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A NEW RELATION IN Q-POCHHAMMER SYMBOL WITH APPLICATIONS

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ABSTRACT. *In this paper, we propose and investigate a new relation in q-Pochhammer symbol. Further, we establish q-bracket, q-factorial and q-binomial coefficient in term of q-Pochhammer symbol using our proposed relation. As an application, we express q-Pochhammer symbols in term of ordinary equations to define new surface graphs.*

Keywords: q-Pochhammer symbol; q-Algebra; Surface Equations

1. **Introduction.** The subject of number theory is mainly divided into algebraic and analytic number theory. The well-known Q-series falls in analytic number theory. In recent years, it has been valued and attracted by many number theorists around the world to paragon its classy results and loveliness. The ζ -series is such a series which contains factors in ζ , expressed as,

$$(\eta;\zeta)_t = (1-\eta)(1-\eta\zeta)(1-\eta\zeta^2)(1-\eta\zeta^{n-1}),\ t\geq 0 \text{and } (\eta;\zeta)_0 = 1$$

can also be written as,

$$(\eta;\zeta)_t = \prod_{k=1}^{t}(1-\eta\zeta^{k-1})$$

When $t\to\infty$, It is denoted by $(\eta;\zeta)_\infty$ and is termed as "$\zeta -$ Pochhammer symbol", introduced by Andrews in 1986. That is,

$$(\eta;\zeta)_\infty = \prod_{k=1}^{\infty}(1-\eta\zeta^{k-1}),\ \ |\zeta|<1$$

The Q series perceives symmetric mathematics especially in the theory of partitions. As well as, it is helpful in enumerating possibilities in Combinatorics, Analysis, Physics and Computer Algebra.

2: Results and Discussions: In this section, we establish a new relation between q-Pochhammer symbols given in Theorem 2.2. Then after, we incorporate Theorem 2.2 to express q-Pochhammer symbols in term of ordinary equations and then define new surface graphs. Finally, we depict these surfaces in fig. 1, fog. 2 and in fig. 3. Indeed, it would be of at most interest to know about these surfaces in a later work. Before proving the desired relation, we need the following simple lemma. Particularly, we use the foll0wing the product term instead of fraction as given in [4] to establish new results for use in the sequel.

Lemma 2.1:

If $\rho,\sigma\in$ R, then

$$\rho\sigma = \prod_{j=1}^{\infty}\frac{(\mathrm{a}+\mathrm{j}-1)(\mathrm{b}+\mathrm{j}-1)}{(\mathrm{b}+\mathrm{j})(\mathrm{a}+\mathrm{j})}$$

Proof

$$\prod_{h=1}^{\infty}\frac{(\rho+h-1)(\sigma+h-1)}{(\sigma+h)(\rho+h)}=\frac{\prod_{h=1}^{\infty}(\rho+h-1)\prod_{h=1}^{\infty}(\sigma+h-1)}{\prod_{h=1}^{\infty}(\sigma+h)\prod_{h=1}^{\infty}(\rho+h)}$$
$$=\frac{\rho(\rho+1)(\rho+2)\sigma(\sigma+1)(\sigma+2)}{(\rho+1)(\rho+2)(\sigma+1)(\sigma+2)}$$
$$=\rho\sigma$$

Theorem 2.2

If $a,b>0$ *and* $q<1$, *then* $(1-\frac{a}{q})(1-\frac{b}{q})(a,q)_\infty(b,q)_\infty=(\frac{a}{q},q)_\infty(\frac{b}{q},q)_\infty$,

where $(z,q)_\infty$ denotes the q-Pochhammer Symbol.

Proof. It is well known that

$$\lim_{q\to-1}\frac{1-q^v}{1-q}=v \tag{1}$$

Instead $a,b,$ use $1-q^a,1-q^b$ in Lemma 2.1 and apply limit defined in equation (1), on either sides, we get

$$\lim_{q\to-1}(1-q^a)(1-q^b)=\lim_{q\to-1}\prod_{j=1}^{\infty}\frac{(1-q^{a+j-1})(1-q^{b+j-1})}{(1-q^{a+j})(1-q^{b+j})}$$
$$=\lim_{q\to-1}\frac{(1-q^a)(1-q^b)(q^{a-1},q)_\infty(q^{b-1},q)_\infty}{(1-q^{a-1})(1-q^{b-1})(q^a,q)_\infty(q^b,q)_\infty}$$
$$=\lim_{q\to-1}\frac{(q^{a-1},q)_\infty(q^{b-1},q)_\infty}{(q^a,q)_\infty(q^b,q)_\infty} \tag{2}$$

Now if we assume that, q^a tends to a and q^b tends to b, then equation (2), can be reduced to

$$(1-\frac{a}{q})(1-\frac{b}{q})=\frac{(\frac{a}{q},q)_\infty(\frac{b}{q},q)_\infty}{(a,q)_\infty(b,q)_\infty}$$

This shows that

$$(1-\frac{a}{q})(1-\frac{b}{q})(a,q)_\infty(b,q)_\infty=(\frac{a}{q},q)_\infty(\frac{b}{q},q)_\infty.$$

Corollary 2.3 *If* $0<q<1$ *and* $n\in N$, *then*

$$(1-q^{n+1})(1-q^n)=\frac{(q^n,q)_\infty}{(q^{n+2},q)_\infty}$$

Proof.

Put $a=q^{n+2}$, $b=q^{n+1}$ in Theorem 2.2, we obtain

$$(1-q^{n+1})(1-q^n)(q^{n+1},q)(q^{n+2},q)=(q^{n+1},q)(q^n$$

Or

$$(1-q^{n+1})(1-q^n)=\frac{(q^n,q)_\infty}{(q^{n+2},q)_\infty}$$

Consequences It is worth mentioning that Corollary 2.3 is very much elegant in producing new surface equations in term of q-Pochhammer Symbols, While its too difficult and seemed to be impossible for otherwise. We first introduce new variables in term of q-Pochhammer Symbols using Corollary 2.3. These are defined as under. Put $n=1,2,3$ and $n=4$ in Corollary 2.3, and we let variables x,y,z and t as,

1. Put $n=1$ in Corollary 2.3, we get

$$(1-q^2)(1-q)=\frac{(q,q)_\infty}{(q^3,q)_\infty}$$

Take

$$x=\frac{(q,q)_\infty}{(q^3,q)_\infty} \tag{3}$$

Using the well known property $(a;q)_\infty=\prod_{k=0}^{n-1}(aq^k;q^n)_\infty$ with $n=2$ in the R.H.S of equation

(3), we have

$$x = \frac{(q;q^2)_\infty (q^2;q^2)_\infty}{(q^{r3};q^2)_\infty (q^4;q^2)_\infty} \tag{4}$$

2. Take $n = 2$ in Corollary 2.3, we get

$$(1-q^3)(1-q^2) = \frac{(q^2,q)_\infty}{(q^4,q)_\infty} \tag{5}$$

$Take$

$$y = \frac{(q^2,q)_\infty}{(q^4,q)_\infty}$$

Using the property $(a;q)_\infty = \prod_{k=0}^{n-1} (aq^k;q^n)_\infty$ with $n = 3$ in the R.H.S of equation (5), we have

$$y = \frac{(q^2;q^3)_\infty (q^3;q^3)_\infty}{(q^5;q^3)_\infty (q^6;q^3)_\infty}$$

3. Similarly, For $n = 3,4$ in Corollary 2.3, we define,

$$z = \frac{(q^3;q^4)_\infty (q^4;q^4)_\infty}{(q^7;q^4)_\infty (q^8;q^4)_\infty} \tag{6}$$

and

$$t = \frac{(q^4;q^5)_\infty (q^5;q^5)_\infty}{(q^9;q^5)_\infty (q^{10};q^5)_\infty} \tag{7}$$

Next we find equations between two or three or three variables using above defined q-Pochhammer symbols in term of variables. The following equations can be verified using any mathematical software like Mathematica, Maple etc. Here, we pause the calculations and write the equations directly. Finally, we also draw graphs of these equations by letting letting left sides of these equations as a new surface. In fact, it is in the form of q-Pochhammer symbols if we back substitute the values of the variables x, y, z and t etc. These surfaces have been depicted in , fig 2 fig 3 and fig 4.

Equation Between x and y

Theorem 2.4 *For* $0 < q < 1$, *let*

$$x = \frac{(q;q^2)_\infty (q^2;q^2)_\infty}{(q^3;q^2)_\infty (q^4;q^2)_\infty} \text{ and } y = \frac{(q^2;q^3)_\infty (q^3;q^3)_\infty}{(q^5;q^3)_\infty (q^6;q^3)_\infty} \quad \textit{then,}$$

$$-x^5 + 6x^4 - 9x^3 - 7xy^2 - 5x^3y + 15x^2y + y^3 = 0$$

Proof. If we substitute $q \to \frac{v}{1+v}$ in consequences 1 and 2, we get

$$x = \frac{1+2v}{(1+v)^3} \quad \text{and} \quad y = \frac{6v^3+9v^2+5v+1}{(1+v)^5}$$

Eliminating v from above two equations, we get the desired result.

$$-x^5 + 6x^4 - 9x^3 - 7x(y)^2 - 5x^3y + 15x^2y + y^3 = 0 \tag{8}$$

and $f(x,y) = -x^5 + 6x^4 - 9x^3 - 7xy^2 - 5x^3y + 15x^2y + y^3$, is depicted below in Fig.1.

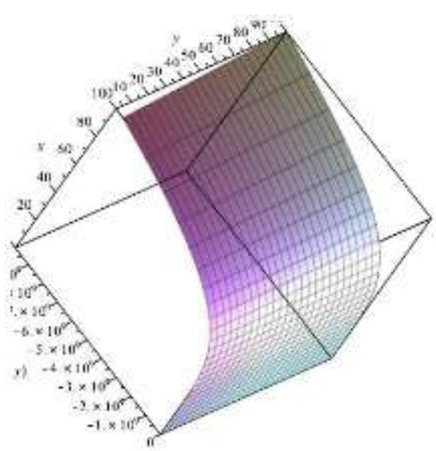

Fig.1. Equation Between y and z

Theorem 2.5 *For* $0 < q < 1$ *and let* $y = \frac{(q^2;q^3)_\infty (q^3;q^3)_\infty}{(q^5;q^3)_\infty (q^6;q^3)_\infty}$ and $z = \frac{(q^3;q^4)_\infty (q^4;q^4)_\infty}{(q^7;q^4)_\infty (q^8;q^4)_\infty}$ *then,*

$$20y^4z - 7y(z)^4 - 2y^5z + z^5 + 19y^2z^3 - y^7 + 4y^6 - 8y^5 - 26y^3z^2 = 0$$

Proof. If we substitute $q \to \frac{v}{1+v}$ in consequences 2 and 3, we get,

$$y = \frac{6v^3+9v^2+5v+1}{(1+v)^5} \quad \text{and} \quad z = \frac{1+7v+21v^2+34v^3+30v^4+12v^5}{(1+v)^7} \tag{9}$$

Eliminating v from above two equations, we get the desired result.

$$20y^4z - 7y(z)^4 - 2y^5z + z^5 + 19y^2z^3 - y^7 + 4y^6 - 8y^5 - 26y^3z^2 = 0 \tag{10}$$

and $f(y,z) = 20y^4z - 7y(z)^4 - 2y^5z + z^5 + 19y^2z^3 - y^7 + 4y^6 - 8y^5 - 26y^3z^2$, is depicted below in Fig.2.

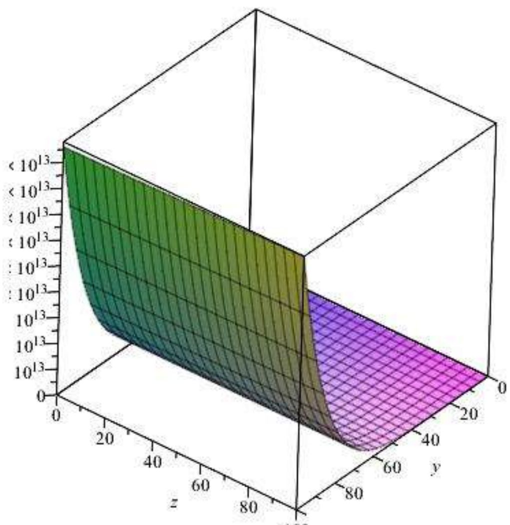

Fig.2. Equation Between z and t

Theorem 2.6 *For* $0 < q < 1$, *if* $z = \frac{(q^3;q^4)_\infty (q^4;q^4)_\infty}{(q^7;q^4)_\infty (q^8;q^4)_\infty}$ and $t = \frac{(q^4;q^5)_\infty (q^5;q^5)_\infty}{(q^9;q^5)_\infty (q^{10};q^5)_\infty}$ *then,*

$$-z^9 - 9t^2z^6 + 7tz^7 + 5z^8 + t^7 - 2t^5z^2 - 18t^4z^3 + 32t^3z^4 + 10t^2z^5$$
$$-11tz^6 - 15z^7 - 6t^6 + 13t^5z + 5t^4z^2 - 4t^3z^3 - 32t^2z^4 + 25z^6 + 9t^5$$
$$-39t^4z + 64t^3z^2 - 64t^2z^3 + 55tz^4 - 25z^5 = 0$$

Proof. If we substitute $q \to \frac{v}{1+v}$ in consequences 3 and 4, we get

$$z = \frac{1+7v+21v^2+34v^3+30v^4+12v^5}{(1+v)^7}$$

and

$$t = \frac{v^9 - v^5(1+v)^4 - v^4(1+v)^5 + (1+v)^9}{(1+v)^9}$$

Eliminating v from above two equations and encounter

$$-z^9 - 9t^2z^6 + 7tz^7 + 5z^8 + t^7 - 2t^5z^2 - 18t^4z^3 + 32t^3z^4 + 10t^2z^5$$
$$-11tz^6 - 15z^7 - 6t^6 + 13t^5z + 5t^4z^2 - 4t^3z^3 - 32t^2z^4 + 25z^6 + 9t^5$$
$$-39t^4z + 64t^3z^2 - 64t^2z^3 + 55tz^4 - 25z^5 = 0$$

and

$$f(z,t) = z^9 - 9t^2z^6 + 7tz^7 + 5z^8 + t^7 - 2t^5z^2 - 18t^4z^3 + 32t^3z^4 + 10t^2z^5 -11tz^6 - 15z^7 - 6t^6 + 13t^5z + 5t^4z^2 - 4t^3z^3 - 32t^2z^4 + 25z^6 + 9t^5 - 39t^4z + 64t^3z^2 - 64t^2z^3 + 55tz^4 - 25z^5 = 0,$$

is depicted below in Fig.3 given below

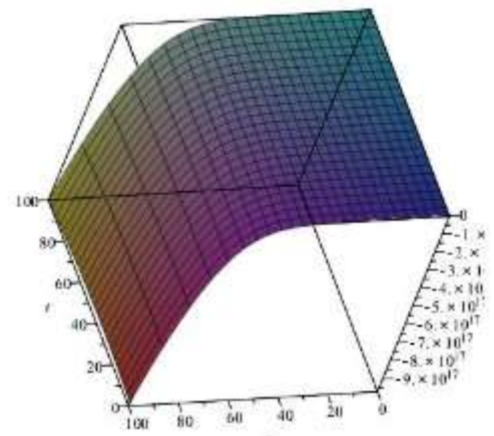

Fig.3.

Corollary 2.7

$$(1-q^{n+1})(1-q^n) = \frac{(q^n,q^{n+1})_\infty(q^{n+1},q^{n+1})_\infty}{(q^{2n+1},q^{n+1})_\infty(q^{2n+2},q^{n+1})_\infty},$$

$0 < q < 1$ *and* $n \in N^+$. (11)

Replacing a by q^2 and b by zq^2 in Theorem 2.2, we have the following useful proposition.

Proposition 2.8 *If* $z > 0$, and $q < 1$ then,

$$(1-q)(1-zq) = \frac{(q,q)_\infty(zq,q)_\infty}{(q^2,q)_\infty(zq^2,q)_\infty}$$

In the following theorems, we use above results to validate the results regarding q-Bracket, q-Factorial and q-Binomial coefficients in term of q-Pochhammer Symbols as given in [4]. We prove (1) and rest of the results can be validated alike.

Theorem 2.9

1. $[k]_q = \frac{1}{(1-q)^2}\frac{(q,q)_\infty(q^k,q)_\infty}{(q^2,q)_\infty(q^{k+1},q)_\infty}$
2. *If* $0 < q < 1$ *and* $n \in N$, *then* $[n]_q! = \frac{(q,q)_\infty^2}{(1-q^2)(q^2,q)_\infty(q^{n+1},q)_\infty}$
   Where, $[n]_q!$ *denotes the* q *Factorial Function*
3. *If* $0 < q < 1$ *and* $n, r \in N$, *Then*

$$\binom{n}{r} = \frac{(q^{r+1},q)_\infty(1-q^2)(q^2,q)_\infty(q^{n-r+1},q)_\infty}{(q^{n+1},q)_\infty(q,q)_\infty^2}$$

Where $\binom{n}{r}$ denotes the q Binomial Coefficient

1.
Proof. Replacing q^{k-1} by z in Proposition 2.8, we find that,

$$(1-q)(1-q^k) = \frac{(q,q)_\infty (q^k,q)_\infty}{(q^2,q)_\infty (q^{k+1},q)_\infty} \tag{17}$$

We know that, $[n]_q = \frac{1-q^n}{1-q}$. This implies that $[k]_q = \frac{1-q^k}{1-q}$. But then

$$[k]_q = \frac{1}{(1-q)^2} \frac{(q,q)_\infty (q^k,q)_\infty}{(q^2,q)_\infty (q^{k+1},q)_\infty} \tag{18}$$

Corollary 2.10 $\frac{1}{(1-q)^n} = \frac{(q;q)_\infty}{(1-q^2)(q^2,q)_\infty}$

Proof. For n, we know that, $[n]_q! = \Gamma_q(n+1)$, then

$$\Gamma_q(n) = \frac{(q;q)_\infty}{(q^n;q)_\infty}(1-q)^{1-n} \tag{21}$$

Replacing n by $n+1$, we have

$$\Gamma_q(n+1) = \frac{(q;q)_\infty}{(q^{n+1};q)_\infty}(1-q)^{-n}$$

$$= \frac{(q;q)_\infty}{(q^{n+1};q)_\infty (1-q)^n} \tag{22}$$

Using q factorial and equation (22), we obtain,

$$\frac{(q;q)_\infty}{(q^{n+1};q)_\infty (1-q)^n} = \frac{(q,q)_\infty^2}{(1-q^2)(q^2,q)_\infty (q^{n+1},q)_\infty}$$

Thus,

$$\frac{1}{(1-q)^n} = \frac{(q;q)_\infty}{(1-q^2)(q^2,q)_\infty}$$

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