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A Study of Fourth Hankel Determinant of Certain Analytic Function

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Abstract

The main motive of this paper is to find an upper bound of the fourth Hankel determinant $H_{4,1}(f)$ for a subclass $\mathcal{S}$, with hyperbolic domain.

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1 Introduction, Definitions and Motivation

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the region $\mathfrak{A} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized at $f(0) = f'(0) - 1 = 0$. Therefore, for $f(z) \in \mathcal{A}$, one has

$$f(z) = z + \sum_{n=2}^{\infty} a_n\, z^n, \quad z \in \mathfrak{A}, \tag{1.1}$$

while $\mathcal{S}$ denotes a subclass of $\mathcal{A}$ which contains normalized univalent functions. For two functions $f(z)$ and $g(z)$ analytic in $\mathfrak{A}$, we say that $f(z)$ is subordinate to $g(z)$, denoted by $f(z) \prec g(z)$, if there is an analytic function $w(z)$ with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. If $g(z)$ is univalent, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathfrak{A}) \subseteq g(\mathfrak{A})$.

We now consider a subclass $\mathcal{SL}$ of analytic functions as follows;

$$\mathcal{SL} = \left\{ f(z) \in \mathcal{A} : \left| f'(z)^2 - 1 \right| < 1, \, z \in \mathfrak{A} \right\}.$$

Related to the coefficient a_n of taylor series of univalent functions,the so called Fekete-Szegö problem is considered to be a major result. Different classified techniques have been used by authors to maximize Fekete-Szegö functional. These studies shows interesting geometric characteristics of hyperbolic domain for different types of functions. we refer the readers to readout [1],[10],[11, 26] and [17] for more details.

We define the subordination of two functions f and g symbolically written as $f \prec g$, and is defined as

$$f(z) = g(w(z)), \quad z \in \mathfrak{A} \tag{1.2}$$

where w is Schwarz function such that $w(0) = 0, |w(z)| < 1$ for $z \in \mathfrak{A}$.

Definition 1. *[19]A function $p(z)$ is said to be in the class $\mathcal{K} - \mathcal{P}(a, b)$, iff,*

$$p(z) \prec (a + b) + (a - b) p_k(z), \tag{1.3}$$

where $\tilde{p}_k(z) = 1 + \frac{2}{1-k^2} \sinh^2 \left[\left(\frac{2}{\Pi} \arccos k \right) \arctan h \sqrt{z} \right]$, $0 < k < 1$, a, b must be chosen accordingly,as:

$$b \in \begin{cases} \left[\frac{1}{2k^2-1} \right), & \text{when } 0 < k < \frac{1}{\sqrt{2}} \\ (-\infty, 1), & \text{when } \frac{1}{\sqrt{2}} \leq k \leq 1 \end{cases} \tag{1.4}$$

and

$$\xi_1(k, b) \leq a < 1 + \xi_1(k, b) \tag{1.5}$$

with

$$\xi_1(k, b) = \frac{k^2(1 - b)}{1 - k^2} - \frac{\sqrt[k]{k^2(1 - b)^2 + (1 - k^2)(1 - b^2)}}{k^2 - 1}.$$

Definition 2. *[19] A function $f(z) \in \mathcal{S}$ is said to be in class $\mathcal{K} - \mathcal{SL}(a, b)$, $0 < k < 1$, a, b satisfy (1.4) and (1.5), if and only if*

$$\left[\Re \left\{ f'(z) - a \right\} \right]^2 > k^2 \left[\left| f'(z) - a + b - 1 \right|^2 + 2b(1 - b) \right].$$

or in other words

$$f'(z) \in \mathcal{K} - \mathcal{P}(a, b). \tag{1.6}$$

We solve the problem for the functions of classes $\mathcal{K} - \mathcal{P}(a, b)$ and $\mathcal{K} - \mathcal{SL}(a, b)$.

It can easily be seen that $f(z) \in \mathcal{K} - \mathcal{SL}(a, b)$ if

$$f'(z) \prec P(z), \; z \in \mathfrak{A}. \tag{1.7}$$

The qth Hankel determinant $H_q(n)$, $q \geq 1$, $n \geq 1$, for a function $f(z) \in \mathcal{S}$ is:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.8}$$

.It is well-known that the Fekete-Szegö functional $\left|a_1 - a_0^2\right|$ is $H_{2,1}(f)$. Fekete-Szegö then further generalized the estimate $\left|a_1 - \mu a_0^2\right|$ with μ real and $f(z) \in \mathcal{S}$. Moreover, we also know that the functional $\left|a_0 a_2 - a_1^2\right|$ is equivalent to $H_{2,1}(f)$.In particular, sharp upper bounds on $H_{2,}(2)$ were obtained by the authors of articles [24, 23] for various classes of functions. In the present investigation, we study the upper bound of $H_{4,1}(f)$ for a subclass $\mathcal{K} - \mathcal{SL}(a,b)$ of analytic functions by using Hankel determinants. For recent investigation on this topic, we may refer to [20, 2, 6, 15, 12, 8, 21, 14, 25, 9]

For our main results we need the following lemmas.

Lemma 1. *[18] If $p(z) \in \mathcal{P}$ and of the form $p(z) = z + \sum_{n=2}^{\infty} a_n\ z^n$, $z \in \mathfrak{A}$, then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & (\nu \leq 0), \\ 2 & (0 \leq \nu \leq 1) \\ 4\nu - 2 & (\nu \geq 1). \end{cases}$$

Lemma 2. *[22] If $p(z) \in \mathcal{P}$ and of the form $p(z) = z + \sum_{n=2}^{\infty} a_n\ z^n$, $z \in \mathfrak{A}$,, then, for all $n, m \in \mathbb{N}$*

$$|\mu c_n c_m - c_{n+m}| \leq \begin{cases} 2 & 0 \leq \mu \leq 1 \\ 2|2\mu - 1| & \textit{otherwise} \end{cases}$$

Lemma 3. *[16] If $p(z) \in \mathcal{P}$ and of the form $p(z) = z + \sum_{n=2}^{\infty} a_n\ z^n$, $z \in \mathfrak{A}$,, then, for all $n, m \in \mathbb{N}$*

$$\begin{aligned} \left|c_1^3 - 2c_1c_2 + c_3\right| &\leq 2\ , \textit{for all } n \geq 1 \\ |c_n| &\leq 2\ , \textit{for all } n \geq 1 \end{aligned}$$

Main Results

Theorem 1. *Let $f(z) \in \mathcal{K} - \mathcal{SL}(a,b)$ where $0 < k < 1$, and a, b are taken according to (1.4) and (1.5) and has the form (1.1). Then*

$$|a_n| \leq \frac{2T^2\,|1-b|}{n\,(1-k^2)},\ \textit{for all } n \geq 2 \tag{1.9}$$

These inqualities is sharp.

Proof. For $p \in P$ and of the form $p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$, we consider

$$p(z) = \frac{1 + w(z)}{1 - w(z)}$$

where $w(z)$ is such that $w(0)=0$ and $|w(z)|<1$.And from (1.7), we have

$$f'(z) \prec P(z)$$

Then it follows easily that

$$\begin{aligned} w(z) &= \frac{p(z)-1}{p(z)+1}. \\ &= \frac{\left(1+c_1z+c_2z^2+c_3z^3+c_4z^4+...\right)-1}{\left(1+c_1z+c_2z^2+c_3z^3+c_4z^4+...\right)+1} \\ &= \frac{1}{2}c_1z+\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)z^2+\left(\frac{1}{2}c_3-\frac{1}{2}c_1c_2+\frac{1}{8}c_1^3\right)z^3 \qquad (1.10) \\ &+\left(\frac{1}{2}c_4-\frac{1}{2}c_1c_3+\frac{3}{8}c_1^2c_2-\frac{1}{4}c_2^2-\frac{1}{16}c_1^4\right)z^4+... \end{aligned}$$

Now, if $\tilde{p}_k(w(z))=1+R_1(k)w(k)+R_2(k)w^2(k)+R_3(k)w^3(k)+R_4(k)w^4(k)+$ ...

then from (1.10), we have

$$\begin{aligned} \tilde{p}_k(w(z)) &= 1+R_1(k)w(k)+R_2(k)w^2(k)+R_3(k)w^3(k)+R_4(k)w^4(k)+..., \\ &= 1+R_1(k)\left(\frac{1}{2}c_1z+\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)z^2+\left(\frac{1}{2}c_3-\frac{1}{2}c_1c_2+\frac{1}{8}c_1^3\right)z^3\right) \\ &+R_2(k)\left(\frac{1}{2}c_1z+\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)z^2+\left(\frac{1}{2}c_3-\frac{1}{2}c_1c_2+\frac{1}{8}c_1^3\right)z^3\right)^2 \\ &+R_3(k)\left(\frac{1}{2}c_1z+\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)z^2+\left(\frac{1}{2}c_3-\frac{1}{2}c_1c_2+\frac{1}{8}c_1^3\right)z^3\right)^3+... \end{aligned}$$

Where $R_1(k), R_2(k)$ and $R_3(k)$ are given by

$$\begin{aligned} R_1(k) &= \frac{2T^2}{1-k^2} \\ R_2(k) &= \frac{2T^2}{3(1-k^2)}\left(2+T^2\right) \\ R_3(k) &= \frac{2T^2}{9(1-k^2)}\left(\frac{23}{5}+4T^2+\frac{2}{5}T^4\right), \end{aligned}$$

and $T=T(k)=\frac{2}{\pi}\arccos(k)$, $0<k<1$, see [13]. By using the above series we have

$$\begin{aligned} \tilde{p}_k(w(z)) &= 1+\frac{T^2}{1-k^2}c_1z+\frac{T^2}{1-k^2}\left(\frac{T^2-1}{6}c_1^2+c_2\right)z^2+ \qquad (1.11) \\ & \frac{T^2}{1-k^2}\left[\left(\frac{2}{45}-\frac{1}{18}T^2+\frac{1}{90}T^4\right)c_1^3+\left(\frac{T^2-1}{3}\right)c_1c_2+c_3\right]z^3 \\ &+\frac{T^2}{1-k^2}\left[\begin{array}{c}\left(\frac{T^6}{2520}-\frac{T^4}{180}+\frac{7T^2}{360}-\frac{1}{70}\right)c_1^4- \\ \frac{T^2-1}{30}\left(T^4-4\right)c_1^2c_2+\frac{T^2-1}{3}c_1c_3+\frac{T^2-1}{6}c_2^2\end{array}\right]z^4+... \qquad (1.12) \end{aligned}$$

Since $p \in \mathcal{K} - \mathcal{P}(a,b)$, $0 < k < 1$, so from $(1.2), (1.3)$ and (1.11), we have

$$\begin{aligned}
p(z) &= (a+b) + (a-b)\, p_k(w(z)), \\
&= 1 + a + \frac{T^2(1-b)}{1-k^2} c_1 z + \frac{T^2(1-b)}{1-k^2}\left(c_2 + \frac{T^2-1}{6} c_1^2\right) z^2 \\
&\quad + \frac{T^2(1-b)}{1-k^2}\left[\left(\frac{2}{45} - \frac{1}{18}T^2 + \frac{1}{90}T^4\right) c_1^3 + \left(\frac{T^2-1}{3}\right) c_1 c_2 + c_3\right] z^3 \\
&\quad + \frac{T^2(1-b)}{1-k^2}\left[\begin{array}{c} \left(\frac{T^6}{2520} - \frac{T^4}{180} + \frac{7T^2}{360} - \frac{1}{70}\right) c_1^4 - \\ \frac{T^2-1}{30}\left(T^4-4\right) c_1^2 c_2 + \frac{T^2-1}{3} c_1 c_3 + \frac{T^2-1}{6} c_2^2 \end{array}\right] z^4 + \ldots.
\end{aligned}$$

As from (1.1) we have

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots.$$

By equating the coefficients of like powers of z, we get

$$a_2 = \frac{T^2(1-b)}{2(1-k^2)} c_1 \tag{1.13}$$

$$\begin{aligned}
a_3 &= \frac{T^2(1-b)}{3(1-k^2)}\left[c_2 - \frac{1-T^2}{6} c_1^2\right] \\
a_4 &= \frac{T^2(1-b)}{4(1-k^2)}\left[\left(\frac{2}{45} - \frac{1}{18}T^2 + \frac{1}{90}T^4\right) c_1^3 + \left(\frac{T^2-1}{3}\right) c_1 c_2 + c_3\right]
\end{aligned}$$

$$a_5 = \frac{T^2(1-b)}{5(1-k^2)}\left[\begin{array}{c} \left(\frac{T^6}{2520} - \frac{T^4}{180} + \frac{7T^2}{360} - \frac{1}{70}\right) c_1^4 - \\ \frac{T^2-1}{30}\left(T^4-4\right) c_1^2 c_2 + \frac{T^2-1}{3} c_1 c_3 + \frac{T^2-1}{6} c_2^2 \end{array}\right]. \tag{1.14}$$

Now we find the coefficients,

$$|a_2| \leq \frac{T^2|1-b|}{2(1-k^2)} |c_1|.$$

Then by Lemma 3, we get

$$|a_2| \leq \frac{T^2|1-b|}{1-k^2}. \tag{1.15}$$

As

$$\begin{aligned}
|a_3| &= \left|\frac{T^2(1-b)}{3(1-k^2)}\left[c_2 - \frac{1-T^2}{6} c_1^2\right]\right| \\
&\leq \frac{T^2|1-b|}{3(1-k^2)}\left|c_2 - \frac{1-T^2}{6} c_1^2\right|.
\end{aligned}$$

Now by Lemma 2, we get

$$|a_3| \leq \frac{2T^2|1-b|}{3(1-k^2)}. \tag{1.16}$$

Also

$$
\begin{aligned}
|a_4| &\le \frac{T^2|1-b|}{4(1-k^2)}\left|\left(\frac{2}{45}-\frac{1}{18}T^2+\frac{1}{90}T^4\right)c_1^3+\left(\frac{T^2-1}{3}\right)c_1c_2+c_3\right| \\
&\le \frac{T^2|1-b|}{4(1-k^2)}\left|\frac{2}{45}-\frac{1}{18}T^2+\frac{1}{90}T^4\right|\left|c_1^3-2c_1c_2+c_3\right| \\
&\quad +\frac{T^2|1-b|}{4(1-k^2)}-\frac{11}{45}+\frac{1}{12}T^2+\frac{2}{90}T^4\left|c_1c_2-c_3\right| \\
&\quad +\frac{T^2|1-b|}{4(1-k^2)}\left|\frac{32}{45}+\frac{5}{24}T^2+\frac{1}{90}T^4\right||c_3|.
\end{aligned}
$$

Again by using Lemma 2 and 3, we get

$$|a_4| \le \frac{T^2|1-b|}{2(1-k^2)}. \tag{1.17}$$

Similarly, by the above methed, we get

$$|a_5| \le \frac{2T^2|1-b|}{5(1-k^2)}. \tag{1.18}$$

This complete the proof of our theorem. □

Theorem 2. *If $f(z) \in \mathcal{K}-\mathcal{SL}(a,b)$ where $0<k<1$, and a, b are taken according to (1.4) and (1.5) . Then for a real number μ, we have*

$$
\left|a_3-\mu a_2^2\right| \le \frac{T^2|1-b|}{3(1-k^2)}\begin{cases}
\frac{4}{3}+\frac{2}{3}T^2-\mu\frac{6T^2(1-b)}{1-k^2}, & \mu \le \frac{(1-k^2)(T^2-1)}{3T^2(1-b)} \\
2, & \frac{(1-k^2)(T^2+7)}{6T^2(1-b)} \le \mu \le \frac{(1-k^2)(T^2-1)}{12T^2(1-b)} \\
-\frac{4}{3}-\frac{2}{3}T^2+\mu\frac{6T^2(1-b)}{1-k^2}, & \mu \ge \frac{(1-k^2)(T^2+5)}{3T^2(1-b)}.
\end{cases}
$$

These inequalities is sharp.

Proof. From (1.13), we have

$$\left|a_3-\mu a_2^2\right| \le \frac{T^2|1-b|}{3(1-k^2)}\left|c_2-\left(\frac{1-T^2}{6}+\mu\frac{3T^2(1-b)}{2(1-k^2)}\right)c_1^2\right|.$$

Put $v=\frac{1-T^2}{6}+\mu\frac{3T^2(1-b)}{2(1-k^2)} \le 0$, by first condition of Lemma 1, we have

$$
\begin{aligned}
\left|c_2-vc_1^2\right| &\le \left\{-4\left(\frac{1-T^2}{6}+\mu\frac{3T^2(1-b)}{2(1-k^2)}\right)+2\right\} \\
&= \frac{4}{3}+\frac{2}{3}T^2-\mu\frac{6T^2(1-b)}{1-k^2},
\end{aligned}
$$

implies

$$\left|a_3-\mu a_2^2\right| \le \frac{T^2|1-b|}{3(1-k^2)}\left(\frac{4}{3}+\frac{2}{3}T^2-\mu\frac{6T^2(1-b)}{1-k^2}\right).$$

Now put $v \in [0,1]$, then by second condition of Lemma 1,we get

$$\left|a_3 - \mu a_2^2\right| \le \frac{2T^2 |1-b|}{3(1-k^2)}.$$

Also, put $v = \frac{1-T^2}{6} + \mu \frac{3T^2(1-b)}{2(1-k^2)} \ge 0$, by third condition of Lemma 1, we have

$$\begin{aligned}\left|c_2 - vc_1^2\right| &\le \left\{4\left(\frac{1-T^2}{6} + \mu\frac{3T^2(1-b)}{2(1-k^2)}\right) - 2\right\} \\ &= -\frac{4}{3} - \frac{2}{3}T^2 + \mu\frac{6T^2(1-b)}{1-k^2}.\end{aligned}$$

Clearly

$$\left|a_3 - \mu a_2^2\right| \le \frac{T^2|1-b|}{2(1-k^2)}\left(-\frac{4}{3} - \frac{2}{3}T^2 + \mu\frac{6T^2(1-b)}{1-k^2}\right).$$

This complete the proof of our theorem. □

Corollary 1. *If $f(z) \in \mathcal{K} - \mathcal{SL}(a,b)$ where $0 < k < 1$ and a, b are taken according to (1.4) and (1.5), then for $\mu = 1$,we have*

$$\left|a_3 - a_2^2\right| \le \frac{2T^2|1-b|}{3(1-k^2)}. \tag{1.19}$$

Theorem 3. *If $f(z) \in \mathcal{K} - \mathcal{SL}(a,b)$ where $0 < k < 1$ and a, b are taken according to (1.4) and (1.5), then*

$$|a_4 - a_2 a_3| \le \frac{T^2|1-b|}{6(1-k^2)}. \tag{1.20}$$

Proof. By using (1.13), we have

$$a_4 - a_2 a_3 = \frac{T^2(1-b)}{12(1-k^2)}\left[\begin{array}{c}\left(\frac{T^2(1-b)\left(1-T^2\right)}{3(1-k^2)} + \frac{2}{15} - \frac{1}{6}T^2 + \frac{1}{30}T^4\right)c_1^3 + \\ \left(T^2 - 1 + \frac{2T^2(1-b)}{(1-k^2)}\right)c_1c_2 + 3c_3\end{array}\right]. \tag{1.21}$$

From (1.21), we can write

$$\begin{aligned}|a_4 - a_2a_3| \le\ & \frac{T^2|1-b|}{12(1-k^2)}\left|\frac{T^2(1-b)\left(1-T^2\right)}{3(1-k^2)} + \frac{2}{15} - \frac{1}{6}T^2 + \frac{1}{30}T^4\right|\left|c_1^3 - 2c_1c_2 + c_3\right| \\ &+ \frac{T^2|1-b|}{12(1-k^2)}\left|\frac{-2T^2(1-b)\left(T^2+2\right)}{3(1-k^2)} - \frac{11}{5} + \frac{2}{3}T^2 + \frac{1}{15}T^4\right|\left|c_1c_2 - c_3\right| \\ &+ \frac{T^2|1-b|}{12(1-k^2)}\left|\frac{-T^2(1-b)\left(T^2+5\right)}{3(1-k^2)} + \frac{32}{15} + \frac{5}{6}T^2 + \frac{1}{30}T^4\right|\left|c_3\right|.\end{aligned}$$

By applying Lemma 2 and Lemma 3 we get

$$|a_4 - a_2a_3| \le \frac{T^2|1-b|}{6(1-k^2)}.$$

□

Corollary 2. *If $f(z) \in \mathcal{K} - \mathcal{SL}(a,b)$ where $0 < k < 1$, and a, b are taken according to* (1.4) *and* (1.5), *then*

$$|a_4 - \mu a_2 a_3| \leq \frac{T^2 |1-b|}{6(1-k^2)} + (1-\mu)\frac{2T^4(1-b)^2}{3(1-k^2)^2}. \tag{1.22}$$

Proof. The proof follows directly from Theorem (3), (1.15) and (1.16) □

Theorem 4. *If $f(z) \in \mathcal{K} - \mathcal{ST}(a,b)$ where $0 < k < 1$, and a, b are taken according to* (1.4) *and* (1.5), *then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{T^4(1-b)^2}{18(1-k^2)^2}. \tag{1.23}$$

Proof. By using (1.13), we have

$$a_2 a_4 - a_3^2 = \frac{T^4(1-b)^2}{72(1-k^2)^2}\left[\begin{array}{c} \left(\frac{8}{45} - \frac{1}{18}T^2 - \frac{31}{10}T^4\right)c_1^4 + \frac{T^2-1}{3}c_1^2 c_2 \\ +9c_1 c_3 - 8c_2^2 \end{array}\right]. \tag{1.24}$$

From (1.24), we can write

$$\begin{aligned} \left|a_2 a_4 - a_3^2\right| \leq\ & \frac{T^4(1-b)^2}{12(1-k^2)^2}\left|\frac{8}{45} - \frac{1}{18}T^2 - \frac{31}{10}T^4\right|\left|c_1^4 - 2c_1^2 c_2 + c_1 c_3\right| \\ & + \frac{T^4(1-b)^2}{12(1-k^2)^2}\left|\frac{1}{45} + \frac{2}{9}T^2 - \frac{31}{5}T^4\right|\left|c_1^2 c_2 - c_2^2\right| \\ & + \frac{T^4(1-b)^2}{12(1-k^2)^2}\left|-\frac{359}{45} + \frac{2}{9}T^2 - \frac{31}{5}T^4\right|\left|c_1 c_3\right| \\ & + \frac{T^4(1-b)^2}{12(1-k^2)^2}\left|\frac{397}{45} + \frac{4}{18}T^2 + \frac{31}{10}T^4\right|\left|c_2\right|^2. \end{aligned}$$

Clearly

$$\begin{aligned} \left|a_2 a_4 - a_3^2\right| \leq\ & \frac{T^4(1-b)^2}{12(1-k^2)^2}\left|\frac{8}{45} - \frac{1}{18}T^2 - \frac{31}{10}T^4\right|\left|c_1\right|\left|c_1^3 - 2c_1 c_2 + c_3\right| \\ & + \frac{T^4(1-b)^2}{12(1-k^2)^2}\left|\frac{1}{45} + \frac{2}{9}T^2 - \frac{31}{5}T^4\right|\left|c_2\right|\left|c_1^2 - c_2\right| \\ & + \frac{T^4(1-b)^2}{12(1-k^2)^2}\left|-\frac{359}{45} + \frac{2}{9}T^2 - \frac{31}{5}T^4\right|\left|c_1\right|\left|c_3\right| \\ & + \frac{T^4(1-b)^2}{12(1-k^2)^2}\left|\frac{397}{45} + \frac{4}{18}T^2 + \frac{31}{10}T^4\right|\left|c_2\right|^2. \end{aligned}$$

By applying Lemma 2 and Lemma 3, we get

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{T^4(1-b)^2}{18(1-k^2)^2}.$$

□

Corollary 3. *If* $f(z) \in \mathcal{K} - \mathcal{SL}(a,b)$ *where* $0 < k < 1$, *and* a, b *are taken according to* (1.4) *and* (1.5), *then*

$$\left|a_2a_4 - \mu a_3^2\right| \leq \frac{T^4(1-b)^2}{2(1-k^2)^2}\left(1 - \frac{4}{9}\mu\right). \tag{1.25}$$

Proof. The proof follows directly from Theorem(4) and (1.16). □

Theorem 5. *If* $f(z) \in \mathcal{K} - \mathcal{ST}(a,b)$ *where* $0 < k < 1$, *and* a, b *are taken according to* (1.4) *and* (1.5), *then*

$$|H_{3,1}(f)| \leq \frac{T^6(1-b)^3}{3(1-k^2)^3} + \frac{17T^4(1-b)^2}{18(1-k^2)^2}. \tag{1.26}$$

Proof. As

$$H_{3,1}(f) = a_3\left(a_2a_4 - a_3^2\right) + a_4(a_4 - a_2a_3) - a_5\left(a_3 - a_2^2\right). \tag{1.27}$$

According to the triangular inequality, we have

$$|H_{3,1}(f)| \leq |a_3|\left|a_2a_4 - a_3^2\right| + |a_4||a_4 - a_2a_3| + |a_5|\left|a_3 - a_2^2\right|.$$

By using Theorem 1, (1.20), $(1.23)$and Corollary 1, we get

$$|H_{3,1}(f)| \leq \frac{T^6(1-b)^3}{27(1-k^2)^3} + \frac{7T^4(1-b)^2}{20(1-k^2)^2}.$$

□

Motivated from the articles [7, 3, 5, 4] on fourth Hankel determinant, we now investigate this determinant for the class $\mathcal{K} - \mathcal{SL}(a,b)$ as below.

Theorem 6. *If* $f(z) \in \mathcal{K} - \mathcal{SL}(a,b)$ *where* $0 < k < 1$, *and* a, b *are taken according to* (1.4) *and* (1.5), *then*

$$\left|a_5 - a_3^2\right| \leq \frac{4T^2(1-b)}{45(1-k^2)}. \tag{1.28}$$

Proof. Proof is abvious as from above metheds. □

Theorem 7. *If* $f(z) \in \mathcal{K} - \mathcal{SL}(a,b)$ *where* $0 < k < 1$, *and* a, b *are taken according to* (1.4) *and* (1.5), *then*

$$|a_2a_5 - a_3a_4| \leq \frac{2T^4(1-b)^2}{15(1-k^2)^2}. \tag{1.29}$$

Proof. Proof is abvious as from above metheds. □

Theorem 8. *If* $f(z) \in \mathcal{K} - \mathcal{SL}(a,b)$ *where* $0 < k < 1$, *and* a, b *are taken according to* (1.4) *and* (1.5), *then*

$$\left|a_3a_5 - a_4^2\right| \leq \frac{T^4(1-b)^2}{15(1-k^2)^2}. \tag{1.30}$$

Proof. Proof is abvious as from above metheds. □

Theorem 9. *If $f(z) \in \mathcal{K} - \mathcal{SL}(a,b)$ where $0 < k < 1$, and a, b are taken according to* (1.4) *and* (1.5), *then*

$$|H_{4,1}(f)| \leq \frac{44T^8(1-b)^4}{1575(1-k^2)^4} + \frac{91T^6(1-b)^3}{750(1-k^2)^3}. \tag{1.31}$$

Proof. As

$$H_{4,1}(f) = a_7 H_{3,1}(f) - a_6\Delta_1 + a_5\Delta_2 - a_4\Delta_3. \tag{1.32}$$

Where each Δ_1, Δ_2 and Δ_3 are determinant of order 3, are given by

$$\Delta_1 = a_6\left(a_3 - a_2^2\right) + a_5(a_2a_3 - a_4) + a_4\left(a_2a_4 - a_3^2\right) \tag{1.33}$$

$$\Delta_2 = a_6(a_4 - a_2a_3) - a_5\left(a_5 - a_3^2\right) + a_4(a_2a_5 - a_3a_4) \tag{1.34}$$

$$\Delta_3 = a_6\left(a_2a_4 - a_3^2\right) - a_5(a_2a_5 - a_3a_4) + a_4\left(a_3a_5 - a_4^2\right) \tag{1.35}$$

$$H_{3,1}(f) = a_3\left(a_2a_4 - a_3^2\right) + a_4(a_4 - a_2a_3) + a_5\left(a_3 - a_2^2\right) \tag{1.36}$$

By applying triangular inequality on (1.33), (1.34), (1.35) and (1.36), we get

$$|\Delta_1| \leq |a_6|\left|a_3 - a_2^2\right| + |a_5||a_2a_3 - a_4| + |a_4|\left|a_2a_4 - a_3^2\right|, \tag{1.37}$$

$$|\Delta_2| \leq |a_6||a_4 - a_2a_3| + |a_5|\left|a_5 - a_3^2\right| + |a_4||a_2a_5 - a_3a_4|, \tag{1.38}$$

$$|\Delta_3| \leq |a_6|\left|a_2a_4 - a_3^2\right| + |a_5||a_2a_5 - a_3a_4| + |a_4|\left|a_3a_5 - a_4^2\right|. \tag{1.39}$$

$$|H_{3,1}(f)| \leq |a_3|\left|a_2a_4 - a_3^2\right| + |a_4||a_4 - a_2a_3| + |a_5|\left|a_3 - a_2^2\right|. \tag{1.40}$$

From Theorem 1, 3, 4, 6, 7 and 8 and from Corollary 1, we can easily obtained:

$$|\Delta_1| \leq \frac{11T^4(1-b)^2}{15(1-K^2)^2} + \frac{T^6(1-b)^3}{36(1-K^2)^3},$$

$$|\Delta_2| \leq \frac{3T^4(1-b)^2}{(1-K^2)^2} + \frac{T^6(1-b)^3}{15(1-K^2)^3},$$

$$|\Delta_3| \leq \frac{44T^6(1-b)^3}{675(1-K^2)^3}.$$

While from(1.26), we conclude that

$$|H_{3,1}(f)| \leq \frac{T^6(1-b)^3}{3(1-k^2)^3} + \frac{17T^4(1-b)^2}{18(1-k^2)^2}.$$

Clearly (1.32) implies

$$|H_{4,1}(f)| \leq |a_7||H_{3,1}(f)| + |a_6||\Delta_1| + |a_5||\Delta_2| + |a_4||\Delta_3| \tag{1.41}$$

By putting all the values in (1.41) ,we get

$$|H_{4,1}(f)| \leq \frac{44T^8(1-b)^4}{1575(1-k^2)^4} + \frac{91T^6(1-b)^3}{750(1-k^2)^3}.$$

□

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