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NOVEL INVESTIHATIONS IN Q-POCHHAMMER SYMBOL

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ABSTRACT. *In this paper, we propose and examine a new relation in q-Pochhammer symbol. Further, we set up q-section, q-factorial and q-binomial coefficient in term of q-Pochhammer symbol utilizing our proposed connection.*

Keywords: q-Pochhammer symbol; q-Algebra; q-binomial coefficient.

1. Introduction: The subject of number theory is fundamentally partitioned into algebraic and analytic number theory. The outstanding Q-SERIES falls in analytic number theory. It has been esteemed and pulled in by many number scholar around the globe to paragon its tasteful outcomes and beauty. The Q-SERIES is such an arrangement which contains factors in q, communicated as,

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1}),\ n \geq 0 \text{ and } (a;q)_0 = 1$$

This can also be written as,

$$(a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1})$$

When $n \to \infty$, It is denoted by $(a;q)_\infty$ and is termed as "$q-$ Pochhammer symbol", introduced by Andrews in 1986. That is,

$$(a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^{k-1}), \qquad |q| < 1$$

Q arrangement sees symmetric mathematics especially in the theory of partitions. And additionally, it is useful in specifying potential outcomes in Combinatorics, Analysis, Physics and Computer Algebra.

2. Results and Discussions: In this area, we build up another connection between q-Pochhammer images given in Theorem 2.2. At that point after, we see Theorem 2.2 to express q-Pochhammer images in term of customary conditions and afterward characterize new surface diagrams. Prior to demonstrating the coveted connection, we require the accompanying basic lemma.

Lemma 2.1 *If $a, b \in R$, then* $\frac{a^2}{b} = \frac{\prod_{j=1}^{\infty}(a+j-1)^2(b+j)}{(a+j)^2(b+j-1)}$

Theorem 2.2 *If $a, b > 0$ and $q < 1$, then* $(1-\frac{a}{q})^2)\frac{(\frac{b}{q},q)_\infty}{(b,q)_\infty} = (1-\frac{b}{q})\frac{(\frac{a}{q},q)_\infty^2}{(a,q)_\infty^2}$,

where $(z,q)_\infty$ denotes the q-Pochhammer Symbol.

Proof.

It is well known that

$$\lim_{q\to-1}\frac{(1-q^v)}{(1-q)} = v \tag{1}$$

Instead $a, b,$ use $1-q^a, 1-q^b$ in Lemma 1.1 and apply limit defined in equation (1), on either sides, we get

$$\lim_{q\to-1}(1-q^a)^2(1-q^b) = \lim_{q\to-1}\frac{\prod_{j=1}^{\infty}(1-q^{a+j-1})^2(1-q^{b+j})}{(1-q^{a+j})^2(1-q^{b+j-1})}$$

$$= \lim_{q\to-1}\frac{(1-q^a)^2(q^{a-1},q)_\infty^2(q^b,q)_\infty(1-q^{b-1})}{(1-q^b)(q^{b-1},q)_\infty(1-q^{a-1})^2(q^a,q)_\infty^2}$$

$$\lim_{q\to-1}\frac{(1-q^{a-1})^2}{(1-q^{b-1})} = \lim\frac{(q^{a-1},q)_\infty^2(q^b,q)_\infty}{(q^a,q)_\infty^2(q^{b-1},q)_\infty}$$

Now if we assume that, q^a tends to a and q^b tends to $b,$ then above equation can be reduced to

$$\frac{(1-\frac{a}{q})^2}{(1-\frac{b}{q})}=\frac{(\frac{a}{q},q)_\infty^2(b,q)_\infty}{(a,q)_\infty^2(\frac{b}{q},q)_\infty} \tag{2}$$

This shows that

$$(1-\frac{a}{q})^2\frac{(\frac{b}{q},q)_\infty}{(b,q)_\infty}=(1-\frac{b}{q})\frac{(\frac{a}{q},q)_\infty^2}{(a,q)_\infty^2}. \tag{3}$$

Corollary 2.3 *If* $0<q<1$ *and* $n\in N$, *then*

$$\frac{(1-q^{n+1})^2}{(1-q^n)}=\frac{(q^{n+1},q)_\infty^3}{(q^n,q)_\infty(q^{n+2},q)_\infty^2}$$

Proof.

Put $a=q^{n+2}$, $b=q^{n+1}$ in Theorem 1.2, we obtain

$$(1-q^{n+1})^2\frac{(q^n,q)_\infty}{(q^{n+1},q)_\infty}=(1-q^n)\frac{(q^{n+1},q)_\infty^2}{(q^{n+2},q)_\infty^2}$$

(or)

$$\frac{(1-q^{n+1})^2}{(1-q^n)}=\frac{(q^{n+1},q)_\infty^3}{(q^n,q)_\infty(q^{n+2},q)_\infty^2}$$

Consequences

It is worth mentioning that Corollary 2.3 is very much elegant in producing new surface equations in term of q-Pochhammer Symbols, While its too difficult and seemed to be impossible for otherwise. We first introduce new variables in term of q-Pochhammer Symbols using Corollary 2.3. These are defined as under. Put $n=1,2,3$ and $n=4$ in Corollary 2.3, and we let variables x,y,z and t as,

1. Take $n=1$ in Corollary 2.3, we get

$$\frac{(1-q^2)^2}{(1-q)}=\frac{(q^2,q)_\infty^3}{(q,q)_\infty(q^3,q)_\infty^2}$$

Take

$$x=\frac{(q^2,q)_\infty^3}{(q,q)_\infty(q^3,q)_\infty^2} \tag{4}$$

Using the well known property $(a;q)_\infty=\prod_{k=0}^{n-1}(a(q)^k;q^n)_\infty$
with $n=2$ in the R.H.S of equation (3), we have

$$x=\frac{(q^2,q^2)_\infty^2(q^3,q^2)_\infty}{(q,q^2)_\infty(q^4,q^2)_\infty^2} \tag{5}$$

2. Take $n=2$ in Corollary 2.3, we get

$$\frac{(1-q^3)^2}{(1-q^2)}=\frac{(q^3,q)_\infty^3}{(q^2,q)_\infty(q^4,q)_\infty^2} \tag{6}$$

Using the well known property $(a;q)_\infty=\prod_{k=0}^{n-1}(a(q)^k;q^n)_\infty$
with $n=3$ in the R.H.S of equation (5), we have

$$y=\frac{(q^3,q^3)_\infty^2(q^5,q^3)_\infty}{(q^2,q^3)_\infty(q^6,q^3)_\infty^2}$$

$$y=\frac{1+q+q^2-q^3-q^4-q^5}{1+q}$$

3. Similarly, For $n=3,4$ in Corollary 2.3, we define,

$$z=\frac{(q^4,q^4)_\infty^2(q^7,q^4)_\infty}{(q^3,q^4)_\infty(q^8,q^4)_\infty^2}$$

$$=\frac{1+q+q^2+q^3-q^4-q^5-q^6-q^7}{1+q+q^2} \tag{7}$$

and

$$t = \frac{(q^5,q^4)_\infty^2(q^9,q^4)_\infty}{(q^4,q^4)_\infty(q^{10},q^4)_\infty^2} = \frac{(1-q^5)^2}{(1-q^4)} \tag{8}$$

Next we discover conditions between a two or three variables utilizing above characterized q-Pochhammer factors in term of variables. The accompanying conditions can be confirmed utilizing any scientific programmer like Mathematica, Maple and so forth. Here, we pause the calculations and compose the conditions straightforwardly. At last, we likewise can draw diagrams of these conditions by letting left sides of these conditions as a new surface. Indeed, it is a q-Pochhammer images in the event that we back substitute the estimations of the factors x, y, z and t and so on.

Equation Between x and y

Theorem 2.4 *If* $0 < q < 1$ *and let* $x = \frac{(q^2,q^2)_\infty^2(q^3,q^2)_\infty}{(q,q^2)_\infty(q^4,q^2)_\infty^2}$ and $y = \frac{(q^3,q^3)_\infty^2(q^5,q^3)_\infty}{(q^2,q^3)_\infty(q^6,q^3)_\infty^2}$ *then,*

$$x^5 - 5x^3y + 4x^2y^2 + xy^3 + 6x^3 - 18x^2y + 11xy^2 - xy + 9x - 8y = 0$$

Proof. If we substitute $q \to \frac{1-v}{1+v}$ in consequences 1 and 2, we get

$$x = \frac{8v}{(1+v)^3} \quad \text{and} \quad y = \frac{v(3+v^2)^2}{(1+v)^4}$$

Eliminating v from above two equations, we get the desired result.

$$x^5 - 5x^3y + 4x^2y^2 + xy^3 + 6x^3 - 18x^2y + 11xy^2 - xy + 9x - 8y = 0 \tag{9}$$

and

$$f(x,y) = x^5 - 5x^3y + 4x^2y^2 + xy^3 + 6x^3 - 18x^2y + 11xy^2 - xy + 9x - 8y$$

Equation Between y and z

Theorem 2.5 *If* $0 < q < 1$ *and let* $y = \frac{(1-q^3)^2}{(1-q^2)} = \frac{(q^3,q^3)_\infty^2(q^5,q^3)_\infty}{(q^2,q^3)_\infty(q^6,q^3)_\infty^2}$ and

$z = \frac{(1-q^4)^2}{(1-q^3)} = \frac{(q^4,q^4)_\infty^2(q^7,q^4)_\infty}{(q^3,q^4)_\infty(q^8,q^4)_\infty^2}$ *then,*

$$\begin{aligned}&-y^7z - 13y^6z + 14y^5z^2 - 5y^4z^3 - y^2z^5 - 17y^5z + 62y^4z^2 - 47y^3z^3 + 2y^2z^4 + 32y^5\\ &- 49y^4z + 112y^3z^2 - 68y^2z^3 - 155y^3z + 142y^2z^2 - 9yz^3 + 64y^3 - 119y^2z\\ &+ 54yz^2 - 3yz + 32y - 27z = 0\end{aligned}$$

Proof. If we substitute $q \to \frac{1-v}{1+v}$ in consequences 2 and 3, we get

$$y = \frac{v(3+v^2)^2}{(1+v)^4} \quad \text{and} \quad z = \frac{32v(1+v^2)^2}{(1+v)^5(3+v^2)} \tag{10}$$

Eliminating v from above two equations, we get the desired result.

$$\begin{aligned}&-y^7z - 13y^6z + 14y^5z^2 - 5y^4z^3 - y^2z^5 - 17y^5z + 62y^4z^2 - 47y^3z^3 + 2y^2z^4\\ &+ 32y^5 - 49y^4z + 112y^3z^2 - 68y^2z^3 - 155y^3z + 142y^2z^2 - 9yz^3 + 64y^3 - 119y^2z\\ &+ 54yz^2 - 3yz + 32y - 27z = 0\end{aligned}$$

Equation Between z and t

Theorem 2.6 *For* $0 < q < 1$, *if* $z = \frac{(1-q^4)^2}{(1-q^3)} = \frac{(q^4,q^4)_\infty^2(q^7,q^4)_\infty}{(q^3,q^4)_\infty(q^8,q^4)_\infty^2}$ and

$t = \frac{(1-q^5)^2}{(1-q^4)} = \frac{(q^5,q^4)_\infty^2(q^9,q^4)_\infty}{(q^4,q^4)_\infty(q^{10},q^4)_\infty^2}$ *then,*

$$\begin{aligned}&- - t^2z^8 + t^7z^2 + 10t^6z^3 + 30t^5z^4 + 24t^4z^5 - 21t^3z^6 - 12t^2z^7 - 2tz^8 -\\ &-5t^6z^2 + 24t^5z^3 + 108t^4z^4 + 60t^3z^5 - 149t^2z^6 - 12tz^7 - z^8 - 71t^5z^2 +\\ &-24t^4z^3 - 14t^3z^4 - 130t^2z^5 - 112tz^6 - 4t^5z - 343t^4z^2 + 128t^3z^3 - 295t^2z^4 +\\ &-140tz^5 - 20z^6 + 18t^4z + 227t^3z^2 + 424t^2z^2 + 236tz^4 + 436t^3z - 663t^2z^2 108tz^3 -\\ &-110z^4 + 8t^3 - 410t^2z + 352tz^2 - 108t^2 + 148tz - 100z^2 + 102t - 25 = 0\end{aligned}$$

Proof. If we substitute $q \to \frac{v}{1+v}$ in consequences 3 and 4, we get

$$z = \frac{32v(1+v^2)^2}{(1+v)^5(3+v^2)}$$

and

$$t = \frac{v(5 + 10v^2 + v^4)^2}{2(1 + v)^6(1 + v^2)}$$

Eliminating v from above two equations and encounter

$$\begin{gathered} - - t^2z^8 + t^7z^2 + 10t^6z^3 + 30t^5z^4 + 24t^4z^5 - 21t^3z^6 - 12t^2z^7 - 2tz^8 - \\ -5t^6z^2 + 24t^5z^3 + 108t^4z^4 + 60t^3z^5 - 149t^2z^6 - 12tz^7 - z^8 - 71t^5z^2 + \\ -24t^4z^3 - 14t^3z^4 - 130t^2z^5 - 112tz^6 - 4t^5z - 343t^4z^2 + 128t^3z^3 - 295t^2z^4 + \\ -140tz^5 - 20z^6 + 18t^4z + 227t^3z^2 + 424t^2z^2 + 236tz^4 + 436t^3z - 663t^2z^2 108tz^3 - \\ -110z^4 + 8t^3 - 410t^2z + 352tz^2 - 108t^2 + 148tz - 100z^2 + 102t - 25 = 0 \end{gathered}$$

Similarly, equation between y and t ca be obtain by taking,

$$y = \frac{(1-q^3)^2}{(1-q^2)} = \frac{(q^3,q^3)_\infty^2(q^5,q^3)_\infty}{(q^2,q^3)_\infty(q^6,q^3)_\infty^2}$$

and

$$t = \frac{(1 - q^5)^2}{(1 - q^4)} = \frac{(q^5, q^4)_\infty^2(q^9, q^4)_\infty}{(q^4, q^4)_\infty(q^{10}, q^4)_\infty^2}$$

Corollary 2.7

$$\frac{(1-q^{n+1})^2}{(1-q^n)} = \frac{(q^{n+1},q^{n+1})_\infty^2(q^{2n+1},q^{n+1})_\infty}{(q^n,q^{n+1})_\infty(q^{2n+2},q^{n+1})_\infty^2}, 0 < q < 1 andn \in N^+. \tag{11}$$

Proof. By Corollary 2.3,

$$\frac{(1-q^{n+1})^2}{(1-q^n)} = \frac{(q^{n+1},q)_\infty^3}{(q^n,q)_\infty(q^{n+2},q)_\infty^2} \tag{12}$$

Using the Elementary property, We get

$$(q^{n+1}, q)_\infty = \prod_{k=0}^{n} (q^{n+k+1}; q^{n+1})_\infty$$

and

$$\begin{aligned} \frac{(q^{n+1}, q)_\infty^3}{(q^n, q)_\infty(q^{n+2}, q)_\infty^2} &= \prod_{k=0}^{n} \frac{(q^{n+k+1}, q^{n+1})_\infty^3}{(q^{n+k}, q^{n+1})_\infty(q^{n+k+2}, q^{n+1})_\infty^2} \\ &= \frac{(q^{n+1},q^{n+1})_\infty^2(q^{2n+1},q^{n+1})_\infty}{(q^n,q^{n+1})_\infty(q^{2n+2},q^{n+1})_\infty^2} \\ &= \frac{(1-q^{n+1})^2}{(1-q^n)} \end{aligned}$$

Hence,

$$\frac{(1 - q^{n+1})^2}{(1 - q^n)} = \frac{(q^{n+1}, q^{n+1})_\infty^2(q^{2n+1}, q^{n+1})_\infty}{(q^n, q^{n+1})_\infty(q^{2n+2}, q^{n+1})_\infty^2}$$

Replacing a by q^2 and b by zq^2 in Theorem 2.2, we have the following useful proposition.

Proposition 2.8 *If* $z > 0$, and $q < 1$ then, $\frac{(1-q)^2}{(1-zq)} = \frac{(q,q)_\infty^2(zq^2,q)_\infty}{(q^2,q)_\infty(zq,q)_\infty}$

Theorem 2.9

$$\frac{(1 - q^3)}{(1 - q^2)} = \frac{(q^3, q)_\infty^2}{(q^2, q)_\infty(q^4, q)_\infty} = \frac{1 + q + q^2}{1 + q}$$

Proof. Replacing q^2 by z in Proposition 2.8, we get

$$\frac{(1-q)^2}{(1-q^3)} = \frac{(q,q)_\infty^2 (q^4,q)_\infty}{(q^2,q)_\infty (q^3,q)_\infty} \tag{13}$$

Also,

$$\frac{(1-q^3)}{(1-q)^2} = \frac{(q^2,q)_\infty (q^3,q)_\infty}{(q,q)_\infty^2 (q^4,q)_\infty} \tag{14}$$

Again replacing q by z in Proposition 2.8, we obtain,

$$\frac{(1-q)^2}{(1-q^2)} = \frac{(q,q)_\infty^2 (q^3,q)_\infty}{(q^2,q)_\infty (q^2,q)_\infty} \tag{15}$$

The above two equations yields that,

$$\frac{(1-q^3)}{(1-q^2)} = \frac{(q^3,q)_\infty^2}{(q^2,q)_\infty (q^4,q)_\infty}$$

$$= \frac{1+q+q^2}{1+q} \tag{16}$$

Consequently,

$$\frac{(1-q^3)}{(1-q^2)} = \frac{(q^3,q)_\infty^2}{(q^2,q)_\infty (q^4,q)_\infty}$$

$$= \frac{1+q+q^2}{1+q}$$

In the following results we express Q-Bracket, Q-Factorial and Q-Binomial coefficients in term of q-Pochhammer Symbols in the accompanying subsections. These results also vlidate results givn inin [4].

Theorem 2.10 *If* $q \in C - 1$, *then* $[k]_q = \frac{(1-q)(q^2,q)_\infty (q^k,q)_\infty}{(q,q)_\infty^2 (q^{k+1},q)_\infty}$

Proof. Replacing q^{k-1} by z in Proposition 2.8, we find that,

$$\frac{(1-q)^2}{(1-q^k)} = \frac{(q,q)_\infty^2 (q^{k+1},q)_\infty}{(q^2,q)_\infty (q^k,q)_\infty} \tag{17}$$

We know that, $[n]_q = \frac{1-q^n}{1-q}$. This implies that $[k]_q = \frac{1-q^k}{1-q}$. But then

$$[k]_q = \frac{(1-q)(q^2,q)_\infty (q^k,q)_\infty}{(q,q)_\infty^2 (q^{k+1},q)_\infty} \tag{18}$$

Theorem 2.11 *If* $0 < q < 1$ *and* $n \in N$, *then* $[n]_q! = \prod_{k=1}^{n} \frac{(1-q)(q^2,q)_\infty}{(q,q)_\infty (q^{n+1},q)_\infty}$

Where, $[n]_q!$ *denotes the q- Factorial Function*

Proof. We know that, $[n]_q! = \prod_{k=1}^{n} [k]_q$. Then by Theorem 2.10, we find that,

$$[n]_q! = \prod_{k=1}^{n} \frac{(1-q)(q^2,q)_\infty}{(q,q)_\infty (q^{n+1},q)_\infty} \tag{19}$$

Theorem 2.12 *If* $0 < q < 1$ *and* $n, r \in N$, *Then*

$$\binom{n}{r} = \frac{(q^{r+1},q)_\infty (q,q)_\infty (q^{n-r+1},q)_\infty}{(q^{n+1},q)_\infty (q^2,q)_\infty (1-q)}$$

Where $\binom{n}{r}$ denotes the q-Binomial Coefficient or Gaussian Binomial Coefficient and $(a;q)_\infty$ denotes the q-Pochhammer Symbol.

Proof. We know that $\binom{n}{r} = \frac{[n]_q!}{[r]_q! [n-r]_q!}$

Then by Theorem (2.11), we get,

$$\binom{n}{r} = \frac{(q^{r+1},q)_\infty (q,q)_\infty (q^{n-r+1},q)_\infty}{(q^{n+1},q)_\infty (q^2,q)_\infty (1-q)}$$

The following corollary is easy to deduce by using the relationship of factorial function with Gammma function.

Corollary 2.13 *We have*

$$\frac{(1-q)(q^2,q)_\infty}{(q,q)_\infty} = \frac{(q,q)_\infty}{(1-q)^n}$$

Proof. For $n \in N$, we know that, $[n]_q! = \Gamma_q(n+1)$, then

$$\Gamma_q(n) = \frac{(q;q)_\infty}{(q^n;q)_\infty}(1-q)^{1-n} \tag{20}$$

Replacing n by $n+1$, we have

$$\begin{aligned}\Gamma_q(n+1) &= \frac{(q;q)_\infty}{(q^{n+1};q)_\infty}(1-q)^{-n} \\ &= \frac{(q;q)_\infty}{(q^{n+1};q)_\infty(1-q)^n}\end{aligned} \tag{21}$$

Using q-factorial and equation (24), we obtain,
$Thus,$

$$\frac{(1-q)(q^2,q)_\infty}{(q,q)_\infty} = \frac{(q,q)_\infty}{(1-q)^n}$$

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