SOME PROPERTIES OF MEROMORPHIC ALPHA-CONVEX FUNCTIONS AND ITS APPLICATIONS

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Abstract. The aim of the present paper is to obtain sufficient condition for the class of meromorphic alpha convex functions of order \( \xi \) and then to study mapping properties of an integral operator. Many known results appear as special consequences of our work.

Keywords: Meromorphic alpha convex functions; Integral operator

1. Introduction. Let \( \Sigma (n) \) denote the class of meromorphic functions \( f(z) \) normalized by

\[
f(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_k z^k, \quad (1.1)
\]

which are analytic in the punctured unit disk \( U^* = \{ z : 0 < |z| < 1 \} \). In particular, \( \Sigma (1) = \Sigma \). For \( \lambda \) is real with \( |\lambda| < \frac{\pi}{2} \), \( \alpha \geq 0 \), \( 0 \leq \xi < 1 \), \( n \in \mathbb{N} \), we denote by \( \Sigma S(\lambda, n, \xi) \), \( \Sigma C(\lambda, n, \xi) \) and \( \Sigma M(\lambda, n, \alpha, \xi) \), the subclasses of \( \Sigma (n) \) consisting of all meromorphic functions of the form (1.1) which are defined, respectively, by

\[
-\text{Re} e^{-i\lambda} \frac{zf'(z)}{f(z)} > \xi \cos \lambda, \quad (z \in U^*), \quad (1.2)
\]

\[
-\text{Re} e^{-i\lambda} \left( \frac{zf''(z)}{f'(z)} \right) > \xi \cos \lambda, \quad (z \in U^*), \quad (1.3)
\]

\[
-\text{Re} e^{-i\lambda} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)} \right\} > \xi \cos \lambda, \quad (z \in U^*). \quad (1.4)
\]

Making \( \lambda = 0 \), \( n = 1 \) in (1.2), (1.3) and (1.4), we get the well-known subclasses of \( \Sigma \) consisting of meromorphic functions which are starlike, convex and alpha convex of order \( \xi \) \( (0 \leq \xi < 1) \) respectively. For detail of the classes defined by (1.2), (1.3), (1.4) and related topics, we refer the work of Rosihan and Ravichandran [1], Goyal and Prajapat [2], Joshi and Srivastava [3], Liu and Srivastava [4], Raina and Srivastava [5], Xu and Yang [6] and Owa et al [7].

For \( f(z) \in \Sigma \), Wang [8] and Nehari and Netanyahu [9] introduced and studied the subclass \( \Sigma_{\alpha}\tau \) of \( \Sigma \) consisting of functions \( f(z) \) satisfying

\[
-\text{Re} \frac{(zf'(z))'}{f'(z)} < \tau, \quad (\tau > 1, \ z \in U^*). \quad (1.5)
\]
We now define a subclass $\Sigma^{N}(\lambda, n, \alpha, \tau)$ of $\Sigma(n)$ consisting of functions $f(z)$ of the form (1.1) satisfying
\[-Re e^{i \lambda} \left(1 - \frac{z}{f(z)}\right) + \frac{z f''(z)}{f'(z)} + \alpha \left(\frac{zf''(z)}{f'(z)}\right)' < \tau \cos \lambda, \quad (\tau > 1, \ z \in \mathbb{U}^*). \quad (1.5)\]

Integral operators for different classes of analytic, univalent functions in the open unit disk are studied by various authors, see [10, 11, 12, 13, 14, 15, 16]. We now consider the following general integral operator of meromorphic functions
\[G_{m}(z) = I_{m}(\delta, \alpha, \tau; f_{m}(z)) = \left\{ \frac{\delta z}{z^{2}} \left(\frac{t^{\delta-1} \alpha}{\alpha} t (f_{m}(t)) \right)^{\alpha} \right\}^{1/\delta}. \quad (1.6)\]
For $\delta = 1$, we obtain the integral operator $I_{m}(f_{m}(z))$ introduced and studied by Mohammed and Darus [17].

Sufficient conditions for the class were studied by various authors for different subclasses of analytic and multivalent functions, for some of the related work see [18, 19, 20, 21]. The object of the present paper is to obtain sufficient conditions for the class $\Sigma M(\lambda, n, \alpha, \xi)$ and then study mapping properties of the integral operator given by (1.6).

We will assume throughout our discussion, unless otherwise stated, that $\lambda$ is real with $|\lambda| < \frac{\pi}{2}$, $0 \leq \xi < 1$, $\tau > 1$, $n \in \mathbb{N}$, $\alpha, \delta > 0$ for $f \in \{1, \ldots, m\}$, $\delta > 0$, $\alpha, \delta > 0$ and
\[J_{n}(f) = (1 - \alpha) \frac{zf''(z)}{f'(z)} + \alpha \left(\frac{zf''(z)}{f'(z)}\right)'. \quad (1.7)\]
To obtain our main results, we need the following Lemma.

**Lemma 1.1** [21]. If $q(z) \in \Sigma(n)$ with $n \geq 1$ and satisfies the condition
\[|z^2 q'(z) + 1| < \frac{n}{\sqrt{n^2 + 1}} \quad (z \in \mathbb{U}^*), \]
then
\[q(z) \in \Sigma S(n). \]

**2. Sufficiency criteria for the class. $\Sigma M(\lambda, n, \alpha, \xi)$**

**Theorem 2.1.** If $f(z) \in \Sigma(n)$ satisfies
\[\left\{ \left( z f(z) \left(\frac{-zf''(z)}{f'(z)}\right)^{\alpha} \right) \right\}^{1/\xi \cos \lambda} e^{i \lambda} J_{n}(f) + \xi \cos \lambda + i \sin \lambda \]
\[+ (1 - \xi) \cos \lambda | < \frac{n}{\sqrt{n^2 + 1}} (1 - \xi) \cos \lambda \quad (z \in \mathbb{U}^*), \quad (2.1)\]
then $f(z) \in \Sigma M(\lambda, n, \alpha, \xi)$, where $J_{n}(f)$ is given by (1.7).

**Proof.** Let us set a function $q(z)$ by
\[q(z) = \frac{1}{z} \left( z f(z) \left(\frac{-zf''(z)}{f'(z)}\right)^{\alpha} \right) \left(\frac{e^{i \lambda}}{1 - \xi} \cos \lambda \right) = \frac{1}{z} + \frac{\alpha e^{i \lambda} \alpha^{n} \lambda^{n}}{(1 - \xi) \cos \lambda} \ldots \quad (2.2)\]
for $f(z) \in \Sigma(n)$. Then clearly (2.2) shows that $q(z) \in \Sigma(n)$.

Logarithmic differentiating of (2.2) gives
\[\frac{q'(z)}{q(z)} = \frac{e^{i \lambda}}{(1 - \xi) \cos \lambda} \left[ (1 - \alpha) \frac{zf''(z)}{f'(z)} + \alpha \left(\frac{zf''(z)}{f'(z)}\right)' + \frac{1}{z} \right] - \frac{1}{z} \quad (2.3)\]
which further implies
\[|z^2 q'(z) + 1| = \left| z f(z) \left(\frac{-zf''(z)}{f'(z)}\right)^{\alpha} \right\}^{1/\xi \cos \lambda} e^{i \lambda} \left(\frac{1}{1 - \xi} \cos \lambda \right) \]
\[\left[ J_{n}(f) + \xi \cos \lambda + i \sin \lambda \right] + 1. \]
Thus using (2.1), we get
\[ |z^2 q'(z) + 1| \leq \frac{n}{\sqrt{n^2 + 1}}, \quad (z \in \mathbb{U}_*). \]
Therefore by Lemma 1.1, we have \( q(z) \in \Sigma S(n) \).

From (2.3), we can write
\[ \frac{zq'(z)}{q(z)} = \frac{1}{(1 - \xi) \cos \lambda} \left[ e^{i\lambda} J_\alpha(f) + \xi \cos \lambda + i \sin \lambda \right]. \]
Since \( q(z) \in \Sigma S(n) \), it implies that \( Re \left( -\frac{zq'(z)}{q(z)} \right) > 0 \). Therefore, we get
\[ \frac{1}{(1 - \xi) \cos \lambda} \left[ -Re e^{i\lambda} J_\alpha(f) - \xi \cos \lambda \right] = Re \left( -\frac{zq'(z)}{q(z)} \right) > 0 \]
or
\[ -Re e^{i\lambda} J_\alpha(f) > \xi \cos \lambda. \]
and therefore \( f(z) \in \Sigma M(\lambda, n, \alpha, \xi). \)

By taking \( \alpha = 0 \) and \( \alpha = 1 \) in Theorem 2.1, we obtain Corollary 2.2 and Corollary 2.3 respectively.

**Corollary 2.2.** If \( f(z) \in \Sigma(n) \) satisfies
\[ \left| \left( zf(z) \right) e^{i\lambda} \left[ e^{i\lambda} z f'(z) + \xi \cos \lambda + i \sin \lambda \right] + (1 - \xi) \cos \lambda \right| < \frac{n(1 - \xi) \cos \lambda}{\sqrt{n^2 + 1}}, \quad (z \in \mathbb{U}_*), \quad (2.4) \]
then \( f(z) \in \Sigma S(\lambda, n, \xi) \).

**Corollary 2.3.** If \( f(z) \in \Sigma(n) \) satisfies
\[ \left| -z^2 f'(z) \right| e^{i\lambda} \left[ e^{i\lambda} \left( \frac{zf'(z)}{f'(z)} + 1 \right) + \xi \cos \lambda + i \sin \lambda \right] + (1 - \xi) \cos \lambda \left( \frac{n}{\sqrt{n^2 + 1}} \right), \quad (z \in \mathbb{U}_*), \quad (2.5) \]
then \( f(z) \in \Sigma S(\lambda, n, \xi) \).

3. **Mapping properties of the integral operator.** \( G_m(z) \).

**Theorem 3.1.** For \( j \in \{1, \ldots, m\} \), let \( f_j(z) \in \Sigma(n) \) and satisfy (2.4). If
\[ \delta z^{2} G_{m}^{\delta^{-1}}(z) - G_{m}^{\delta}(z) = \frac{2m}{\delta + m} (z f_j(z))^m, \]
then \( G_m(z) \in \Sigma S(\lambda, n, \delta, \eta) \) with \( \eta > 1 \) and \( G_m(z) \) is given by (1.6).

**Proof.** From (1.6), we obtain
\[ \delta z^2 G_{m}^{\delta^{-1}}(z) - G_{m}^{\delta}(z) = \frac{2m}{\delta + m} (z f_j(z))^m. \]

Divide both sides by \( z G_{m}^{\delta^{-1}}(z) \), we have
\[ \frac{\delta z G_{m}(z) + (p + 1) G_{m}(z) = \delta z^{2} G_{m}^{\delta^{-1}}(z) - G_{m}^{\delta}(z)}{\delta z G_{m}(z) + 2 G_{m}(z)} \]
\[ = (\delta - 2) \left( 1 + (1 - \delta) \frac{G_{m}(z)}{G_{m}(z)} + \frac{m}{\delta + m} \frac{f_j(z)}{f_j(z)} + \frac{1}{z} \right) \quad (3.2) \]

Now by simple computation, we get
\[ \left( 1 - \frac{1}{\delta} \right) \frac{G_{m}(z)}{G_{m}(z)} + \frac{1}{\delta} \frac{G_{m}(z)}{G_{m}(z)} = \frac{1}{\delta + 1} \frac{G_{m}(z)}{G_{m}(z)} + \frac{1}{\delta} \left( 1 - \frac{1}{\delta} \right) \]
\[ + \frac{G_{m}(z)}{z G_{m}(z)} \left( \frac{f_j(z)}{f_j(z)} + 1 \right) + 2 (\delta - 2) \],
or, equivalently we have

$$-e^{i\lambda} \left\{ \left( 1 - \frac{1}{\delta} \right) \frac{zG_m'(z)}{G_m(z)} + \frac{1}{\delta} \frac{(zG_m'(z))'}{G_m'(z)} \right\} = \frac{1}{\delta} \sum_{j=1}^{m} \alpha_j \left( -e^{i\lambda} \frac{zf_j'(z)}{f_j(z)} - e^{i\lambda} \right)$$

$$+ \frac{1}{\delta} (4 - \delta) e^{i\lambda} + \frac{G_m(z)}{zG_m'(z)} \left[ 2 - \delta \sum_{j=1}^{m} \alpha_j \left( \frac{zf_j'(z)}{f_j(z)} + 1 \right) \right] (1 + 1) e^{i\lambda},$$

By taking real part on both sides, we obtain

$$-\text{Re} e^{i\lambda} \left\{ \left( 1 - \frac{1}{\delta} \right) \frac{zG_m'(z)}{G_m(z)} + \frac{1}{\delta} \frac{(zG_m'(z))'}{G_m'(z)} \right\} = \frac{1}{\delta} \sum_{j=1}^{m} \alpha_j \left( -\text{Re} e^{i\lambda} \frac{zf_j'(z)}{f_j(z)} - \cos \lambda \right)$$

$$+ \frac{1}{\delta} (4 - \delta) \cos \lambda + \text{Re} \left[ \frac{G_m(z)}{zG_m'(z)} \left[ (2 - \delta) \sum_{j=1}^{m} \alpha_j \left( \frac{zf_j'(z)}{f_j(z)} + 1 \right) \right] (1 + 1) e^{i\lambda},$$

which further implies that

$$-\text{Re} e^{i\lambda} \left\{ \left( 1 - \frac{1}{\delta} \right) \frac{zG_m'(z)}{G_m(z)} + \frac{1}{\delta} \frac{(zG_m'(z))'}{G_m'(z)} \right\} \leq \frac{1}{\delta} \sum_{j=1}^{m} \alpha_j \left( -\text{Re} e^{i\lambda} \frac{zf_j'(z)}{f_j(z)} - \cos \lambda \right)$$

$$+ \frac{1}{\delta} (4 - \delta) \cos \lambda + \frac{G_m(z)}{zG_m'(z)} \left[ (2 - \delta) \sum_{j=1}^{m} \alpha_j \left( \frac{zf_j'(z)}{f_j(z)} + 1 \right) \right] 2e^{i\lambda},$$

Let

$$\eta = \left| \frac{G_m(z)}{zG_m'(z)} \left[ (2 - \delta) \sum_{j=1}^{m} \alpha_j \left( \frac{zf_j'(z)}{f_j(z)} + 1 \right) \right] 2e^{i\lambda} \right|$$

$$+ \frac{1}{\delta} (4 - \delta) + \frac{1}{\delta} \sum_{j=1}^{m} \alpha_j \left( -\frac{1}{\cos \lambda} \text{Re} e^{i\lambda} \frac{zf_j'(z)}{f_j(z)} - \cos \lambda \right).$$

Clearly we have

$$\eta > \frac{1}{\delta} (4 - \delta) + \frac{1}{\delta} \sum_{j=1}^{m} \alpha_j \left( -\text{Re} e^{i\lambda} \frac{zf_j'(z)}{f_j(z)} - \cos \lambda \right).$$

Then by using (3.1) and Theorem 2.1 with \( \alpha = 0 \), we obtain

$$\eta > \frac{1}{\delta} \sum_{j=1}^{m} \alpha_j \left( \xi - 1 + (4 - \delta) \right) > 1.$$

Therefore \( G_m(z) \in \Sigma_N(\lambda, n, \delta, \eta) \) with \( \eta > 1. \)

REFERENCES


