A(r)^2 + (I_A(r))^2 + (N_A(r))^2 \le 1)$$

For spherical fuzzy set $\{[r, P_A(r), I_A(r), N_A(r)|r \in R]\}$, having three components

$$P_A(r), I_A(r), N_A(r)$$

These components are said to spherical fuzzy numbers and every spherical fuzzy number can be expressed by $e = (P_e, I_e, N_e)$,Where$P_e, I_(e)$and $n_e \in [0, 1]$, with condition that $0 \leq P_e^2 + I_e^2 + N_e^2 \leq 1$.

2.0.6 Definition

Assuming that e_j = <$P_{(e_j)}, I_{(e_j)}, N_{(e_j)}$> and e_k = <$P_{(e_k)}, I_{(e_k)}, N_{(e_k)}$> be any two spherical fuzzy numbers. Then their intersection, union and compliment are describes as;

$$\begin{aligned}
&(1)\ \ e_j \subseteq e_k \ \ iff \ \ \forall r \in R, P_{e_j} \leq P_{e_k}, I_{e_j} \leq I_{e_k} \ \ and \ \ N_{e_j} \geq N_{e_k} \\
&(2)\ \ e_j = e_k \ \ iff \ \ e_j \subseteq e_k \ \ and \ \ e_k \subseteq e_j \\
&(3)\ \ e_j \cup e_k \ \ = \langle max(P_{e_j}, P_{e_k}), min(I_{e_j}, I_{e_k}), min(N_{e_j}, N_{e_k})\rangle \\
&(4)\ \ e_j \cap e_k \ \ = \langle min(P_{e_j}, P_{e_k}), min(I_{e_j}, I_{e_k}), max(N_{e_j}, N_{e_k})\rangle, \\
&(5)\ \ e_j^c = \langle N_{e_j}, I_{e_j}, P_{e_j}\rangle
\end{aligned} \tag{1}$$

2.0.7 Definition

Suppose that $e_j = \langle P_{(e_j)}, I_{(e_j)}, N_{(e_j)}\rangle$ and $e_k = \langle P_{(e_k)}, I_{(e_k)}, N_{(e_k)}\rangle$ be any two spherical fuzzy numbers and $\tau \geq 0$.Then the operations of spherical fuzzy numbers can be denoted as;

$$(1)\tau e_j = \langle \sqrt{1 - (1 - P_{e_j}^2)^\tau}, (I_{e_j})^\tau, (N_{e_j})^\tau\rangle$$

$$e_j + e_k = \langle \sqrt{P_{e_j}^2 + P_{e_j}^2 - P_{e_j}^2.P_{e_k}^2}, I_{e_j}.I_{e_k}, N_{e_j}.N_{e_k}\rangle$$

$$e_j \times e_k = \langle P_{e_j}.P_{e_k}, I_{e_j}.I_{e_k}, \sqrt{N_{e_j}^2 + N_{e_j}^2 - N_{e_j}^2.N_{e_k}^2}\rangle$$

$$(1)e_j^\tau = \langle (P_{e_j})^\tau, (I_{e_j})^\tau, \sqrt{1 - (1 - N_{e_j}^2)^\tau}\rangle$$

2.0.8 Definition

Let $e_k = \langle P_{(e_k)}, I_{(e_k)}, N_{(e_k)}\rangle$ are any spherical fuzzy numbers. Then
$(1)sc(e_k) = \dfrac{P_{e_j} + 1 - I_{e_k} + 1 - N_{e_k}}{3} = \dfrac{1}{3}(2 + P_{e_k} - I_{e_k} - N_{e_k})$ which is score function.
$(2)ac(e_k) = P_{e_k} - N_{e_k}$ which denoted as accuracy function.
$(3)cr(e_k) = P_{e_j}$ which denoted as certainty function.

2.0.9 Definition

Let $e_g = \langle P_{(e_g)}, I_{(e_g)}, N_{(e_g)} \rangle$ and $e_h = \langle P_{(e_h)}, I_{(e_h)}, N_{(e_h)} \rangle$ are any two spherical fuzzy numbers. Then (*a*) $sc(e_g) > sc(e_h)$ then $e_g > e_h$.
(*b*) $sc(e_g) = sc(e_h)$ and $ac(e_g) > ac(e_h)$ then $e_g > e_h$.
(*c*) $sc(e_g) = sc(e_h)$, $ac(e_g) = ac(e_h)$ and $cr(e_g) > cr(e_h)$ then $e_g > e_h$.
(*d*) $sc(e_g) = sc(e_h)$, $ac(e_g) = ac(e_h)$ and $cr(e_g) = cr(e_h)$ then $e_g = e_h$.

2.1 The Cubic Linguistic Spherical Fuzzy Sets

Firstly the idea of cubic linguistic spherical fuzzy sets is proposed by combining linguistic spherical fuzzy sets with uncertain linguistic spherical fuzzy sets. Examples are also given to illustrate score function and operational rules. Accuracy function is also explained.

2.1.1 Definition

Consider X as a finite universe of discourse. Let$\tilde{S} = \{s_\beta | s_\theta \leq s_\beta \leq s_t, \beta \in [0, t]\}$ be a linguistic term set that is continuous. CLSFS (Cubic Linguistic Spherical Fuzzy set) 'A' defined on X is described as,

$$A = \{(x, [S_\theta(x), S_\tau(x)], [S_\phi(x), S_\psi(x)], [S_\sigma(x), S_\varepsilon(x)], [S_\delta(x), S_\mu(x), S_\varsigma(x)])x \in X\},$$

where
$([S_\theta(x), S_\tau(x)], [S_\phi(x), S_\psi(x)], [S_\sigma(x), S_\varepsilon(x)])$is a ULSFV and$(S_\delta(x), S_\mu(x), S_\varsigma(x))$ is an LSFV. For convenience, $\langle([S_\theta(x), S_\tau(x)], [S_\phi(x), S_\psi(x)], [S_\sigma(x), S_\varepsilon(x)])\rangle$is called cubic linguistic spherical Fuzzy value (CLSFV). Also we can write it as
$\eta = \langle([S_\theta, S_\tau], [S_\phi, S_\psi], [S_\sigma, S_\varepsilon]), (S_\delta, S_\mu, S_\varsigma)\rangle$ for simplicity.

2.1.2 Definition

Let $\eta_1 = \langle([S_{\theta_1}, S_{\tau_1}], [S_{\phi_1}, S_{\psi_1}], [S_{\sigma_1}, S_{\varepsilon_1}]), (S_{\delta_1}, S_{\mu_1}, S_{\varsigma_1})\rangle$, $\eta_2 = \langle([S_{\theta_2}, S_{\tau_2}], [S_{\phi_2}, S_{\psi_2}], [S_{\sigma_2}, S_{\varepsilon_2}]), (S_{\delta_2}, S_{\mu_2}, S_{\varsigma_2})\rangle$ and $\eta = \langle([S_\theta, S_\tau], [S_\phi, S_\psi], [S_\sigma, S_\varepsilon]), (S_\delta, S_\mu, S_\varsigma)\rangle$ be any three cubic linguistic spherical fuzzy variables (CLSFVs) defined on the given linguistic term set that is
$\tilde{S} = \{s_\beta | s_\theta \leq s_\beta \leq s_t, \beta \in [0, t]\}$ and λ be a positive real number

$$(1)\ \eta_1 \oplus \eta_2 = \left\langle \left(\left[S_{t\sqrt{\frac{\theta_1^2}{t^2}+\frac{\theta_2^2}{t^2}-\frac{\theta_1^2\theta_2^2}{t^4}}}, S_{t\sqrt{\frac{\tau_1^2}{t^2}+\frac{\tau_2^2}{t^2}-\frac{\tau_1^2\tau_2^2}{t^4}}} \right], \left[S_{t\left(\frac{\phi_1\phi_2}{t^2}\right)} + S_{t\left(\frac{\psi_1\psi_2}{t^2}\right)} \right], \left[S_{t\left(\frac{\sigma_1\sigma_2}{t^2}\right)} + S_{t\left(\frac{\varepsilon_1\varepsilon_2}{t^2}\right)} \right] \right) \left(S_{t\sqrt{\frac{\delta_1^2}{t^2}+\frac{\delta_2^2}{t^2}-\frac{\delta_1^2\delta_2^2}{t^4}}}, S_{t\left(\frac{\mu_1\mu_2}{t^2}\right)}, S_{t\left(\frac{\varsigma_1\varsigma_2}{t^2}\right)} \right) \right\rangle$$

$$(2)\ \eta_1 \otimes \eta_2 = \left\langle \left(\left[S_{t\left(\frac{\theta_1\theta_2}{t^2}\right)} + S_{t\left(\frac{\tau_1\tau_2}{t^2}\right)} \right], \left[S_{t\sqrt{\frac{\phi_1^2}{t^2}+\frac{\phi_2^2}{t^2}-\frac{\phi_1^2\phi_2^2}{t^4}}}, S_{t\sqrt{\frac{\psi_1^2}{t^2}+\frac{\psi_2^2}{t^2}-\frac{\psi_1^2\psi_2^2}{t^4}}} \right] \right) \right.$$
$$\left. \left[S_{t\sqrt{\frac{\sigma_1^2}{t^2}+\frac{\sigma_2^2}{t^2}-\frac{\sigma_1^2\sigma_2^2}{t^4}}}, S_{t\sqrt{\frac{\varepsilon_1^2}{t^2}+\frac{\varepsilon_2^2}{t^2}-\frac{\varepsilon_1^2\varepsilon_2^2}{t^4}}} \right] \left(S_{t\sqrt{\frac{\delta_1^2}{t^2}+\frac{\delta_2^2}{t^2}-\frac{\delta_1^2\delta_2^2}{t^4}}}, S_{t\left(\frac{\mu_1\mu_2}{t^2}\right)}, S_{t\left(\frac{\varsigma_1\varsigma_2}{t^2}\right)} \right) \right\rangle$$

$$(3)\ \lambda\eta = \left\langle \left(\left[S_{t\sqrt{1-\left(1-\frac{\theta^2}{t^2}\right)^\lambda}}, S_{t\sqrt{1-\left(1-\frac{\tau^2}{t^2}\right)^\lambda}} \right], \left[S_{t\left(\frac{\phi}{t}\right)^\lambda} + S_{t\left(\frac{\psi}{t}\right)^\lambda} \right], \left[S_{t\left(\frac{\sigma}{t}\right)^\lambda} + S_{t\left(\frac{\varepsilon}{t}\right)^\lambda} \right] \right), \right.$$
$$\left. \left(S_{t\sqrt{1-\left(1-\frac{\delta^2}{t^2}\right)^\lambda}}, S_{t\left(\frac{\mu}{t}\right)^\lambda}, S_{t\left(\frac{\varsigma}{t}\right)^\lambda} \right) \right\rangle$$

$$(3)\ \eta^\lambda = \left\langle \left(\left[S_{t\left(\frac{\theta}{t}\right)^\lambda} + S_{t\left(\frac{\tau}{t}\right)^\lambda} \right], \left[S_{t\sqrt{1-\left(1-\frac{\delta^2}{t^2}\right)^\lambda}}, S_{t\sqrt{1-\left(1-\frac{\phi^2}{t^2}\right)^\lambda}} \right], \left[S_{t\sqrt{1-\left(1-\frac{\sigma^2}{t^2}\right)^\lambda}}, S_{t\sqrt{1-\left(1-\frac{\mu^2}{t^2}\right)^\lambda}} \right] \right) \right.$$
$$\left. \left(S_{t\left(\frac{\delta}{t}\right)^\lambda}, S_{t\sqrt{1-\left(1-\frac{\mu^2}{t^2}\right)^\lambda}}, S_{t\sqrt{1-\left(1-\frac{\mu^2}{t^2}\right)^\lambda}} \right) \right\rangle$$

2.1.3 Definition

Let $\eta = \langle([S_\theta, S_\tau], [S_\phi, S_\psi], [S_\sigma, S_\varepsilon]), (S_\delta, S_\mu, S_\varsigma)\rangle$ be a cubic linguistic spherical fuzzy value (CLSFV) defined on the linguistic term set
$\tilde{S} = \{s_\beta | s_\theta \leq s_\beta \leq s_t, \beta \in [0, t]\}$,as a result the score function of η is described as

$$S(\eta) = S_{(\sqrt{(2t^2+\theta^2+\tau^2-\phi^2-\psi^2-\sigma^2-\varepsilon^2)/6}+\sqrt{(t^2+\delta^2-\mu^2-\varsigma^2)/3})/2},$$

Then the accuracy function of η is described as

$$H(\eta) = S_{(\sqrt{(\theta^2+\tau^2+\phi^2+\psi^2+\sigma^2+\varepsilon^2)/2}+\sqrt{(\delta^2+\mu^2+\varsigma^2)/3})},$$

Let $\eta_1 = \langle([S_{\theta_1}, S_{\tau_1}], [S_{\phi_1}, S_{\psi_1}], [S_{\sigma_1}, S_{\varepsilon_1}]), (S_{\delta_1}, S_{\mu_1}, S_{\varsigma_1})\rangle$
and $\eta_2 = \langle([S_{\theta_2}, S_{\tau_2}], [S_{\phi_2}, S_{\psi_2}], [S_{\sigma_2}, S_{\varepsilon_2}]), (S_{\delta_2}, S_{\mu_2}, S_{\varsigma_2})\rangle$ be any two CLPFVs,

$$(1) \quad If \quad S(\eta_1) > S(\eta_2), \quad then \ \eta_1 > \eta_2,$$

$$(2) \quad If \quad S(\eta_1) = S(\eta_2), \quad then \quad \eta_1 = \eta_2,$$

$$If \quad H(\eta_1) > H(\eta_2), \quad then \quad \eta_1 > \eta_2,$$

$$If \quad H(\eta_1) = H(\eta_2), then \quad \eta_1 = \eta_2.$$

2.1.4 Definition

Let $\eta_1 = \langle([S_{\theta_1}, S_{\tau_1}], [S_{\phi_1}, S_{\psi_1}], [S_{\sigma_1}, S_{\varepsilon_1}]), (S_{\delta_1}, S_{\mu_1}, S_{\varsigma_1})\rangle$
and $\eta_2 = \langle([S_{\theta_2}, S_{\tau_2}], [S_{\phi_2}, S_{\psi_2}], [S_{\sigma_2}, S_{\varepsilon_2}]), (S_{\delta_2}, S_{\mu_2}, S_{\varsigma_2})\rangle$be any two cubic linguistic spherical fuzzy values (CLS-FVs) defined on the linguistic term set $\tilde{S} = \{s_\beta | s_\theta \leq s_\beta \leq s_t, \beta \in [0, t]\}$, then the distance between η_1 and $\eta_2$is expressed as $d(\eta_1, \eta_2) = \dfrac{|\theta_1^2 - \theta_2^2| + |\tau_1^2 - \tau_2^2| + |\phi_1^2 - \phi_2^2| + |\psi_1^2 - \psi_2^2| + |\sigma_1^2 - \sigma_2^2| + |\varepsilon_1^2 - \varepsilon_2^2| + |\delta_1^2 - \delta_2^2| + |\mu_1^2 - \mu_2^2| + |\zeta_1^2}{8t^2}$

2.1.5 Definition

For any CLTS $S = \{S_\varepsilon | \varepsilon \in [0, t]\}$where t is a positive integer and three LqROFNs
$\gamma = (S_a, S_b)$, $\gamma_1 = (S_{a_1}, S_{b_1})$ and $\gamma_2 = (S_{a_2}, S_{b_2})$ with $S_{a_1}, S_{b_1}, S_{a_2}, S_{b_2} \in S$;then

$$(1)\ \gamma_1 \oplus Y_2 = (S_{a_1}, S_{b_1}) \oplus (S_{a_2}, S_{b_2}) = \left(S_{t\left(1-\sqcap_{i=1}^{2}\left(1-\frac{a_j^q}{t^q}\right)\right)^{\frac{1}{q}}}, S_{t(\frac{b_1 b_2}{t^2})} \right)$$

$$(1)\ \gamma_1 \otimes Y_2 = (S_{a_1}, S_{b_1}) \otimes (S_{a_2}, S_{b_2}) = \left(S_{t(\frac{a_1 a_2}{t^2})}, S_{t\left(1-\sqcap_{i=1}^{2}\left(1-\frac{b_j^q}{t^q}\right)\right)^{\frac{1}{q}}} \right)$$

$$(3)\ \ \lambda\gamma = \lambda(S_a, S_b) = \left(S_{t\left(1-\left(1-\frac{a^q}{t^q}\right)^{\lambda}\right)^{\frac{1}{q}}}, S_{t\left(\frac{b}{t}\right)^{\lambda}} \right)$$

$$(4)\ \ \gamma^\lambda = (S_a, S_b) = \left(S_{t\left(\frac{b}{t}\right)^{\lambda}}, S_{t\left(1-\left(1-\frac{a^q}{t^q}\right)^{\lambda}\right)^{\frac{1}{q}}} \right)$$

2.2 Some basic average operators AOs

2.2.1 Definition

Let $\alpha_i(i = 1, 2, \ldots, n)$be n crisp numbers and$k = 1, 2, 3, \ldots, n$, if

$$HM^{(k)}(a_1, a_2, a_3, ...a_n) = \frac{\sum_{1 \leq i_1 < ... < i_k \leq n} (\sqcap_j^k a_{i_j})^{1/k}}{C_n^k}$$

Then $HM^(k)$is known as the Hamy mean operator and C_n^k is known as binomial coefficient.

2.2.2 Definition

Let $\alpha_i(i = 1, 2, 3, \ldots, n)$be n crisp numbers. Then the PA operator is described as

$$PA(a_1, a_2, a_3, ...a_n) = \frac{\sum_{i=1}^n (1 + T(a_i))a_i}{\sum_{i=1}^n (1 + T(a_i))}$$

Where $T(a_i) = \sum_{i=1;j \neq i}^n Sup(a_i, a_j)$, $Sup(a_i, a_j)$ expressing support for a_i from a_j satisfying the given conditions as follows

(1) $0 \leq Sup(a_i, a_j) \leq 1$

(2) $Sup(a_i, a_j) = Sup(a_j, a_i)$

(3) $Sup(a, b) \leq Sup(c, d), |a, b| \geq |c, d|$.

2.2.3 Definition

Let $\alpha_i(i = 1, 2, 3, \ldots, n)$ be n crisp numbers and $k = 1, 2, 3, \ldots, n$, then the Power Hamy Operator is described as:

$$PHM^{(k)}(a_1, a_2, a_3, ...a_n) = \frac{1}{c^k_{n=1 \leq i_1 < ... < i_k \leq n}} \left(\sqcap_{j=1}^K \frac{n(1 + T(a_{ij}))a_{i_j}}{\sum_{i=1}^n (1 + T(a_{ij}))} \right)^{1/k}$$

Where $T(a_i) = \sum_{i=1;j \neq i}^n Sup(a_i, a_j)$, $Sup(a_i, a_j)$ denotes the support for a_j from a_i

Some Average Operators for CLSFVs with specific properties The cubic linguistic spherical fuzzy power average operator [5]

2.2.4 Definition

Suppose $\eta_i(i = 1, 2, 3, 4, \ldots, n)$ be any collection of CLSFVs. Then the CLSFPA operator is given by

$$CLSFPA(\eta_1, \eta_2, \ldots, \eta_n) = \sum_{i=1}^n \frac{(1 + T(\eta_i))}{\sum_{i=1}^n (1 + T(\eta_i))} \eta_i,$$

Where$T(\eta_i) = \sum_{i=1;j \neq i}^n Sup(\eta_i, \eta_j)$, $Sup(\eta_i, \eta_j)$ denotes the support for η_j from η_i satisfying the given conditions

(1) $0 \leq Sup(\eta_i, \eta_j) \leq 1$

(2) $Sup(\eta_i, \eta_j) = Sup(\eta_j, \eta_i)$

(3) $Sup(\eta_1, \eta_2) \leq Sup(\eta_3, \eta_4)$,$d(\eta_1, \eta_2) \leq d(\eta_3, \eta_4)$.

The cubic linguistic spherical fuzzy power weighted average operator [20].

2.2.5 Definition

Let $\eta_i(i = 1, 2, 3, 4, \ldots, n)$be a finite collection of CLSFVs and let$w = (w_1, w_2, w_3, \ldots, w_n)$ be the corresponding weight vector, satisfying the condition that $0 \leq w_i \leq 1$ and $\sum_{i=1}^{n} w_i = 1$. The cubic linguistic spherical fuzzy power weighted average (CLSFPWA) operator is expressed as

$$CLSFPWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{\oplus_{i=1}^{n}(1 + T(\alpha_i))}{\sum_{i=1}^{n}(1 + T(\alpha_i))}\alpha_i,$$

Where$T(\alpha_i) = \sum_{i=1;j\neq i}^{n} Sup(\alpha_i, \alpha_j), Sup(\alpha_i, \alpha_j)$ denotes the support for α_j from α_i.

The cubic linguistic spherical fuzzy power hamy mean operator [10].

The CLSFPA (cubic linguistic spherical fuzzy power average) and CLSFPWA (cubic linguistic spherical fuzzy power weighted average) operators both have the same capability to diminish the negative influence of unreasonable cubic linguistic spherical fuzzy values but they are not useable for considering the interrelationship between aggregated CLSFVs. So, in next section some hybrid average operators for CLSFVs are proposed.

2.2.6 Definition

Suppose $\eta_i(i = 1, 2, 3, \ldots, n)$ be a finite collection of CLSFVs and let $k = (1, 2, 3, \ldots, n)$. Then the CLSFPHM operator is described as

$$CLSFPHM^{(k)}(\eta_1, \eta_2, \ldots, \eta_n) = \frac{1}{c^k_{n=1 \leq i_1 < \ldots < i_k \leq n}}\left(\left[\otimes_{i=1}^{k} \frac{n(1 + T(\eta_{i_j}))\eta_{i_j}}{\sum_{i=1}^{n}(1 + T(\eta_i))}\right]^{1/k}\right)$$

Where$T(\eta_i) = \sum_{i=1;j\neq i}^{n} Sup(\eta_i, \eta_j), Sup(\eta_i, \eta_i)$ denotes the support for η_j from η_i.To simplify previous equation we assume

$$r_i = \frac{1 + T(\eta_i)}{\sum_{i=1}^{n}(1 + T(\eta_i))}$$

Then equation 21 can be written as

$$CLSFPHM^{(k)}(\eta_1, \eta_2, \ldots, \eta_n) = \frac{1}{c^k_{n=1 \leq i_1 < \ldots < i_k \leq n}}\left(\otimes_{i=1}^{k} n r_{i_j} \eta_{i_j}\right)^{1/k}$$

Where $0 \leq r_i \leq 1$ and $\sum_{i=1}^{n} r_i = 1$.

The cubic linguistic spherical fuzzy power weighted hamy mean operator [17].

2.2.7 Definition

Suppose $\eta_i(i = 1, 2, 3, 4, \ldots, n)$ be a finite collection of CLSFVs(cubic linguistic spherical fuzzy values) where $k = 1, 2, 3, \ldots, n$. Consider $w = (w_1, w_2, w_3 \ldots, w_n)^T$ as corresponding weight vector, satisfying the condition $0 \leq w_i \leq 1$ and $\sum_{i=1}^{n} w_i = 1$. The (CLSFPWHM) operator is described as

$$CLSFPWHM^{(k)}(\eta_1, \eta_2, \ldots, \eta_n) = \frac{1}{c^k_{n=1 \leq i_1 < \ldots < i_k \leq n}}\left(\left[\otimes_{i=1}^{k} \frac{n w_{i_j}(1 + T(\eta_{i_j}))\eta_{i_j}}{\sum_{i=1}^{n} w_i(1 + T(\eta_i))}\right]^{1/k}\right)$$

Where$T(\eta_i) = \sum_{i=1;j\neq i}^{n} Sup(\eta_i, \eta_j), Sup(\eta_i, \eta_j)$ denotes the support for η_j.To simplify previous equation we assume

$$g_i = \frac{w_i(1 + T(\eta_i))}{\sum_{i=1}^{n} w_i(1 + T(\eta_i))}$$

Then equation 32 can be written as

$$CLSFPWHM^{(k)}(a_1, a_2, \ldots, a_n) = \frac{1}{c^{k}_{n=1\leq i_1<\ldots<i_k\leq n}} \left(\otimes_{i=1}^{k} n g_{i_j} \eta_{i_j}\right)^{1/k}$$

Where $0 \leq g_i \leq 1$ and $\sum_{i=1}^{n} g_i = 1$.

2.2.8 Definition

Consider we have a $CLTSS = \{s_\varepsilon | \varepsilon \in [0, t]\}$. Where t is a positive integer and forthree any $LqROFNs\gamma = (s_a, s_b), \gamma_1 = (s_{a_1}, s_{b_1})$ and $\gamma_2 = (s_{a_2}, s_{b_2})$ with $s_{a_1}, s_{b_1}, s_{a_2}, s_{b_2} \in S$, we have

$$(1)\ \gamma_1 \oplus \gamma_2 = (S_{a_1}, S_{b_1}) \oplus (S_{a_2}, S_{b_2}) = \left(S_{t\left(1-\prod_{i=1}^{2}\left(1-\frac{a_i^q}{t^q}\right)\right)}, S_{t\left(1-\prod_{i=1}^{2}\left(1-\frac{a_i^q}{t^q}\right)-\prod_{i=1}^{2}\left(1-\frac{a_i^q}{t^q}-\frac{b_i^q}{t^q}\right)\right)^{\frac{1}{q}}} \right)$$

$$(1)\ \gamma_1 \otimes \gamma_2 = (S_{a_1}, S_{b_1}) \otimes (S_{a_2}, S_{b_2}) = \left(S_{t\left(1-\prod_{i=1}^{2}\left(1-\frac{b_i^q}{t^q}\right)-\prod_{i=1}^{2}\left(1-\frac{a_i^q}{t^q}-\frac{b_i^q}{t^q}\right)\right)^{\frac{1}{q}}}, S_{t\left(1-\prod_{i=1}^{2}\left(1-\frac{b_i^q}{t^q}\right)\right)} \right)$$

$$(3)\ \lambda\gamma = \lambda(S_a, S_b) = \left(S_{t\left(1-\left(1-\frac{a^q}{t^q}\right)^{\lambda}\right)^{\frac{1}{q}}}, S_{t\left(1-\prod_{i=1}^{2}\left(1-\frac{a}{t^q}\right)^{\lambda}-\prod_{i=1}^{2}\left(1-\frac{a}{t^q}-\frac{b}{t^q}\right)^{\lambda}\right)^{\frac{1}{q}}} \right)$$

$$(4)\ \gamma^{\lambda} = (S_a, S_b) = \left(S_{t\left(\left(1-\frac{b^q}{t^q}\right)^{\lambda}-\left(1-\frac{a}{t^q}-\frac{b^q}{t^q}\right)^{\lambda}\right)^{\frac{1}{q}}}, S_{t\left(1-\left(1-\frac{b^q}{t^q}\right)^{\lambda}\right)^{\frac{1}{q}}} \right)$$

2.3 Theorems

2.3.1 Theorems

Let η, η_1 and η_2 be any three CLSFVs and $\lambda_1, \lambda_2, \lambda_3 > 0$, then

$$(1)\eta_1 \oplus \eta_2 = \eta_2 \oplus \eta_1$$
$$(2)\eta_1 \otimes \eta_2 = \eta_2 \otimes \eta_1$$
$$(3)\lambda(\eta_1 \oplus \eta_2) = \lambda\eta_2 \oplus \lambda\eta_1$$
$$(4)\lambda_1\eta \oplus \lambda_2\eta = (\lambda_1 + \lambda_2)\eta$$
$$(5)\eta^{\lambda_1} \otimes \eta^{\lambda_2} = \eta^{(\lambda_1+\lambda_2)}$$
$$(6)\eta_1^{\lambda} \otimes \eta_2^{\lambda} = (\eta_1 \otimes \eta_2)^{\lambda}$$

The following comparison method for cubic linguistic spherical fuzzy values (CLSFVs) is provided for the ranking of any two CLSFVs.

2.3.2 Theorems

Let $\eta_i = \langle([S_{\theta_i}, S_{\tau_i}], [S_{\sigma_i}, S_{\varepsilon_i}], [S_{\delta_i}, S_{\varsigma_i}])\rangle(i = 1, 2, 3, \ldots, n)$be a collection of CLSFV defined on $\tilde{S} \cong \{s_\beta | s_\theta \leq s_\beta \leq s_t, \beta \in [0, t]\}$, that is linguistic term set, and then the result by the CLSFPA operator is still a CLSFV.

$$(1)CLSFPA(\eta_1, \eta_2, \ldots, \eta_n) =$$

$$\left\langle\left(\left[S_{t\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i^2}{t^2}\right)\right)^{\frac{(1+T(\eta_i))}{\sum_{i=1}^{n}(1+T(\eta_i))}}}}, S_{t\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\frac{\tau_i^2}{t^2}\right)\right)^{\frac{(1+T(\eta_i))}{\sum_{i=1}^{n}(1+T(\eta_i))}}}}\right],\right.\right.$$
$$\left[S_{t\prod_{i=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\frac{(1+T(\eta_i))}{\sum_{i=1}^{n}(1+T(\eta_i))}}}, S_{t\prod_{i=1}^{n}\left(\frac{\varepsilon_i}{t}\right)^{\frac{(1+T(\eta_i))}{\sum_{i=1}^{n}(1+T(\eta_i))}}}\right],$$
$$\left[S_{t\prod_{i=1}^{n}\left(\frac{\phi_i}{t}\right)^{\frac{(1+T(\eta_i))}{\sum_{i=1}^{n}(1+T(\eta_i))}}}, S_{t\prod_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\frac{(1+T(\eta_i))}{\sum_{i=1}^{n}(1+T(\eta_i))}}}\right],$$
$$\left.\left(S_{t\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i^2}{t^2}\right)\right)^{\frac{(1+T(\eta_i))}{\sum_{i=1}^{n}(1+T(\eta_i))}}}}, S_{t\prod_{i=1}^{n}\left(\frac{\mu_i}{t}\right)^{\frac{(1+T(\eta_i))}{\sum_{i=1}^{n}(1+T(\eta_i))}}}, S_{t\prod_{i=1}^{n}\left(\frac{\varsigma_i}{t}\right)^{\frac{(1+T(\eta_i))}{\sum_{i=1}^{n}(1+T(\eta_i))}}}\right)\right)\right\rangle$$

2.3.3 Theorems

Idempotency [16]

Suppose $\eta_i(i = 1, 2, 3, 4, \ldots, n)$ be a collection of CLSFVs, if $\eta_i = \eta$ = <([S_θ, S_τ], [S_ϕ, S_ψ], [S_σ, s_ε]), ($S_\delta, S_\mu, s_\varsigma$)> holds for all i, then,

$$CLSFPA(\eta_1, \eta_2, \ldots, \eta_n) = \eta$$

.

2.3.4 Theorems

Boundedness)[16]

Suppose $\eta_i(i = 1, 2, 3, \ldots, n)$ be a collection of CLSFVs, then

$$CLSFPA(\eta^-, \eta^-, \ldots, \eta^-) \leq CLSFPA(\eta_1, \eta_2, \ldots, \eta_n) \leq CLSFPA(\eta^+, \eta^+, \ldots, \eta^+)$$

where,

$$\eta^- = \eta_i = \langle([S_{min_{i=1}^n \theta_i}, S_{min_{i=1}^n \tau_i}], [S_{min_{i=1}^n \phi_i}, S_{min_{i=1}^n \psi_i}], [S_{max_{i=1}^n \sigma_i}, S_{max_{i=1}^n \varepsilon_i}])$$

$$(S_{min_{i=1}^n \delta_i}, S_{min_{i=1}^n \mu_i}, S_{max_{i=1}^n \varsigma_i})\rangle$$

and

$$\eta^+ = \langle([S_{max_{i=1}^n \theta_i}, S_{max_{i=1}^n \tau_i}], [S_{min_{i=1}^n \phi_i}, S_{min_{i=1}^n \psi_i}], [S_{min_{i=1}^n \sigma_i}, S_{min_{i=1}^n \varepsilon_i}])$$

$$(S_{max_{i=1}^n \delta_i}, S_{min_{i=1}^n \mu_i}, S_{min_{i=1}^n \varsigma_i})\rangle$$

2.3.5 Theorems

Let $\eta_i = \langle([S_{\theta_i}, S_{\tau_i}], [S_{\sigma_i}, S_{\varepsilon_i}]), (S_{\delta_i}, S_{\varsigma_i})\rangle(i = 1, 2, 3, 4, \ldots, n)$ be a collection of CLSFVs defined on $\tilde{S} = \{s_\beta | s_\theta \leq s_\beta \leq s_t, \beta \in [0, t]\}$,, that is linguistic term set, then the result by the CLSFPWA operator is still a CLSFV.

(1)$CLSFPA(\eta_1, \eta_2, \ldots, \eta_n)$ =

$$
\left\{
\begin{array}{l}
\Bigg\langle \Bigg(\left[S_{t\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i^2}{t^2}\right)\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}}, S_{t\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\frac{\tau_i^2}{t^2}\right)\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}} \right], \\
\left[S_{t\prod_{i=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}, S_{t\prod_{i=1}^{n}\left(\frac{\varepsilon_i}{t}\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}} \right], \\
\left[S_{t\prod_{i=1}^{n}\left(\frac{\phi_i}{t}\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}, S_{t\prod_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}} \right], \\
\left(S_{t\sqrt{\left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i^2}{t^2}\right)\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}}, S_{t\prod_{i=1}^{n}\left(\frac{\mu_i}{t}\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}, S_{t\prod_{i=1}^{n}\left(\frac{\varsigma_i}{t}\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}} \right) \Bigg) \Bigg\rangle
\end{array}
\right.
$$

Theorem 1. *Let* $\eta_i = \langle([S_{\theta_i}, S_{\tau_i}], [S_{\sigma_i}, S_{\varepsilon_i}]), (S_{\delta_i}, S_{\varsigma_i})\rangle (i = 1, 2, 3, 4, \ldots, n)$ *be a collection of CLSFVs defined on* $\tilde{S} = \{s_\beta | s_\theta \leq s_\beta \leq s_t, \beta \in [0, t]\}$,, *that is linguistic term set, then the result by the CLSFPWA operator is still a CLSFV.*

(1)$CLSFPA(\eta_1, \eta_2, \ldots, \eta_n) =$

$$\left\langle\left(\left[S_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\theta_i^2}{t^2}\right)\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}}, S_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\tau_i^2}{t^2}\right)\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}}\right],\right.\right.$$

$$\left[S_{t\sqcap_{i=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}, S_{t\sqcap_{i=1}^{n}\left(\frac{\varepsilon_i}{t}\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}\right],$$

$$\left[S_{t\sqcap_{i=1}^{n}\left(\frac{\phi_i}{t}\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}, S_{t\sqcap_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}\right],$$

$$\left.\left.\left(S_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\theta_i^2}{t^2}\right)\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}}, S_{t\sqcap_{i=1}^{n}\left(\frac{\mu_i}{t}\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}, S_{t\sqcap_{i=1}^{n}\left(\frac{\varsigma_i}{t}\right)^{\frac{w_i(1+T(\eta_i))}{\sum_{i=1}^{n} w_i(1+T(\eta_i))}}}\right)\right)\right)\right\rangle$$

2.3.6 Theorems

Theorem 2. *Let* $\eta_i = \langle([S_{\theta_i}, S_{\tau_i}], [S_{\sigma_i}, S_{\varepsilon_i}]), (S_{\delta_i}, S_{\varsigma_i})\rangle (i = 1, 2, 3, 4, \ldots, n)$ *be a collection of CLSFVs defined on* $\tilde{S} = \{s_\beta | s_\theta \leq s_\beta \leq s_t, \beta \in [0, t]\}$*,, that is linguistic term set, then the result by the CLSFHM operator is still a CLSFV.*

$$CLSFHM^{(k)}(\eta_1, \eta_2, \ldots, \eta_n) =$$

$$
\left\{
\begin{array}{l}
\left[S\sqrt[t]{1-\prod^{n}_{1\le i_1<\ldots<i_k\le i_n}\left(1-\left(\prod_{j=1}^{n}\left(\frac{\theta_{i_j}}{t}\right)^{\frac{1}{k}nr_{ij}}\right)^{2}\right)^{\frac{1}{c_n^k}}},\ S\sqrt[t]{1-\prod^{n}_{1\le i_1<\ldots<i_k\le i_n}\left(1-\left(\prod_{j=1}^{n}\left(\frac{\tau_{i_j}}{t}\right)^{\frac{1}{k}nr_{ij}}\right)^{2}\right)^{\frac{1}{c_n^k}}} \right], \\
\left[S\sqrt[t]{1-\prod^{n}_{1\le i_1<\ldots<i_k\le i_n}\left(1-\left(\prod_{j=1}^{n}\left(\frac{\phi_{i_j}}{t}\right)^{\frac{1}{k}nr_{ij}}\right)^{2}\right)^{\frac{1}{c_n^k}}},\ S\sqrt[t]{1-\prod^{n}_{1\le i_1<\ldots<i_k\le i_n}\left(1-\left(\prod_{j=1}^{n}\left(\frac{\psi_{i_j}}{t}\right)^{\frac{1}{k}nr_{ij}}\right)^{2}\right)^{\frac{1}{c_n^k}}} \right], \\
\left[S\sqrt[t]{1-\prod^{n}_{1\le i_1<\ldots<i_k\le i_n}\left(1-\left(\prod_{j=1}^{n}\left(\frac{\sigma_{i_j}}{t}\right)^{\frac{1}{k}nr_{ij}}\right)^{2}\right)^{\frac{1}{c_n^k}}},\ S\sqrt[t]{1-\prod^{n}_{1\le i_1<\ldots<i_k\le i_n}\left(1-\left(\prod_{j=1}^{n}\left(\frac{\varepsilon_{i_j}}{t}\right)^{\frac{1}{k}nr_{ij}}\right)^{2}\right)^{\frac{1}{c_n^k}}} \right], \\
,\left(S\sqrt[t]{1-\prod^{n}_{1\le i_1<\ldots<i_k\le i_n}\left(1-\left(\prod_{j=1}^{n}\left(\frac{\delta_{i}}{t^2}\right)^{\frac{1}{k}nr_{ij}}\right)^{2}\right)^{\frac{1}{c_n^k}}},\ S\sqrt[t]{1-\prod^{n}_{1\le i_1<\ldots<i_k\le i_n}\left(1-\left(\prod_{j=1}^{n}\left(\frac{\mu_{i}}{t^2}\right)^{\frac{1}{k}nr_{ij}}\right)^{2}\right)^{\frac{1}{c_n^k}}}\right. \\
\left. S\sqrt[t]{1-\prod^{n}_{1\le i_1<\ldots<i_k\le i_n}\left(1-\left(\prod_{j=1}^{n}\left(\frac{\varsigma_{i}}{t^2}\right)^{\frac{1}{k}nr_{ij}}\right)^{2}\right)^{\frac{1}{c_n^k}}} \right)
\end{array}
\right.
$$

$$CLSFPA(\eta_1,\eta_2,\ldots,\eta_n)=$$

$$\left\{\left(\left[S_t\sqrt{1-\sqcap_{1\le i_1<\ldots<i_k\le i_n}^{n}\left(1-\left(\sqcap_{j=1}^{n}\left(\frac{\theta_i}{t}\right)^{\frac{1}{k}nr_{ij}}\right)^2\right)}^{\frac{1}{c_n^k}},S_t\sqrt{1-\sqcap_{1\le i_1<\ldots<i_k\le i_n}^{n}\left(1-\left(\sqcap_{j=1}^{n}\left(\frac{\tau_i}{t}\right)^{\frac{1}{k}nr_{ij}}\right)^2\right)}^{\frac{1}{c_n^k}}\right],\right.\right.$$

$$\left.,\left[S_t\sqrt{1-\sqcap_{1\le i_1<\ldots<i_k\le i_n}^{n}\left(1-\left(\sqcap_{j=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\frac{1}{k}nr_{ij}}\right)^2\right)}^{\frac{1}{c_n^k}},S_t\sqrt{1-\sqcap_{1\le i_1<\ldots<i_k\le i_n}^{n}\left(1-\left(\sqcap_{j=1}^{n}\left(\frac{\varepsilon_i}{t}\right)^{\frac{1}{k}nr_{ij}}\right)^2\right)}^{\frac{1}{c_n^k}}\right]\right),$$

$$\left(S_t\sqrt{1-\sqcap_{1\le i_1<\ldots<i_k\le i_n}^{n}\left(1-\left(\sqcap_{j=1}^{n}\left(\frac{\delta_i}{t^2}\right)^{\frac{1}{k}nr_{ij}}\right)^2\right)}^{\frac{1}{c_n^k}},S_t\sqrt{1-\sqcap_{1\le i_1<\ldots<i_k\le i_n}^{n}\left(1-\left(\sqcap_{j=1}^{n}\left(\frac{\mu_i}{t^2}\right)^{\frac{1}{k}nr_{ij}}\right)^2\right)}^{\frac{1}{c_n^k}}\right.$$

$$\left.\left.S_t\sqrt{1-\sqcap_{1\le i_1<\ldots<i_k\le i_n}^{n}\left(1-\left(\sqcap_{j=1}^{n}\left(\frac{\varsigma_i}{t^2}\right)^{\frac{1}{k}nr_{ij}}\right)^2\right)}^{\frac{1}{c_n^k}}\right),\right.$$

Proof.

$$\eta\gamma_{ij}\eta_{ij} = \left\{ \begin{array}{l} \left[S_{t\left(\frac{\theta_i}{t}\right)^{\eta\gamma_{ij}}}, S_{t\left(\frac{\tau_i}{t}\right)^{\eta\gamma_{ij}}} \right], \left[S_{t\sqrt{\left(1-\left(1-\frac{\phi^2}{t^2}\right)\right)^{\eta\gamma_{i_j}}}}, S_{t\sqrt{\left(1-\left(1-\frac{\psi^2}{t^2}\right)\right)^{\eta\gamma_{i_j}}}} \right], \\ , \left[S_{t\sqrt{\left(1-\left(1-\frac{\phi^2}{t^2}\right)\right)^{\eta\gamma_{i_j}}}}, S_{t\sqrt{\left(1-\left(1-\frac{\psi^2}{t^2}\right)\right)^{\eta\gamma_{i_j}}}} \right], \\ \left(S_{t\left(\frac{\delta_i}{t}\right)^{\eta\gamma_{ij}}}, S_{t\sqrt{\left(1-\left(1-\frac{\mu^2}{t^2}\right)\right)^{\eta\gamma_{i_j}}}}, S_{t\sqrt{\left(1-\left(1-\frac{\varepsilon^2}{t^2}\right)\right)^{\eta\gamma_{i_j}}}} \right) \end{array} \right.$$

and,

$$\otimes_{j=1}^{k} \eta\gamma_{ij}\eta_{ij} = \left\{ \begin{array}{l} \left[S_{t\left(\sqcap_{j=1}^{k}\left(\frac{\theta_i}{t}\right)\right)^{\eta\gamma_{ij}}}, S_{t\left(\sqcap_{j=1}^{k}\left(\frac{\tau_i}{t}\right)\right)^{\eta\gamma_{ij}}} \right], \left[S_{t\sqrt{\left(1-\sqcap_{j=1}^{k}\left(1-\frac{\phi^2}{t^2}\right)\right)^{\eta\gamma_{i_j}}}}, S_{t\sqrt{\left(1-\sqcap_{j=1}^{k}\left(1-\frac{\psi^2}{t^2}\right)\right)^{\eta\gamma_{i_j}}}} \right], \\ , \left[S_{t\sqrt{\left(1-\sqcap_{j=1}^{k}\left(1-\frac{\phi^2}{t^2}\right)\right)^{\eta\gamma_{i_j}}}}, S_{t\sqrt{\left(1-\sqcap_{j=1}^{k}\left(1-\frac{\psi^2}{t^2}\right)\right)^{\eta\gamma_{i_j}}}} \right], \\ \left(S_{t\left(\sqcap_{j=1}^{k}\left(\frac{\delta_i}{t}\right)\right)^{\eta\gamma_{ij}}}, S_{t\sqrt{\left(1-\sqcap_{j=1}^{k}\left(1-\frac{\mu^2}{t^2}\right)\right)^{\eta\gamma_{i_j}}}}, S_{t\sqrt{\left(1-\left(1-\sqcap_{j=1}^{k}\frac{\varepsilon^2}{t^2}\right)\right)^{\eta\gamma_{i_j}}}} \right) \end{array} \right.$$

Then,

$$(\otimes_{j=1}^{k}\eta\gamma_{ij}\eta_{ij})^{1/k} = \left\{ \begin{array}{l} \left[S_{t\left(\sqcap_{j=1}^{k}\left(\frac{\theta_i}{t}\right)\right)^{\frac{1}{k}\eta\gamma_{ij}}}, S_{t\left(\sqcap_{j=1}^{k}\left(\frac{\tau_i}{t}\right)\right)^{\frac{1}{k}\eta\gamma_{ij}}} \right], \\ \left[S_{t\sqrt{\left(1-\sqcap_{j=1}^{k}\left(1-\frac{\phi^2}{t^2}\right)\right)^{\frac{1}{k}\eta\gamma_{i_j}}}}, S_{t\sqrt{\left(1-\sqcap_{j=1}^{k}\left(1-\frac{\psi^2}{t^2}\right)\right)^{\frac{1}{k}\eta\gamma_{i_j}}}} \right], \\ , \left[S_{t\sqrt{\left(1-\sqcap_{j=1}^{k}\left(1-\frac{\phi^2}{t^2}\right)\right)^{\frac{1}{k}\eta\gamma_{i_j}}}}, S_{t\sqrt{\left(1-\sqcap_{j=1}^{k}\left(1-\frac{\psi^2}{t^2}\right)\right)^{\frac{1}{k}\eta\gamma_{i_j}}}} \right], \\ \left(S_{t\left(\sqcap_{j=1}^{k}\left(\frac{\delta_i}{t}\right)\right)^{\frac{1}{k}\eta\gamma_{ij}}}, S_{t\sqrt{\left(1-\sqcap_{j=1}^{k}\left(1-\frac{\mu^2}{t^2}\right)\right)^{\frac{1}{k}\eta\gamma_{i_j}}}}, S_{t\sqrt{\left(1-\left(1-\sqcap_{j=1}^{k}\frac{\varepsilon^2}{t^2}\right)\right)^{\frac{1}{k}\eta\gamma_{i_j}}}} \right) \end{array} \right.$$

Furthermore

$$\oplus_{1\leq i_1<\ldots<i_k\leq i_n}(\otimes_{j=1}^{k}\eta\gamma_{ij}\eta_{ij})^{1/k}$$

$$
= \left\{ \begin{array}{l}
\left[S_{t}\sqrt{\left(1-\sqcap_{1\le i_1<\ldots<i_k\le in}\left(1-\sqcap_{j=1}^{k}\frac{\theta_{i_j}^{2}}{t^{2}}\right)\right)^{\frac{1}{k}^{\eta\gamma_{i_j}}}}, S_{t}\sqrt{\left(1-\sqcap_{1\le i_1<\ldots<i_k\le in}\left(1-\sqcap_{j=1}^{k}\frac{\tau_{i_j}^{2}}{t^{2}}\right)\right)^{\frac{1}{k}^{\eta\gamma_{i_j}}}} \right], \\
\left[S_{t}\sqrt{\left(1-\sqcap_{1\le i_1<\ldots<i_k\le in}\left(1-\sqcap_{j=1}^{k}\frac{\phi_{i_j}^{2}}{t^{2}}\right)\right)^{\frac{1}{k}^{\eta\gamma_{i_j}}}}, S_{t}\sqrt{\left(1-\sqcap_{1\le i_1<\ldots<i_k\le in}\left(1-\sqcap_{j=1}^{k}\frac{\psi_{i_j}^{2}}{t^{2}}\right)\right)^{\frac{1}{k}^{\eta\gamma_{i_j}}}} \right], \\
, \left[S_{t}\sqrt{\left(1-\sqcap_{1\le i_1<\ldots<i_k\le in}\left(1-\sqcap_{j=1}^{k}\frac{\sigma_{i_j}^{2}}{t^{2}}\right)\right)^{\frac{1}{k}^{\eta\gamma_{i_j}}}}, S_{t}\sqrt{\left(1-\sqcap_{1\le i_1<\ldots<i_k\le in}\left(1-\sqcap_{j=1}^{k}\frac{\varepsilon_{i_j}^{2}}{t^{2}}\right)\right)^{\frac{1}{k}^{\eta\gamma_{i_j}}}} \right], \\
\left(S_{t}\sqrt{\left(1-\sqcap_{1\le i_1<\ldots<i_k\le in}\left(1-\sqcap_{j=1}^{k}\frac{\delta^{2}}{t^{2}}\right)\right)^{\frac{1}{k}^{\eta\gamma_{i_j}}}}, S_{t}\sqrt{\left(1-\sqcap_{1\le i_1<\ldots<i_k\le in}\left(1-\sqcap_{j=1}^{k}\frac{\mu^{2}}{t^{2}}\right)\right)^{\frac{1}{k}^{\eta\gamma_{i_j}}}}, \right. \\
\left. , S_{t}\sqrt{\left(1-\sqcap_{1\le i_1<\ldots<i_k\le in}\left(1-\sqcap_{j=1}^{k}\frac{\varepsilon^{2}}{t^{2}}\right)\right)^{\frac{1}{k}^{\eta\gamma_{i_j}}}} \right)
\end{array} \right.
$$

$$
\frac{\oplus_{1\le i_1<\ldots<i_k\le i_n}\left(\otimes_{j=1}^{k}\eta\gamma_{ij}\eta_{ij}\right)^{1/k}}{C_n^k}
$$

$$
=\left\{\begin{array}{l}
\left[S_{\sqrt[t]{\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\frac{\theta_{i_j}^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}},S_{\sqrt[t]{\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\frac{\tau_{i_j}^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}}\right],\\
\left[S_{\sqrt[t]{\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\frac{\phi_{i_j}^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}},S_{\sqrt[t]{\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\frac{\psi_{i_j}^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}}\right],\\
,\left[S_{\sqrt[t]{\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\frac{\sigma_{i_j}^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}},S_{\sqrt[t]{\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\frac{\varepsilon_{i_j}^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}}\right],\\
\left(S_{\sqrt[t]{\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\frac{\delta^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}},S_{\sqrt[t]{\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\frac{\mu^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}},\right.\\
\left.,S_{\sqrt[t]{\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\frac{\varepsilon^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}}\right)
\end{array}\right.
$$

□

2.3.7 Theorems

Theorem 3. *Suppose* $\eta_i = (i = 1, 2, 3, ...n)$ *be a collection of CLSFVs,iff* $\eta_i = \langle([S_{\theta_i}, S_{\tau_i}], [S_{\sigma_i}, S_{\varepsilon_i}]), (S_{\delta_i}, S_{\mu_i}, S_{\varsigma_i})\rangle (i = 1, 2, 3, 4, \ldots, n) n$ *for all i,then*

$$CLSFHM^k(\eta_1, \eta_2, ..., \eta_n) = \eta$$

Proof. Since $\eta_i = \eta = \langle([S_{\theta_i}, S_{\tau_i}], [S_{\sigma_i}, S_{\varepsilon_i}]), (S_{\delta_i}, S_{\mu_i}, S_{\varsigma_i})\rangle$ holds for $i = (1, 2, 3, ...n)$, so $Sup(\eta_i, \eta_j) = 1$ for$i = 1, 2, 3, ...n$ and $j = 1, 2, 3, ...n$.Then $r_i = (1, 2, 3, ...n)$ by using the statement of above theorem, we have, □

2.3.8 Theorems

Theorem 4. *Let* $\eta_i = \eta = \langle([S_{\theta_i}, S_{\tau_i}], [S_{\sigma_i}, S_{\varepsilon_i}]), (S_{\delta_i}, S_{\mu_i}, S_{\varsigma_i})\rangle$ *where* $i = 1, 2, 3, \ldots, n$*be a collection of CLSFVs, then*

$$CLSFPHM(\eta^-, \eta^-, \ldots, \eta^-) \leq CLSFPHM(\eta_1, \eta_2, \ldots, \eta_n) \leq CLSFPHM(\eta^+, \eta^+, \ldots, \eta^+)$$

Where,

$$\eta^- = \eta_i = \langle([S_{min_{i=1}^n \theta_i}, S_{min_{i=1}^n \tau_i}], [S_{min_{i=1}^n \phi_i}, S_{min_{i=1}^n \psi_i}], [S_{max_{i=1}^n \sigma_i}, S_{max_{i=1}^n \varepsilon_i}]) \\ (S_{min_{i=1}^n \delta_i}, S_{min_{i=1}^n \mu_i}, S_{max_{i=1}^n \varsigma_i})\rangle$$

and

$$\eta^+ = \langle([S_{max_{i=1}^n \theta_i}, S_{max_{i=1}^n \tau_i}], [S_{min_{i=1}^n \phi_i}, S_{min_{i=1}^n \psi_i}], [S_{min_{i=1}^n \sigma_i}, S_{min_{i=1}^n \varepsilon_i}]) \\ (S_{max_{i=1}^n \delta_i}, S_{min_{i=1}^n \mu_i}, S_{min_{i=1}^n \varsigma_i})\rangle$$

Proof. We have

$$S_{t\sqrt{\left(1-\sqcap_{1\leq i_1<\ldots<i_k\leq in}\left(1-\sqcap_{j=1}^{k}\frac{\theta_{i_j}^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}} \leq S_{t\sqrt{\left(1-\left(1-\frac{c\theta_{i_j}^2}{t^2}\right)^{\frac{1}{k}k\frac{1}{n}n}\right)^{\frac{1}{C_n^k}C_n^k}}} = S_{max_{i=1}^n \theta_i}$$

Similarly, we can get

$$S_{t\sqrt{\left(1-\sqcap_{1\leq i_1<\ldots<i_k\leq in}\left(1-\sqcap_{j=1}^{k}\frac{\phi_{i_j}^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}} \geq min_{i=1}^n \phi_i,$$

$$S_{t\sqrt{\left(1-\sqcap_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\sqcap_{j=1}^{k}\frac{\psi_{i_j}^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}} \leq S_{max_{i=1}^n \psi_i}$$

And

$$S_{t\sqrt{\left(1-\sqcap_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\sqcap_{j=1}^{k}\frac{\varepsilon_{i_j}^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}} \geq min_{i=1}^n \varepsilon_i$$

$$S_{t\sqrt{\left(1-\sqcap_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\sqcap_{j=1}^{k}\frac{\delta_{i_j}^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}} \leq S_{max_{i=1}^n \delta_i}$$

$$S_{t\sqrt{\left(1-\sqcap_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\sqcap_{j=1}^{k}\frac{\varsigma_{i_j}^2}{t^2}\right)^{\frac{1}{k}\eta\gamma_{i_j}}\right)^{\frac{1}{C_n^k}}}} \geq min_{i=1}^n \varsigma_i$$

According to definition $CLSFPHM^{(k)}(\eta_1, \eta_2, \ldots, \eta_n) \leq \eta^+$an be obtained. Similarly, we can get $\eta^- \leq CLSFPHM^{(k)}(\eta_1, \eta_2, \ldots, \eta_n)$. Thus according to Theorem 7,
$CLSFPHM(\eta^+, \eta^+, \ldots, \eta^+) \geq CLSFPHM(\eta_1, \eta_2, \ldots, \eta_n) \geq CLSFPHM(\eta^-, \eta^-, \ldots, \eta^-)$ is proved. For different values of the parameter k, we have

Case 1.

For k=1, the CLSFPHM operator reduces the CLSFPA
$CLSFPHM^{(1)}(\eta_1, \eta_2, \ldots, \eta_n)$ =

$$\left\langle\left(\left[S_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\theta_i^2}{t^2}\right)\right)^{\gamma_i}}}, S_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\tau_i^2}{t^2}\right)\right)^{\gamma_i}}}\right], \left[S_{t\sqcap_{i=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\gamma_i}}, S_{t\sqcap_{i=1}^{n}\left(\frac{\varepsilon_i}{t}\right)^{\gamma_i}}\right],\right.\right.$$

$$\left.\left.\left[S_{t\sqcap_{i=1}^{n}\left(\frac{\phi_i}{t}\right)^{\gamma_i}}, S_{t\sqcap_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\gamma_i}}\right], \left(S_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\theta_i^2}{t^2}\right)\right)^{\gamma_i}}}, S_{t\sqcap_{i=1}^{n}\left(\frac{\mu_i}{t}\right)^{\gamma_i}}, S_{t\sqcap_{i=1}^{n}\left(\frac{\varsigma_i}{t}\right)^{\gamma_i}}\right)\right)\right\rangle$$

$= \oplus_{i=1}^{n} \gamma_i \eta_i = CLSFPA(\eta_1, \eta_2, \ldots, \eta_n)$.
In this case, if $Sup(\eta_i, \eta_j) = d (d > 0)$ for $i = 1, 2, 3, 4, \ldots, n$ and $j = 1, 2, 3, 4, \ldots, n (i \neq j)$, then the CLSFPHM operator reduces to the cubic linguistic spherical fuzzy geometric (CLSFG) operator, that is
$CLSFPHM(\eta_1, \eta_2, \ldots, \eta_n) =$

$$\left\langle \left(\left[s_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\theta_i^2}{t^2}\right)\right)^{\frac{1}{n}}}}, s_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\tau_i^2}{t^2}\right)\right)^{\frac{1}{n}}}} \right], \left[s_{t\sqcap_{i=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\frac{1}{n}}}, s_{t\sqcap_{i=1}^{n}\left(\frac{\varepsilon_i}{t}\right)^{\frac{1}{n}}} \right], \right.\right.$$

$$\left.\left. \left[s_{t\sqcap_{i=1}^{n}\left(\frac{\phi_i}{t}\right)^{\frac{1}{n}}}, s_{t\sqcap_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\frac{1}{n}}} \right], \left(s_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\theta_i^2}{t^2}\right)\right)^{\frac{1}{n}}}}, s_{t\sqcap_{i=1}^{n}\left(\frac{\mu_i}{t}\right)^{\frac{1}{n}}}, s_{t\sqcap_{i=1}^{n}\left(\frac{\varsigma_i}{t}\right)^{\frac{1}{n}}} \right) \right) \right\rangle$$

$= \frac{1}{n} \oplus_{i=1}^{n} \eta_i = CLSFA(\eta_1, \eta_2, \ldots, \eta_n)$

Case 2.

For $k = n$, the CLSFPHM operator reduces to the following:
$CLSFPHM^{(1)}(\eta_1, \eta_2, \ldots, \eta_n) =$

$$\left\langle \left(\left[s_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\theta_i^2}{t^2}\right)\right)^{\gamma_i}}}, s_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\tau_i^2}{t^2}\right)\right)^{\gamma_i}}} \right], \left[s_{t\sqcap_{i=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\gamma_i}}, s_{t\sqcap_{i=1}^{n}\left(\frac{\varepsilon_i}{t}\right)^{\gamma_i}} \right], \right.\right.$$

$$\left.\left. \left[s_{t\sqcap_{i=1}^{n}\left(\frac{\phi_i}{t}\right)^{\gamma_i}}, s_{t\sqcap_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\gamma_i}} \right], \left(s_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\theta_i^2}{t^2}\right)\right)^{\gamma_i}}}, s_{t\sqcap_{i=1}^{n}\left(\frac{\mu_i}{t}\right)^{\gamma_i}}, s_{t\sqcap_{i=1}^{n}\left(\frac{\varsigma_i}{t}\right)^{\gamma_i}} \right) \right) \right\rangle$$

$= \oplus_{i=1}^{n} \gamma_i \eta_i = CLSFPA(\eta_1, \eta_2, \ldots, \eta_n)$ In this case, if $Sup(\eta_i, \eta_j) = d (d > 0)$for $i = 1, 2, 3, 4, \ldots, n$ and $j = 1, 2, 3, 4, \ldots, n (i \neq j)$, then the CLSFPHM operator reduces to the cubic linguistic spherical fuzzy geometric (CLSFG) operator, that is

$CLSFPHM(\eta_1, \eta_2, \ldots, \eta_n) =$

$$\left\langle \left(\left[S_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\theta_i^2}{t^2}\right)\right)^{\frac{1}{n}}}}, S_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\tau_i^2}{t^2}\right)\right)^{\frac{1}{n}}}} \right], \left[S_{t\sqcap_{i=1}^{n}\left(\frac{\sigma_i}{t}\right)^{\frac{1}{n}}}, S_{t\sqcap_{i=1}^{n}\left(\frac{\varepsilon_i}{t}\right)^{\frac{1}{n}}} \right], \right.\right.$$

$$\left.\left. \left[S_{t\sqcap_{i=1}^{n}\left(\frac{\phi_i}{t}\right)^{\frac{1}{n}}}, S_{t\sqcap_{i=1}^{n}\left(\frac{\psi_i}{t}\right)^{\frac{1}{n}}} \right], \left(S_{t\sqrt{\left(1-\sqcap_{i=1}^{n}\left(1-\frac{\theta_i^2}{t^2}\right)\right)^{\frac{1}{n}}}}, S_{t\sqcap_{i=1}^{n}\left(\frac{\mu_i}{t}\right)^{\frac{1}{n}}}, S_{t\sqcap_{i=1}^{n}\left(\frac{\varsigma_i}{t}\right)^{\frac{1}{n}}} \right) \right) \right\rangle$$

$= \frac{1}{n} \oplus_{i=1}^{n} \eta_i = CLSFA(\eta_1, \eta_2, \ldots, \eta_n)$ □

2.3.9 Theorems

Theorem 5. *Let* $\eta_i = \langle([S_{\theta_i}, S_{\tau_i}], [S_{\sigma_i}, S_{\varepsilon_i}]), (S_{\delta_i}, S_{\varsigma_i})\rangle (i = 1, 2, 3, 4, \ldots, n)$ *be a collection of CLSFVs defined on* $\tilde{S} = \{s_\beta | s_\theta \leq s_\beta \leq s_t, \beta \in [0, t]\}$*,, that is linguistic term set,aggregated value by the CLSFPWHM operator is still a CLSFV and*

2.3.10 Theorems

Boundedness

Theorem 6. *Let* $\eta_i = \eta = \langle([S_{\theta_i}, S_{\tau_i}], [S_{\sigma_i}, S_{\varepsilon_i}]), (S_{\delta_i}, S_{\mu_i}, S_{\varsigma_i})\rangle$ *where* $i = 1, 2, 3, \ldots, n$*be a collection of CLSFVs, then*

$$CLSFPA(\eta^-, \eta^-, \ldots, \eta^-) \leq CLSFPA(\eta_1, \eta_2, \ldots, \eta_n) \leq CLSFPA(\eta^+, \eta^+, \ldots, \eta^+)$$

Where,

$$\eta^- = \eta_i = \langle([S_{min_{i=1}^n \theta_i}, S_{min_{i=1}^n \tau_i}], [S_{min_{i=1}^n \phi_i}, S_{min_{i=1}^n \psi_i}], [S_{max_{i=1}^n \sigma_i}, S_{max_{i=1}^n \varepsilon_i}]) \\ (S_{min_{i=1}^n \delta_i}, S_{min_{i=1}^n \mu_i}, S_{max_{i=1}^n \varsigma_i})\rangle$$

and

$$\eta^+ = \langle([S_{max_{i=1}^n \theta_i}, S_{max_{i=1}^n \tau_i}], [S_{min_{i=1}^n \phi_i}, S_{min_{i=1}^n \psi_i}], [S_{min_{i=1}^n \sigma_i}, S_{min_{i=1}^n \varepsilon_i}]) \\ (S_{max_{i=1}^n \delta_i}, S_{min_{i=1}^n \mu_i}, S_{min_{i=1}^n \varsigma_i})\rangle$$

2.3.11 Theorems

Theorem 7. *Consider we have a CITSs* $= \{S_\varepsilon | \varepsilon \in [0, 1]\}$*. Where t is a positive integer and n LqROFNs* $\alpha_i = (s_{(a_i)}, s_{(b_i)})(i = 1, 2, 3, \ldots, n)$ *havings* $s_{(a_i)} \in S$*and* $s_{(b_i)} \in S$*. if* $\alpha_i = \alpha = (s_a, s_(b))$ *then*

$$LqROFWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha$$

.

Proof. According to the NIOLa, we have

$$
\begin{aligned}
LqROFWA(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \left(S_{t\left(1-\sqcap_{i=1}^{n}\left(1-\frac{a_i}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}}, S_{t\sqcap_{i=1}^{m}\left(\frac{b_i}{t}\right)^{w_i}} \right) \\
&= \left(S_{t\left(1-\sqcap_{i=1}^{n}\left(1-\frac{a_i}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}}, S_{t\sqcap_{i=1}^{m}\left(\frac{b_i}{t}\right)^{w_i}} \right) \\
&= \left(S_{t\left(1-\sqcap_{i=1}^{n}\left(1-\frac{a_i}{t^q}\right)^{\Sigma_{i=1}^{n} w_i}\right)^{\frac{1}{q}}}, S_{t\sqcap_{i=1}^{m}\left(\frac{b_i}{t}\right)^{\Sigma_{i=1}^{n} w_i}} \right) \\
&= \left(S_a, S_b\right) = \alpha
\end{aligned}
$$

□

2.3.12 Theorems

Theorem 8. *Consider we have a CLTSs = $\{S_\varepsilon | \varepsilon \in [0,1]\}$. Where t is a non-negative integer and n LqROFNs $\alpha_i = (s_{(a_i)}, s_{(b_i)})(i = 1, 2, 3, \ldots, n)$ havings $s_{(a_i)} \in$ Sand $s_{(b_i)} \in S$. if $\alpha_i = \alpha = (s_a, s_(b))$ then*

$$
LqROFWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(S_{t\left(1-\sqcap_{i=1}^{n}\left(1-\frac{a_i}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}}, S_{t\sqcap_{i=1}^{m}\left(\frac{b_i}{t}\right)^{w_i}} \right)
$$

Proof. When $n = 1$, we have

$$
LqROFWA(\alpha_1) = w_i\alpha_1 = w_1(s_{a_i}, s_{b_i}) = \left(S_{t\left(1-\left(1-\frac{a_1}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}}, S_{t\left(\frac{b_i}{t}\right)^{w_i}} \right)
$$

When n = 1, we have

$$w_1\alpha_1 = \left(S_{t\left(1-\left(1-\frac{a_1}{t^q}\right)^{w_1}\right)^{\frac{1}{q}}}, S_{t\left(\frac{b_i}{t}\right)^{w_1}}\right)$$

and

$$w_2\alpha_1 = \left(S_{t\left(1-\left(1-\frac{a_2}{t^q}\right)^{w_2}\right)^{\frac{1}{q}}}, S_{t\left(\frac{b_i}{t}\right)^{w_2}}\right)$$

According to NIOLs, we have

$$LqROFWA = \left(S_{t\left(\frac{\left(t\left(1-\left(1-\frac{a_1}{t^q}\right)^{w_1}\right)^{\frac{1}{q}}\right)^q}{t^q}+\frac{\left(t\left(1-\left(1-\frac{a_2}{t^q}\right)^{w_2}\right)^{\frac{1}{q}}\right)^q}{t^q}\right)}\right)$$

$$LqROFWA(\alpha_1, \alpha_1) = w_1\alpha_1 \oplus w_2\alpha_2 =$$

$$\left(S_{t\left(t\left(1-\left(1-\frac{a_1}{t^q}\right)^{w_1}\right)^{\frac{1}{q}}\right)}, S_{t\left(\frac{b_i}{t}\right)^{w_2}}\right)$$

□

2.3.13 Theorems

Theorem 9. *Consider we have a CLTS* $\{S = s_{\varepsilon}|\varepsilon \in [0,t]\}$ *where* t *is positive and two sets of* $\alpha_i = (s_{a_i}, S_{b_i})$ *and* $\alpha_i^{'} = (s_{a_i^{'}}, S_{b_i^{'}})(i = 1, 2, ..., n)$.
If $s_{a_i} \geq s_{a_i^{'}}; s_{b_i} \geq s_{b_i^{'}}$ *the we have*

$$LqROFWG(\alpha_1, \alpha_2, ...\alpha_n) \geq LqROFWG(\alpha_1^{'}, \alpha_2^{'}, ...\alpha_n^{'})$$

.

Proof. For all i, $s_{a_i} \geq s_{a_i^{'}}; s_{b_i} \geq s_{b_i^{'}}$. Then we have,

$$1 - \frac{a_i^q}{t^q} \leq 1 - \frac{a_i^{'q}}{t^q}$$

This implies that $\left(1 - \sqcap_{i=1}^{n}\left(1 - \frac{a_i^q}{t^q}\right)^{w_i}\right)^{\frac{1}{q}} \geq \left(1 - \sqcap_{i=1}^{n}\left(1 - \frac{a_i^{'q}}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}$

and

$$\sqcap_{i=1}^{n}\left(\frac{b_i}{t}\right)^{w_i} \leq \sqcap_{i=1}^{n}\left(\frac{b_i^{'}}{t}\right)^{w_i}$$

Thus,

$$t^q + t^q\left(\left(1 - \sqcap_{i=1}^{n}\left(1 - \frac{a_i^q}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}\right)^q - t^q\left(\sqcap_{i=1}^{n}\left(\frac{b_i}{t}\right)^{w_i}\right)^q \geq$$

$$t^q + t^q\left(\left(1 - \sqcap_{i=1}^{n}\left(1 - \frac{a_i^{'q}}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}\right)^q - t^q\left(\sqcap_{i=1}^{n}\left(\frac{b_i^{'}}{t}\right)^{w_i}\right)^q$$

$$\left(\frac{t^q + t^q\left(\left(1 - \sqcap_{i=1}^{n}\left(1 - \frac{a_i^{'q}}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}\right)^q - t^q\left(\sqcap_{i=1}^{n}\left(\frac{b_i^{'}}{t}\right)^{w_i}\right)^q}{2}\right)^{\frac{1}{q}} \geq$$

$$\left(\frac{t^q + t^q\left(\left(1 - \sqcap_{i=1}^{n}\left(1 - \frac{a_i^q}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}\right)^q - t^q\left(\sqcap_{i=1}^{n}\left(\frac{b_i}{t}\right)^{w_i}\right)^q}{2}\right)^{\frac{1}{q}}$$

Namely,

$$S\left(\frac{t^q + t^q\left(\left(1 - \sqcap_{i=1}^{n}\left(1 - \frac{a_i^q}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}\right)^q - t^q\left(\sqcap_{i=1}^{n}\left(\frac{b_i}{t}\right)^{w_i}\right)^q}{2}\right)^{\frac{1}{q}} \geq$$

$$S\left(\frac{t^q + t^q\left(\left(1 - \sqcap_{i=1}^{n}\left(1 - \frac{a_i'^q}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}\right)^q - t^q\left(\sqcap_{i=1}^{n}\left(\frac{b_i'}{t}\right)^{w_i}\right)^q}{2}\right)^{\frac{1}{q}}$$

Let $a = LqROFWG(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $a' = LqROFWG(\alpha_1', \alpha_2', \ldots, \alpha_n')$; then we have

$$D(a) = S\left(\frac{t^q + t^q\left(\left(1 - \sqcap_{i=1}^{n}\left(1 - \frac{a_i^q}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}\right)^q - t^q\left(\sqcap_{i=1}^{n}\left(\frac{b_i}{t}\right)^{w_i}\right)^q}{2}\right)^{\frac{1}{q}}$$

;

$$\geq D(a') = S\left(\frac{t^q + t^q\left(\left(1 - \sqcap_{i=1}^n\left(1 - \frac{a_i'^q}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}\right)^q - t^q\left(\sqcap_{i=1}^n\left(\frac{b_i'}{t}\right)^{w_i}\right)^q}{2}\right)^{\frac{1}{q}}$$

If$D(a) \geq D(a')$ then, $LqROFWG(\alpha_1, \alpha_2, \ldots, \alpha_n) \geq a' = LqROFWG(\alpha_1', \alpha_2', \ldots, \alpha_n')$
If$D(a) = D(a')$,then

,

$$S\left(\frac{t^q + t^q\left(\left(1 - \sqcap_{i=1}^n\left(1 - \frac{a_i^q}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}\right)^q - t^q\left(\sqcap_{i=1}^n\left(\frac{b_i}{t}\right)^{w_i}\right)^q}{2}\right)^{\frac{1}{q}}$$

$$= S\left(\frac{t^q + t^q\left(\left(1 - \sqcap_{i=1}^n\left(1 - \frac{a_i'^q}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}\right)^q - t^q\left(\sqcap_{i=1}^n\left(\frac{b_i'}{t}\right)^{w_i}\right)^q}{2}\right)^{\frac{1}{q}}$$

According to he condition of $s_{a_i} \geq s_{a_i'}$ and $s_{b_i} \geq s_{b_i'}$, we have,

$$\left(1 - \sqcap_{i=1}^n\left(1 - \frac{a_i^q}{t^q}\right)^{w_i}\right)^{\frac{1}{q}} = \left(1 - \sqcap_{i=1}^n\left(1 - \frac{a_i'^q}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}; \sqcap_{i=1}^n\left(\frac{b_i}{t}\right)^{w_i} = \sqcap_{i=1}^n\left(\frac{b_i'}{t}\right)^{w_i}$$

Thus

$$j(a) = S\left(\frac{t^q + t^q\left(\left(1-\sqcap_{i=1}^{n}\left(1-\frac{a_i^q}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}\right)^q - t^q\left(\sqcap_{i=1}^{n}\left(\frac{b_i}{t}\right)^{w_i}\right)^q}{2}\right)^{\frac{1}{q}}$$

$$j(a') = S\left(\frac{t^q + t^q\left(\left(1-\sqcap_{i=1}^{n}\left(1-\frac{a_i'^q}{t^q}\right)^{w_i}\right)^{\frac{1}{q}}\right)^q - t^q\left(\sqcap_{i=1}^{n}\left(\frac{b_i'}{t}\right)^{w_i}\right)^q}{2}\right)^{\frac{1}{q}}$$

Therefore,

$LqROFWG(\alpha_1, \alpha_2, \ldots, \alpha_n) \geq LqROFWG(\alpha_1', \alpha_2', \ldots, \alpha_n')$. That proofs the claimed statement. □

3 A novel approach to MAGDM based on the proposed operators

We presented a newly made method to MAGDM, which is based on the CLSFPWA and the CLSFPWHM. Suppose $A = \{A_1, A_2, A_3, \ldots, A_m\}$ be a collection of alternatives and $C = \{C_1, C_2, C_3, \ldots, C_n\}$ be a set of attributes. Let $D = \{D_1, D_2, D_3, \ldots, D_l\}$ be a collection of degrees of membership. For attribute $C_j (j = 1, 2, 3, \ldots, n)$ of alternative $D_h (h = 1, 2, 3, \ldots, l)$ expresses his assessment by $\eta_{ij}^h = \langle [S_{\theta_{ij}h}, S_{\tau_{ij}h}], [S_{\sigma_{ijh}}, S_{\varepsilon_{ij}h}], [S_{\delta_{ij}h}, S_{\varsigma_{ij}h}] \rangle$, which is a cubic linguistic spherical fuzzy value (CLSFV) defined on $\tilde{S} \cong \{s_\beta | s_\theta \leq s_\beta \leq s_t, \beta \in [0, t]\}$, which is also a cubic linguistic spherical fuzzy value (CLSFV). Thus, for each degree of membership an unique cubic linguistic spherical fuzzy decision matrix can be obtained, such as $R^h = (\eta_{ijh})_{(mn)}$. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_l)$ is the weight vector of degrees of membership, which satisfies the condition that $\lambda_h \in [0, 1]$ and $\sum_{h=1}^{l} \lambda_h = 1$ suppose $w = (w_1, w_2, w_3, \ldots, w_n)$ is the weight of attributes, satisfying $w_j \in [0, 1]$ and $\sum_{j=1}^{l} w_j = 1$ To solve MAGDM problem related to CLSFSs, the main steps are given in the following. Step1. The original decision matrices must be standardized. There are two types of attributes in real decision-making problems.

1 : benefit attributes (I_1)

2 : cost attributes (I_2)

By using the following formula, the original decision matrices should be normalized:

$$\eta_{ij}^h = \begin{cases} \langle [S_{\theta_{ij}h}, S_{\tau_{ij}h}], [S_{\sigma_{ijh}}, S_{\varepsilon_{ij}h}], [S_{\delta_{ij}h}, S_{\varsigma_{ij}h}] \rangle & c_j \in I_1 \\ \langle [S_{\theta_{ij}h}, S_{\tau_{ij}h}], [S_{\sigma_{ijh}}, S_{\varepsilon_{ij}h}], [S_{\delta_{ij}h}, S_{\varsigma_{ij}h}] \rangle & c_j \in I_2 \end{cases}$$

Step2: Compute $Sup(\eta_{ij}^{h}, \eta_{ij}^{z})$ according to the following formula $Sup(\eta_{ij}^{h}, \eta_{ij}^{z}) = 1 - d(\eta_{ij}^{h}, \eta_{ij}^{z})$ $(h, z = 1, 2, 3, \ldots, l; h \neq z)$, Where $1 - d(\eta_{ij}^{h}, \eta_{ij}^{z})$ is the distance between η_{ij}^{h} and η_{ij}^{z}
Step3: ompute the overall supports $T(\eta_{ij}^{h})$ by

$$T(\eta_{ij}^{h}) = \sum_{h=1;h\neq z}^{l} Sup(\eta_{ij}^{h}, \eta_{ij}^{h})$$

Step4: Use following equation to compute the power weight of η_{ij}^{h}provided by membership degree D_h.

$$\gamma(\eta_{ij}^{h}) = \frac{\lambda_n(1 + T(\alpha_{ij}^{h}))}{\sum_{h=1;h\neq z}^{1} \lambda_n(1 + T(\alpha_{ij}^{h}))}$$

Step5: Use the CLSFPWA operator to calculate aggregated individual decision matrix.
$R^h = (\eta_{ij}^{h})_(mn)..(h = 1, 2, \ldots, l)$to obtain the comprehensive decision matrix $R^h = (\eta_{ij})_{(m\times n)}$:

$$\eta_{ij} = CLPEPWA(\eta_{ij}^{1}, \eta_{ij}^{2}, \ldots, \eta_{ij}^{h}).$$

Step6: Compute $Sup(\eta_{ij}, \eta_i s)$ by
$Sup(\eta_{ij}^{h}, \eta_{is}^{z}) = 1 - d(\eta_{ij}^{h}, \eta_{ij}^{z})$ where$(i = 1, 2, 3, \ldots, m)(j, s1, 2, 3, 4 \ldots, n; j \neq s.)$
Step7: Compute the all supports $T(\eta_{ij}^{h})$ by using the formula:

$$T(\eta_{ij}) = \sum_{j=1;j\neq s}^{l} Sup(\eta_{ij}, \eta_{is})$$

Step8: To compute the power weight of η_{ij}^{h} provided by degree of membershi D_h, use following equation:

$$w_{ij}^{h} = \frac{w_j(1 + T(\alpha_{ij}^{h}))}{\sum_{h=1;h\neq z}^{1} w_j(1 + T(\alpha_{ij}^{h}))}$$

Step9: Use the CLSEPWHM operator to aggregate attribute values $\eta_i = CLPEPWHM(\eta_{i1}, \eta_{i2}, \ldots . \eta_{in})$.
Step10: Compute the score of $\eta_i (i = 1, 2, 3, \ldots, m)$.

4 Conclusion

We proposed the cubic linguistic spherical fuzzy sets (CLSFSs) by using previous theories of fuzzy sets. Our generalization is very useful and reliable to describe the uncertainty as well as hesitation of membership degree in MAGDM problems. The concept of CLSFSs is constructed by LSFSs and uncertain LSFSs which absorbs the advantages of both. It is obvious that uncertain linguistic spherical fuzzy sets (CLSFSs) are more effective and reliable than linguistic spherical fuzzy sets (LSFSs). So, cubic linguistic spherical fuzzy sets(CLSFSs) are more reliable to describe the uncertainty as well as hesitation of membership degree in the MAGDM. Also we proposed the fundamental operation law for CLSFSs and a series of their average operators and presented their aggregation properties.

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