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On Fixed Points of Digraphs Over Lambert Type Map

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Abstract

Define $f(y) = y^2h^y$, where $h \in (Z/mZ)$, the Discrete Lambert Type Map (DLTM). For a set of vertices and edges over DLTM, diagraphs are obtained in which the vertices are from a whole range of residues modulo a fixed integer s, and edges are obtained when $f(y) \equiv v(mods^k)$ is solvable in Y and in terms of diophantine equation as well. In this paper we proposed new results for for fixed point of digraphs over DLTM.

Keywords: Fixed Point, Discrete Lambert Type Map, Residues.

Mathematics subject classification 2000: 05C25, 11E04, 20G15.

1 Introduction

It is well-known that the solution of exponential type function under congruences is not always easy. In fact, it is always a challenging problem in number theory. To find the solution of equations in which unknown appearing in exponential terms, the Lambert function We^W is helpful and have been used frequently by many researchers. These functions already have been enjoyed in [4,5] and in [17] as well. This is, sometime defined as well, $z = W(z)e^{W(z)}$, where z is a complex number. Before finding the fixed point, one should understand the meanings of fixed point with respect to the environment, we are using for. For a detail understanding of fixed points and integer classes, we suggest to read the references [1-3], [4,5,17],[18-20] and then [6-16], so that the reader could enjoy the reading of our proposed results.
A graph is an ordered pair $G(V, E)$ which consists on two sets V and E, where V is set of points named as set of vertices taking as the residues of any given fixed integer and E is a set of edges, obtained by using DLTM and a graph in which edges have direction is called directed graphs or digraphs.

2 Fixed Point.

A number w is said to be fixed point of DLTM iff $w^2g^w \equiv w(modp)$ for a positive integer p. In this paper, we attempt to find fixed points of graphs arising from DLTM. The following results are elaborating fixed points and image structures

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for specific numbers using DLTM graphs.

Theorem 2.1. *If $h \equiv 1 \bmod s$ and $h + x = s^2 + 2$ then x is fixed point of the graph (h, s^2). In this case, 0 and x are the only two fixed points.*

Proof.

$$\begin{aligned} \text{Let } h + x &= s^2 + 2 & (1) \\ \text{then } x &= s^2 + 2 - h & (2) \end{aligned}$$

As $h \equiv 1 \bmod s$, so there must exist an integer t such that $h = 1 + ts$. Putting in equation (2),

$$\begin{aligned} x &= s^2 + 2 - (1 + ts) \\ &= s^2 + 2 - 1 - ts \\ &\equiv 1 - ts \ (\text{mod } s^2) \\ \text{or } x &\equiv 1 - ts \ (mod\ s^2) & (3) \end{aligned}$$

Now using equation (3) in $f(x) = x^2 h^x \bmod s^2$, we get

$$\begin{aligned} f(x) &\equiv ((1 - ts)^2(1 + ts)^{(1-ts)}) \bmod s^2 \\ &\equiv ((1 + t^2s^2 - 2ts)(1 + ts(1 - ts) + terms \ \ involving \ \ s^2)) \bmod s^2 \\ &\equiv ((1 + t^2s^2 - 2ts)(1 + ts - t^2s^2 + terms \ \ involving \ \ s^2)) \bmod s^2 \\ &\equiv ((1 - 2ts)(1 + ts)) \bmod s^2 \\ &\equiv (1 - 2ts + ts - 2t^2s^2) \bmod s^2 \\ &\equiv (1 - ts) \bmod s^2 \\ &\equiv x \bmod s^2 \end{aligned}$$

This completes the proof. □

The Figure 1., depicts the above result.

Theorem 2.2. *Let f be a discrete Lambert Type Map. For an odd prime s, if $h = s - 2$, then h is always a fixed point of f.*

Proof. Let $f(t) = t^2 h^t (mod\ s)$ and $h = s - 2$.

$$\begin{aligned} f(s-2) &\equiv ((s-2)^2(s-2)^{s-2}) \ (mod\ s) \\ &\equiv (s-2)^{2+s-2})mod\ s) \\ &\equiv (s-2)^s (mod\ s). & (4) \end{aligned}$$

Now by Euler's Theorem, we know that $a^s \equiv a(mod\ s)$. But then the equation (4) yields,

$$f(s-2) \equiv (s-2)(mod\ s). \qquad (5)$$

Hence $s - 2$ is a fixed point. □

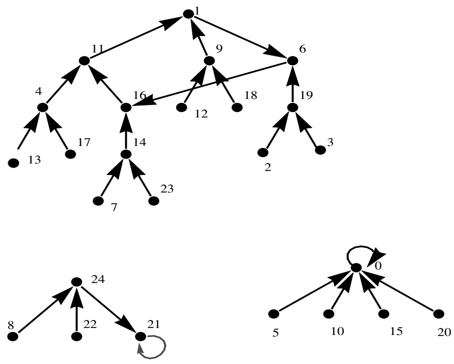

Figure 1: Shows the diagraph $G(6, 5^2)$

3 Multiplicative Order

Let g be any fixed integer and integer $m > 0$. Recall that a multiplicative order modulo m is a least positive integer β such that $g^\beta \equiv 1 \bmod m$. This is denoted by $Ord_m g = \beta$. By incorporating this definition, some other foxed points of the DLTM can be calculated. These are given in the following theorems as under:

Theorem 3.1. *Let f be a DLTM. If $Ord_{s^k} h = s^{k-1},\ k > 1$, then the diagraph $G(h, s^k)$ have two fixed points namely 0 and $s^2 - s + 1$. Also, all multiples of s maps on zero and make an independent component.*

Proof. Since $Ord_{s^k} h = s^{k-1},\ k > 1$, so in particular $Ord_{s+1^2} h = s+1$. Consider

$$f(s^2 - s + 1) = ((s^2 - s + 1)^2 (s+1)^{(s^2-s+1)}) \bmod s^2 \tag{6}$$

As $(s+1, s^2) = 1$, so by Eular's Theorem, we have $(s+1)^{\phi(s^2)} \equiv 1 \bmod s^2$. Putting in equation (6), we get

$$\begin{aligned} f(s^2 - s + 1) &\equiv ((s^2 - s + 1)^2 (s+1)) \bmod s^2 \\ &\equiv (1-s)^2(1+s) \bmod s^2 \\ &\equiv ((1 - 2s + s^2)(s+1)) \bmod s^2 \\ &\equiv (1-2s)(1+s) \bmod s^2 \\ &\equiv s + 1 - 2s^2 - 2s \bmod s^2 \\ &\equiv 1 - 2s^2 - s \bmod s^2 \\ &\equiv (1-s) \ \bmod s^2 \\ &\equiv (1 - s + s^2) \bmod s^2 \end{aligned}$$

so $(s^2 - s + 1)$ is fixed point. Also, it can easily be seen that $f(s^2\beta) \equiv (s^2\beta)^2 g^{s^2\beta} \equiv 0 \bmod s^2$ □

Figure 2., depicts the digraph $G(12, 11^2)$,where $|12| = 11$ and all multiple of 11<121 maps on zero. It has 4 cycles and the numbers 0 and 111 are the only fixed points.

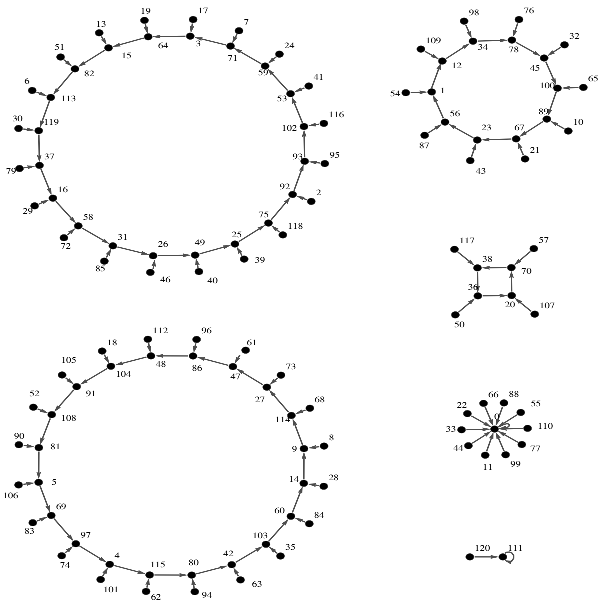

Figure 2: Shows the diagraph $G(12, 11^2)$

Theorem 3.2. *Let f be a DLTM and s be any odd prime. $Ord_{s^k}h = 2$ if and only if $h = s^k - 1$ for any integer k. In this case, 0 is only fixed point of the map.*

Proof. Let $Ord_{s^k}h = 2$. Then 2 is the least positive integer such that $h^2 = 1\ (mod\ s^k)$. This means that $h^2 = 1\ (mod\ s^k)$ or

$$\begin{aligned} h^2 &\equiv (-1)^2 (mod\ s^k) \\ &\equiv s^k - 1)^2\ (mod\ s^k) \\ \text{or}\ \ h^2 &\equiv (s^k - 1)^2\ (mod\ s^k) \end{aligned}$$

This surely gives $h = s^k - 1$.
Conversely, if we assume $h = s^k - 1$, then

$$\begin{aligned} h^2 &\equiv (s^k - 1)^2\ (mod\ s^k) \\ &\equiv (-1)^2\ (mod\ s^k) \\ &\equiv 1\ (mod\ s^k) \end{aligned}$$

Also, note that for any vertex h, $h^2 g^h \equiv g^h \not\equiv h\ (mod\ s^k)$. Thus, 0 is the only fixed point. □

If we take $s = 3, k = 3, h = 26$, then order of 26 is 2 mod 27, so Figure 3 elaborate Theorem 3.2.

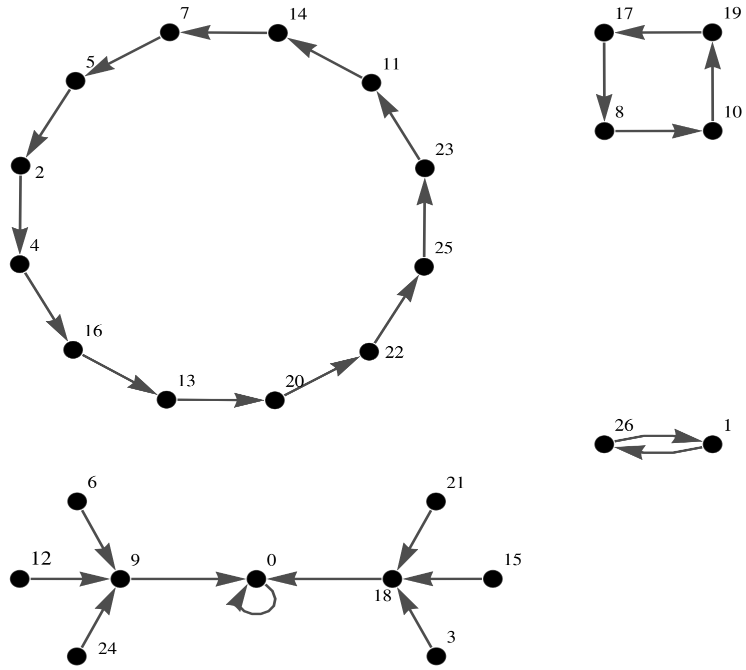

Figure 3: Shows the diagraph $G(26, 27)$

Theorem 3.3. *If t is a fixed point of n, then there must be some y in $G(n)$ such that*

$$y \equiv h^t \ (mod\ n) \quad and \quad ty \equiv 1 \ (mod\ n).$$

Proof. Let t be a fixed point of $G(n)$

$$\begin{aligned} f(t) &= t^2h^t \\ t &\equiv t^2h^t \ (mod\ n) \\ t^2h^t - t &\equiv 0 \ (mod\ n) \\ t(tg^t - 1) &\equiv 0 \ (mod\ n) \\ t &\equiv 0 \ (mod\ n) \\ \text{or} \quad tg^t - 1 &\equiv 0 \ (mod\ n). \\ \text{or} \quad tg^t &\equiv 1 \ (mod\ n). \end{aligned}$$

This clearly shows that t and g^t are the multiplicative inverse modulo n. Thus there must exist some integer y such that $y \equiv h^t \ (mod\ n)$ and this implies that

$ty \equiv 1\ (mod\ n)$. This means that if t is a non-zero fix point then t^{-1} is also a fix point of n. Moreover, all units of m are the fixed of m. □

Corollary 3.4. *Let α be a fixed point of n, then either $\alpha \equiv 0$ or α is a unit of n.*

Proof. We know that, $\alpha, \beta \in RRS\ \ of\ \ s$ are units if and only if

$$\alpha\beta \equiv 1\ (mod\ s),$$

$f(t) = t^2h^t$, when α is fixed point

$$\begin{aligned} \alpha^2 h^\alpha &= \alpha\ (mod\ s) \\ \alpha(\alpha h^\alpha - 1) &= 0\ (mod\ s), \end{aligned}$$

$\alpha \equiv 0\ \ (mod\ s)$ or $\alpha h^\alpha \equiv 1\ \ (mod\ s)$.
Now if $\alpha h^\alpha \equiv 1\ \ (mod\ s)$, we take $h^\alpha = \beta$ for sake of convenience.
There $\alpha h^\alpha \equiv 1\ \ (mod\ s)$ yields the $\alpha\beta \equiv 1\ \ (mod\ s)$ or α is a unit of s. □

Corollary 3.5. *An integer t is a fixed of $G(n) \Leftrightarrow\ th^t \equiv 1\ (mod\ n)$*

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